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Form Factors in $K_{\mu 3}$ and K_{e3} Decay*

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An investigation of the form factors in $K_{\mu 3}$ and K_{e3} decay amplitudes, on the assumption of local production of the fermion pair, is carried out by means of dispersion relation techniques. The approximation is made of taking explicit account of K-π scattering in the unitarity condition and incorporating all the other contributions into a single arbitrary parameter. The validity of this approximation is discussed. The results are expressed in terms of S and P phase shifts for K-π scattering.

1. INTRODUCTION

A SYSTEMATIC investigation of decay processes has been lately carried out within the framework of a universal Fermi interaction with V-A coupling.[1] However, with regard to $K_{\mu 3}$ and K_{e3} one has not yet come to a definite conclusion. The theoretical analysis predicts that for each given type of coupling the dependence on the lepton energy is well determined, whereas the effects of strong interactions appear as form factors which depend only on the pion energy.[2] There have been some arguments to the effect that these form factors are rather slowly varying functions of the pion energy over the physical region for the decay processes. It is the purpose of this work to investigate this question in some more detail making use of dispersion theory.

2. DETERMINATION OF THE FORM FACTORS FROM DISPERSION RELATIONS

We consider the processes

$$K_{\mu 3}\rightarrow \pi+\mu+\nu, \qquad \text{(I)}$$

$$K_{e3}\rightarrow \pi+e+\nu, \qquad \text{(II)}$$

in which the K meson has momentum p_K, the outgoing particles have momenta p_π, p, and q, respectively. The matrix element for this decay process on the assumption of local production of the fermion pair may be written in the form:

$$\begin{aligned}(4p_{\pi 0}p_{K0}p_0q_0)^{\frac{1}{2}}\langle\pi\mu\nu|H_{\text{weak}}|K\rangle &= (2\pi)^4\delta(p_\pi+p+q-p_K)(4p_{\pi 0}p_{K0})^{\frac{1}{2}}\\ &\quad\times\langle\pi|J_\alpha(0)|K\rangle\bar{u}(p)\gamma_\alpha(1-\gamma_5)v(q). \qquad (1)\end{aligned}$$

We have, in addition, restricted to V or A the type of coupling and used the two-component theory of the neutrino. Taking the Λ parity, by convention, to be the same as that of the nucleons, it follows from conservation of parity in strong interactions that the vector or pseudovector character of the baryon current is associated with negative or positive parity of K-mesons.

The strong interaction effects are contained in the factor $\langle\pi|J_\alpha(0)|K\rangle$ which, in the K-meson rest system, depends only on the pion energy.

From Lorentz invariance it follows that

$$\begin{aligned}(4p_{\pi 0}p_{K0})^{\frac{1}{2}}\langle\pi|J_\alpha{}^I(0)|K\rangle &= \tfrac{1}{2}(p_K+p_\pi)_\alpha f_+{}^I+\tfrac{1}{2}(p_K-p_\pi)_\alpha f_-{}^I, \qquad (2)\end{aligned}$$

where the index $I=\frac{1}{2},\frac{3}{2}$ indicates that $J_\alpha{}^I(0)$ transforms like an isospinor of the respective rank, and the f's are functions of

$$\omega=-p_\pi\cdot p_K/m_Km_\pi. \qquad (3)$$

We assume that each covariant amplitude has a representation of the form

$$f(\omega)=\frac{1}{\pi}\int_1^\infty\frac{\rho(\omega')}{\omega'-\omega}d\omega', \qquad (4)$$

where

$$\begin{aligned}\tfrac{1}{2}(p_K+p_\pi)_\alpha\rho_+(\omega)+\tfrac{1}{2}(p_K-p_\pi)_\alpha\rho_-(\omega) &= (2p_{K0})^{\frac{1}{2}}\textstyle\sum_n\delta(p_n+p_\pi-p_K)\\ &\quad\times\langle 0|J_\alpha(0)|n\rangle\langle n|j_\pi(0)|k\rangle. \qquad (5)\end{aligned}$$

The lower limit $\omega'=1$ in (4) comes from the threshold energy of the least massive state $|n\rangle=|K\pi\rangle$.

