found in the present experiment fall into a region in which the Weizsäcker-Williams picture is not valid.

Furthermore, the present results are not necessarily in disagreement with those electron experiments in which the measured cross section has appeared to be larger than that predicted by the theory. The theory takes into account only the type of interaction represented by the Feynman diagram of Fig. 4(A); the process portrayed by the diagram of Fig. 4(B) is neglected. Process B is certainly of negligible importance for incident particles as heavy as muons, but may make a considerable contribution to the probability of direct pair production by electrons. For completeness, it should also be noted that no direct pair theory has yet included the effect of exchange on the electron cross section, which would probably reduce the theoretical value.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the help of Dr. K. Greisen who suggested the problem and whose continuous help and advice were invaluable. They also wish to acknowledge the help of W. Pak who assisted in keeping the apparatus running during the latter half of the run and who scanned almost half of the cloud chamber pictures. Mrs. M. Nielsen scanned most of the hodoscope pictures.

PHYSICAL REVIEW VOLUME 116, NUMBER 4 NOVEMBER 15, 1959

General Relativistic Fluid Spheres

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(Received June 16, 1959)

In Part I of this paper certain well known results concerning the Schwarzschild interior solution are generalized to more general static fluid spheres in the form of inequalities comparing the boundary value of g_{44} with certain expressions involving only the mass concentration and the ratio of the central energy density to the central pressure. A minimal theorem appropriate to the relativistic domain is derived for the central pressure, corresponding to a well-known classical result. Inequalities involving the proper energy and the potential energy are also considered, as is the introduction of the physical radius in place of the coordinate radius. A singularity-free elementary algebraic solution of the field equations is presented and exact values obtained from it compared with the limits prescribed by some of the inequalities. In Part II an answer is given to the question whether the total amount of radiation emitted during the symmetrical gravitational contraction of an amount of matter whose initial energy, at complete dispersion, is W_0 can ever exceed W_0.

PART I

1. Introduction

VARIOUS questions concerning fluid spheres in static (thermodynamic) equilibrium in the context of the general relativity theory are treated not infrequently on the basis of special models, i.e., of special explicit solutions of the field equations, the best known of these probably being the so-called Schwarzschild interior solution.[1] In particular, it is known for this solution that—in terms of the usual coordinate system (see Appendix I)—the ratio of the total mass M to the (coordinate) radius R of the sphere cannot have a value greater than 4/9, or 5/18 if the trace of the energy-momentum tensor is postulated to be non-negative. In other words, although the quantity

$$\Delta = 1 - 2M/R \tag{1.1}$$

certainly must not be negative, this particular solution does not allow Δ to approach zero but prescribes the minimum value $\frac{1}{9}$. This limitation arises essentially from the condition that g_{44} must not be negative anywhere. It should be noted that the result $M \leq 4R/9$ depends upon the coordinates used. However, one may replace R by the physical radius R^* of the sphere, and then in this case one has $M \lesssim 0.3404R^*$, which represents an *invariant* limitation upon the mass of the sphere (invariantly defined in terms of the motion of a test particle "at infinity") if its radius be prescribed. Again, one may calculate quantities such as g_{44} at the center of that Schwarzschild sphere whose central pressure is just one third of the central energy density, and this turns out to be $\frac{1}{4}$, which shows clearly the strongly non-Galilean character of the metric; and so on.

Now, instead of having results of this kind available for a few special models it seems desirable to establish analogous general results for arbitrary static fluid spheres in the form of inequalities, the distributions being subject only to some limitations of a general kind. The present paper accordingly makes a simple attack on this problem. In Secs. 3 and 8 of Part I certain limitations on Δ are established, it being supposed alternatively that the density does not increase or does not decrease outwards; and the possibility is taken into

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[1] R. C. Tolman, Phys. Rev. 55, 364 (1939).

account of an *a priori* restriction upon the pressure to density ratio, such as that which arises from the postulate that the trace of the energy-momentum tensor be non-negative. Section 4 generalizes the result concerning the central value of g_{44} stated above for the Schwarzschild sphere to more general distributions. In Sec. 5, a minimal theorem is established for the central pressure, corresponding to a well-known classical result, while Sec. 6 deals with inequalities involving proper or potential energies. An explicit, entirely elementary algebraic, solution of the field equations, free from singularities, is dealt with in Sec. 7, and the exact numerical values computed from it compared with the limits given by some of the general inequalities, by way of example.

