depending on its (unaccelerated) motion. Effectively, then, the Lorentz contraction is invisible. Any hopes of seeing the contraction in a rapidly moving space vehicle or astronomical body must be discarded.

Although apparent distortion due to rapid motion can be seen by means of steroscopic vision or photography, it is not of the same type as one might expect from the Lorentz contraction.

None of the statements here should be construed as casting any doubt on either the observability or the reality of the Lorentz contraction, as all the results given are derived from the special theory of relativity.

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Active Gravitational Mass*

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Tolman states that ". . .disordered radiation in the interior of a fluid sphere contributes roughly speaking twice as much to the gravitational field of the sphere as the same amount of energy in the form of matter." The gravitational pull exerted by a system on a distant test particle might therefore at first sight be expected to increase if within the system a pair of oppositely charged electrons annihilate to produce radiation. This apparent paradox is analyzed here in the case where gravitational effects internal to the system are unimportant. It is shown that tensions in the wall of the container compensate the effect mentioned by Tolman so that the net gravitational pull exerted by the system does not change.

I. INTRODUCTION

IN Newtonian mechanics the equivalence of active and passive gravitational mass, that is of mass as a quantity which gives rise to, and as a quantity acted upon by, gravitational fields, is made obvious in the form of the familiar equation for the gravitational potential ϕ, namely $\nabla\phi=4\pi\rho$, where ρ is the density of inertial mass.

However, in relativity theory where the field equations take the form $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}$, the inference can sometimes not be drawn so easily. Here not only does the source term include stresses and momenta as well as energy, but the equations are nonlinear. The question presents itself, therefore, to what extent are the distant gravitational fields as calculated by classical and special relativity theory the same as those calculated using general relativity?

The following statement by Tolman suggests that there are important differences: ". . .disordered radiation in the interior of a fluid sphere contributes roughly speaking twice as much to the gravitational field of the sphere as the same amount of energy in the form of matter."[1]

Such a result would seem to lead to certain paradoxes. Consider the conversion of a gamma ray, enclosed in a box, into mass, say an electron-positron pair. This transformation might be thought to halve the contribution of the mass energy to distant gravitational fields.

However, we shall show here that the active gravitational mass of a system is made up of the energy of the walls and other material plus the energy of radiation, divided by c^2, without the added factor of two, provided that the gravitational fields internal to the system are weak.

II. ENCLOSED RADIATION

Tolman's argument is based upon an expression for the distant gravitational field which involves only the classical stress-energy tensor $T_{\mu\nu}$. The reasoning applies to a wide class of cases roughly describable as quasistatic. Included in such cases are those in which the matter is confined to some limited region. This region is considered to be small as compared to the distance at which its gravitational field is to be measured. Moreover, within this region the behavior of the system is not significantly influenced by its own gravitational field. When these conditions are satisfied, and when the distant metric field is expressed in a form,

$$ds^2=-(1+2m^*/r)(dx^2+dy^2+dz^2)+(1-2m^*/r)dt^2, \quad (1)$$

which reveals the mass of the system, $m=(c^2/G)m^*$, or its energy $E=mc^2=(c^4/G)m^*$, then Tolman's arguments[2] give for the energy of the system the value

$$mc^2=(c^4/G)m^*=\int(T_4{}^4-T_1{}^1-T_2{}^2-T_3{}^3)(-g)^{\frac{1}{2}}d^3x. \quad (2)$$

Since the electromagnetic stress-energy tensor has zero trace, it follows that $T_4{}^4$ equals $-(T_1{}^1+T_2{}^2+T_3{}^3)$. Therefore according to (2), Tolman argues, the system

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[1] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, 1934), p. 272.
[2] See Tolman, reference 1, p. 235, Eq. (92.3).

exerts twice the long-distance gravitational pull that would be expected if T_4^4 alone contributed.

However, the terms $T_1^1+T_2^2+T_3^3$ describe the radiation pressure. If this pressure is confined by nongravitational means, there must be walls under tension to counterbalance the pressure. The integrated value of these negative pressure terms or tensions will contribute to (2) a term which just counterbalances the increased gravitational effect of the confined radiation.