As in similar problems,[3] it is practically impossible to take into account the contributions from intermediate states with more than two particles. Therefore we shall neglect in the sum all but the terms which come from $|n\rangle=|K\pi\rangle$ and $|n\rangle=|N\bar{Y}\rangle$,[4] in the hope that the contribution from other states is relatively small. On the other hand, the matrix elements involved in (5) vanish except for those intermediate states with total angular momentum $j=0$, 1. The equations for $f(\omega)$ become decoupled when expressed in terms of transition amplitudes in states of given angular momentum, in the center of mass of the (K,π) system. The connection between $f_\pm$ and f_j is

$$f_0=f_-+\frac{m_K{}^2-m_\pi{}^2}{W^2}f_+;\quad f_1=f_+, \qquad (6)$$

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[3] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); **111**, 355 (1958). Federbush, Goldberger, and Treiman, Phys. Rev. **112**, 642 (1958).
[4] Throughout this paper Y stands for Λ or Σ.

and

$$f_{-}=f_0-\frac{m_K^2-m_\pi^2}{W^2}f_1. \tag{7}$$

We obtain likewise for f_0 and f_1 a representation of the form (4) except that f_0 has a pole at $W^2=0$ ($\omega=\omega_0$) with residue $(m_K^2-m_\pi^2)f_1(\omega_0)$. The relation between W^2 and ω is

$$W^2=m_K^2+m_\pi^2+2m_Km_\pi\omega. \tag{8}$$

In the two-particle approximation the weight functions are given by

$$\rho_0(\omega)=kf_0(\omega)\phi_0(\omega)^*\theta[W^2-(m_K+m_\pi)^2] +k'F_0(\omega)\psi_0(\omega)^*\theta[W^2-(M_Y+M)^2], \tag{9}$$

$$\rho_1(\omega)=kf_1(\omega)\phi_1(\omega)^*\theta[W^2-(m_K+m_\pi)^2] +k'F_1(\omega)\psi_1(\omega)^*\theta[W^2-(M_Y+M)^2], \tag{10}$$

where θ is the step function; k and k' are the momenta in the center of mass of the systems (K,π) and $(N,\bar{Y})$, respectively, related to the total energy W by

$$k^2=[W^2-(m_K+m_\pi)^2][W^2-(m_K-m_\pi)^2]/4W^2,$$
$$k'^2=[W^2-(M_Y+M)^2][W^2-(M_Y-M)^2]/4W^2;$$

$\phi_{0,1}$ and $\psi_{0,1}$ are the (S,P) amplitudes for the transitions

$$K+\pi\rightarrow K+\pi,$$
$$N+\bar{Y}\rightarrow K+\pi,$$

respectively; $F_0(\omega)$ is the scalar part of $\langle 0|J_\alpha(0)|N\bar{Y}\rangle$, and $F_1(\omega)$ is the projection of its vector part in the direction of $\mathbf{k}$.

Within the framework of a primary Fermi interaction the last term in (9, 10) is the source of the decay process while the first term gives an induced contribution to the decay amplitudes due to K-π interaction. In order to get some insight of the details of the decay mechanism, let us investigate the general structure of the term with two-baryon intermediate state.

We can write

$$\left(\frac{p_{N0}p_{\bar{Y}0}}{MM_Y}\right)^{\frac{1}{2}}\langle 0|J_\alpha(0)|N\bar{Y}\rangle =\bar{u}(-p_{\bar{Y}})\gamma_\alpha(G_V-\gamma_5G_A)u(p_N), \tag{11}$$

and

$$\left(\frac{2p_{K0}p_{N0}p_{\bar{Y}0}}{MM_Y}\right)^{\frac{1}{2}}\langle N\bar{Y}|j_\pi(0)|K\rangle =\bar{u}(p_N)O(A+QB)\,u(-p_{\bar{Y}}), \tag{12}$$

where $O=1$ for pseudoscalar K-mesons and $O=\gamma_5$ for scalar K-mesons,[5] $Q=\frac{1}{2}(p_K+p_\pi)$, and $Q=\sum_\mu Q_\mu\gamma_\mu$. Therefore

$$\rho_0^{(2)}=\frac{2k'}{W^2}G\int_1^1[(p_{N0}M_Y\mp p_{\bar{Y}0}M)A -kk'z(p_{N0}-p_{\bar{Y}0})B]2\pi dz, \tag{13}$$

$$\rho_1^{(2)}=\frac{k'}{kW}G\int_{-1}^1[k'z(M_Y\pm M)A+k(-2k'^2z^2 +\tfrac{1}{2}(W^2-M_Y^2-M^2)\pm M_YM)B]2\pi dz, \tag{14}$$

and the upper (lower) signs stand for pseudoscalar (scalar) K-mesons. Since, in perturbation theory, for fixed z, A and B behave like $1/W^2$ for $W^2\rightarrow\infty$, it follows that $\rho_0\sim 1/W^2$ and $\rho_1\sim$const. It then follows that at least one subtraction is necessary in the spectral representation of f_1 but f_0 may still have none.

On the other hand, for small values of k' one has the following behavior of $\rho_0^{(2)}$ and $\rho_1^{(2)}$:

For pseudoscalar K-mesons,

$$\rho_0^{(2)}\sim k'^3(M_Y-M)/(M_Y+M);\quad \rho_1^{(2)}\sim k'MM_Y.$$

For scalar K-mesons,

$$\rho_0^{(2)}\sim k'MM_Y;\quad \rho_1^{(2)}\sim k'^3.$$

Therefore, on kinematic grounds only, one might expect a small contribution to f_0 from $\rho_0^{(2)}$ as compared to the contribution of $\rho_1^{(2)}$ to f_1, while the opposite holds for scalar K-mesons.

Assuming that K-mesons are indeed pseudoscalar, we are then justified in neglecting the contribution from $\rho_0^{(2)}$. Moreover we can cast the contribution from $\rho_1^{(2)}$ into the arbitrary constant which appears as the result of one subtraction in the spectral representation for f_1. This arbitrary constant represents the global result of all dynamical effects which cannot be taken into account individually.

We thus arrive at the following representations for f_0 and f_1:

$$f_0(\omega)=\frac{m_K^2-m_\pi^2}{W^2}f_1(\omega_0)+\frac{1}{\pi}\int_1^\infty\frac{\rho_0^{(1)}(\omega')}{\omega'-\omega}d\omega', \tag{15}$$

$$f_1(\omega)=f_1(\omega_0)+\frac{1}{\pi}\int_1^\infty\frac{W^2}{W'^2}\frac{\rho_1^{(1)}(\omega')}{\omega'-\omega}d\omega'. \tag{16}$$

These equations are of the Omnès type,[6] since $\rho^{(1)}(\omega)$ is given by the product of $f(\omega)$ by the amplitude for K-π scattering and kinematical factors. Therefore one can write down explicit solutions for f_0 and f_1 in terms of the S and P wave phase shifts for K-π scattering, defined by

$$\tan\delta_0=\frac{k\,\mathrm{Re}\phi_0}{1-k\,\mathrm{Im}\phi_0};\quad \tan\delta_1=\frac{k\,\mathrm{Re}\phi_1}{1-k\,\mathrm{Im}\phi_1}. \tag{17}$$

The solutions are

$$f_0(\omega)=\frac{m_K^2-m_\pi^2}{W^2}f_1(\omega_0) \times\exp\left(-\frac{1}{\pi}\int_1^\infty\frac{W^2}{W'^2}\frac{\delta_0(\omega')}{\omega'-\omega}d\omega'\right), \tag{18}$$

[5] We are assuming that Λ and Σ have the same parity. If they have opposite parities, then for $Y=\Sigma$ one should take $O=\gamma_5$ for pseudoscalar and $O=1$ for scalar K-mesons.

[6] R. Omnès, Nuovo cimento 8, 316 (1958).

$$f_1(\omega)=f_1(\omega_0)\exp\left(-\frac{1}{\pi}\int_1^\infty \frac{W^2}{W'^2}\frac{\delta_1(\omega')}{\omega'-\omega}d\omega'\right)=f_+(\omega), \quad (19)$$

which substituted into (7) gives

$$f_-(\omega)=\frac{m_K^2-m_\pi^2}{W^2}f_1(\omega_0)\left\{\exp\left(-\frac{1}{\pi}\int_1^\infty \frac{W^2}{W'^2}\frac{\delta_0(\omega')}{\omega'-\omega}d\omega'\right)-\exp\left(-\frac{1}{\pi}\int_1^\infty \frac{W^2}{W'^2}\frac{\delta_1(\omega')}{\omega'-\omega}d\omega'\right)\right\}. \quad (20)$$

The physical content of the approximations made is that the direct contribution from the primary weak interaction is taken into account in the determination of $f_+(\omega)$ but $f_-(\omega)$ is obtained with the neglect of those direct effects and therefore entirely induced by the K-π interaction. It is this circumstance that led us to a result which depends only on one arbitrary parameter $f_1(\omega_0)$ and is such that the ratio of the two amplitudes f_-/f_+ is independent of this parameter. If, however, the experiments do not confirm this remarkable feature of our results, one concludes that a subtraction is indeed necessary in (15) and that the direct contributions to f_- are not negligible.