III. GENERAL CASE

Quite generally, if the conservation laws $(\partial/\partial x^\nu)\times(-g)^{\frac{1}{2}}T_\mu{}^\nu=0$ hold, (as they do in most cases except for small gravitational effects which we have assumed negligible) and if the system is quasi-static, we can show that the integral of $T_1^1+T_2^2+T_3^3$ is always zero. In physical terms, the pressure in the region where the radiation is contained balances the tension in the region of the walls. In mathematical terms, consider an x^2, x^3 plane, a_1, passing through the system. The rate of change of momentum, T_1^4 on either side of the plane is given by $\pm\int T_1^1(-g)^{\frac{1}{2}}da_1$. As the system is quasi-static, the time average of this integral must be zero. Consequently the time average of the integral $\int T_1^1(-g)^{\frac{1}{2}}d^3x=\int dx^1\int T_1^1(-g)^{\frac{1}{2}}da_1$ also vanishes.

Thus we can see that where conservation laws apply to the electromagnetic stress-energy tensor in and by itself, as they do when gravitational effects are negligible, then the above rule (2) from general relativity leads to the same result for the distant gravitational field as does the classical formula

$$mc^2=\int T_4^4(-g)^{\frac{1}{2}}d^3x. \qquad (3)$$

IV. COULOMB BINDING

In a concrete example let us see that Eqs. (2) and (3) give the same result. Consider the Coulomb binding of, say, an electron in an atom. We cannot evaluate integrals of the electromagnetic stress energy tensor directly, because of the self-energy difficulty, but we can evaluate changes in these integrals when the electron loses energy and spirals in from one state of motion to another. The virial theorem tells us that the field energy, Δ, which is lost, and which describes the change in potential energy of interaction of the particles, goes half into changing the kinetic energy of the particles and half into energy radiated away.

Thus when the energy due to the fields $\int(T_4^4)_F\times(-g)^{\frac{1}{2}}d^3x$ goes down by Δ, the energy due to the particles, $\int(T_4^4)_p(-g)^{\frac{1}{2}}d^3x$ goes up by $\frac{1}{2}\Delta$. Now as $(T_1^1+T_2^2+T_3^3)_F=-(T_4^4)_F$, the integrated field pressure term changes by $+\Delta$. The integral of the corresponding term for the particle, $(T_1^1+T_2^2+T_3^3)_p$, equals $-mv^2$, and therefore changes by the amount Δ also. For the total stress-energy tensor $T_\mu{}^\nu=(T_\mu{}^\nu)_F+(T_\mu{}^\nu)_p$, changes in the integral of $(T_1^1+T_2^2+T_3^3)$ just cancel, as required (Table I).

TABLE I. Changes in energy of particles and fields when radiation escapes from the system.

	Particles	Fields	Total
T_4^4	$+\frac{1}{2}\Delta$	$-\Delta$	$-\frac{1}{2}\Delta$ = change in inertial mass
$T_1^1+T_2^2+T_3^3$	$-\Delta$	$+\Delta$	0
		Total	$-\frac{1}{2}\Delta$ = change in active mass

V. THE QUASI-STATIC REQUIREMENT

Our results are based on Eq. (2) which holds, according to Tolman, when the integral

$$\frac{1}{8\pi}\int \mathfrak{g}^{\alpha\beta}\frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4{}^{\alpha\beta}}\right)d^3x,$$

on the time average equals zero.[2]

It is apparent that when gravitation effects are not important, then, if we write $\mathfrak{g}^{\alpha\beta}=\mathfrak{g}_c{}^{\alpha\beta}+\mathfrak{g}_t{}^{\alpha\beta}$, where $\mathfrak{g}_c{}^{\alpha\beta}$ is the time-independent part and $\mathfrak{g}_t{}^{\alpha\beta}$ is the small time-dependent part, the integral, can be rewritten as the sum of two terms, the term linear in $\mathfrak{g}_t{}^{\alpha\beta}$ giving zero on the time average, and the term quadratic in $\mathfrak{g}_t{}^{\alpha\beta}$ being negligible.

The relation (2), however, has a larger potential applicability. If

$$\mathfrak{g}^{\alpha\beta}\frac{\partial}{\partial t}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}_4{}^{\alpha\beta}}\right)=0 \qquad (4)$$

is introduced as a coordinate condition, then in these special coordinates (2) may always be applied. However, care must be exercised in interpreting the result. The boundary conditions on the distant field (1) are compatible with (4) as a coordinate condition, and therefore the integral (2) can be applied to determine the constant m^* in (1). However, it may be, and in general will be the case that (4) will be incompatible with $(\partial/\partial x^\nu)[(-g)^{\frac{1}{2}}T_\mu{}^\nu]=0$, because in the odd new system of coordinates gravitational forces will *not* any longer be negligible in comparison to the electromagnetic stresses. In such instances it will no longer be true that the integral $\int(T_1^1+T_2^2+T_3^3)(-g)^{\frac{1}{2}}d^3x$ equals zero.