3. EVALUATION OF THE FORM FACTORS

The physical range of values of ω in the decay process (I) is from $\omega_1=-1$ at the lower end of the pion spectrum to $\omega_2=-(m_K^2+m_\pi^2-m_\mu^2)/2m_Km_\pi$ at the upper end.

We shall now estimate the variation of $f_+(\omega)$ in this range. We find

$$\frac{1}{f_1}\frac{df_1}{d\omega}=\frac{\delta_1(\bar\omega)}{\pi}\frac{1}{1-\omega}, \quad (21)$$

where $\bar\omega>1$ depends on ω. The variation of $f_+(\omega)$ between the two ends of the spectrum is then approximately given by

$$\frac{\Delta f_+}{f_+(-1)}\simeq\frac{\delta_1(\bar\omega)}{\pi}\frac{\Delta\omega}{2}=\frac{\delta_1(\bar\omega)}{\pi}\frac{(m_K-m_\pi)^2-m_\mu^2}{4m_Km_\pi}=0.44\frac{\delta_1(\bar\omega)}{\pi}. \quad (22)$$

If there is a P-wave resonance at low enough energy an appreciable variation of $f_+(\omega)$ is expected. If however the P-wave is small, the variation is negligible.

We turn now to the analysis of the behavior of $f_-(\omega)$. In order to simplify the discussion we assume $\delta_1(\omega)$ negligible and $\delta_0(\omega)$ given by

$$\tan\delta_0=a(W^2-(m_K+m_\pi)^2)^{\frac{1}{2}}. \quad (23)$$

The integrations can then be performed yielding the following results:

(1) $a>0$ (repulsive S wave):

$$\frac{f_-(\omega)}{f_+(\omega_0)}=\frac{(m_K^2-m_\pi^2)a}{1+a(m_K+m_\pi)}\frac{1}{W^2}\times\{[(m_K+m_\pi)^2-W^2]^{\frac{1}{2}}-m_K-m_\pi\}<0. \quad (24)$$

The ratio

$$\frac{f_-(\omega_2)}{f_-(\omega_1)}\simeq\frac{1}{2}\left[1+\left(\frac{2m_Km_\pi}{m_K^2+m_\pi^2}\right)^{\frac{1}{2}}\right]=0.85 \quad (25)$$

is independent of the scattering length.

(2) $a<0$ (attractive S wave):

$$\frac{f_-(\omega)}{f_+(\omega_0)}=\frac{(m_K^2-m_\pi^2)a}{1-a[(m_K+m_\pi)^2-W^2]^{\frac{1}{2}}}\frac{1}{W^2}\times\{[(m_K+m_\pi)^2-W^2]^{\frac{1}{2}}-m_K-m_\pi\}>0. \quad (26)$$

As a varies from 0 to $-\infty$ the ratio $f_-(\omega_2)/f_-(\omega_1)$ decreases from 0.85 to 0.6.

So far we have discussed the behavior of the individual amplitudes ignoring isotopic spin dependence. In general f_+ and f_- are linear combinations of amplitudes with isospin indices $I=\frac{1}{2}$ and $I=\frac{3}{2}$ which in turn depend on the K-π scattering amplitudes in the corresponding isostates. If, however, we adhere to the selection rule $I=\frac{1}{2}$ (which, for instance, arises from a Fermi interaction involving ΛP as the baryon pair), then there will be only one amplitude, namely that with $I=\frac{1}{2}$.

4. FINAL REMARKS

The analysis we have made is also valid if the weak interaction proceeds via an intermediate vector field of mass m. The only difference is that the transition amplitude (1) must then be multiplied by the boson propagator $(W^2-m^2)^{-1}[g_{\alpha\beta}-p_\alpha p_\beta/m^2]$.

The K_{e3} decay is naturally more appropriate for an investigation of the form factor $f_+(\omega)$ since $f_-(\omega)$ will not contribute in this case. On the other hand, information concerning $f_-(\omega)$ can only be obtained from data on $K_{\mu 3}$ decay. From (24) and (26) one can see that $f_-(\omega_1)/f_+(\omega_0)$ increases as $|a|\rightarrow\infty$ to a maximum absolute value 0.3 and 0.25, respectively. In addition, f_- appears multiplied by m_μ/m_K in the matrix element for the decay process. Therefore, from this analysis, one can conclude that the effect of f_- on the spectrum is practically negligible.

The results we have obtained and the assumptions underlying them can then be checked when these data become available.

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