Part II deals with a problem of a rather different kind. It answers the question whether in the symmetrical gravitational contraction of matter which is originally in a state of complete dispersion—in which state its energy is W_0—the total amount of energy radiated can ever exceed W_0.

2. Auxiliary Equations

(a) It is convenient to introduce a number of auxiliary quantities in terms of which various equations take on an easily surveyable form. Define

$$\mu(r)=4\pi\int_0^r r^2\rho dr, \tag{2.1}$$

$$\bar{\rho}(r)=3\mu(r)/4\pi r^3. \tag{2.2}$$

These quantities correspond, respectively, to the mass and mean density interior to r as commonly used in astrophysics. It must, however, be understood that in the present context they are mere formal definitions which have no invariant significance. (The true mean density would be defined as $\int_0^r r^2e^{\lambda/2}\rho dr/\int_0^r r^2e^{\lambda/2}dr$.) With this in mind their classical names will nevertheless be retained. To avoid the continual recurrence of certain symbols one may also introduce the quantities

$$P=(4\pi k/c^4)p, \tag{2.3}$$

$$w=(4\pi k/3c^2)\bar{\rho}; \tag{2.4}$$

and

$$r^2=x, \tag{2.5}$$

$$e^{\nu/2}=\zeta, \tag{2.6}$$

$$y=(1-2r^2w)^{\frac{1}{2}}. \tag{2.7}$$

Equation (I.3) of Appendix I now shows at once that if e^λ has no singularity at the origin then

$$e^{-\lambda}=y^2. \tag{2.8}$$

(I.2) then becomes

$$P=2y^2\zeta_{,x}/\zeta-w, \tag{2.9}$$

where the subscript following a comma denotes differentiation with respect to x, while (I.4) becomes

$$P_{,x}=-(P+rw'+3w)\zeta_{,x}/\zeta. \tag{2.10}$$

$\zeta_{,x}/\zeta$ may be eliminated between (2.9) and (2.10), giving

$$P'=-ry^{-2}(P+w)(P+rw'+3w). \tag{2.11}$$

Alternatively one may eliminate P, in which case one has

$$(1-2xw)\zeta_{,xx}-(xw_{,x}+w)\zeta_{,x}-\tfrac{1}{2}w_{,x}\zeta=0. \tag{2.12}$$

It will be seen that the last equation is linear in ζ if w, that is ρ, be prescribed, and that it is also linear in w if ζ be prescribed. It is therefore a very convenient starting point for finding solutions of the field equations in terms of known functions. For instance, it is not difficult to choose w in such a way that (2.12) becomes of the form of the hypergeometric equation of Gauss or one of its concomitants (see for example Sec. 7). If one introduces a new variable

$$\xi=\int_0^x dx/y, \tag{2.13}$$

(2.12) takes the form

$$\zeta_{,\xi\xi}-g(\xi)\zeta=0, \tag{2.14}$$

where $g(\xi)$ stands for $\frac{1}{2}w_{,x}$ expressed as a function of ξ.

(b) If one (invariantly) imposes the condition that ρ does not increase outwards, then it follows that

$$w'\leq 0. \tag{2.15}$$

If subscripts c and b refer to the center and the boundary of the sphere, respectively, then one has for the "mass concentration" δ,

$$\delta=w_b/w_c\leq 1. \tag{2.16}$$

Note that[2]

$$w_b=M/R^3, \tag{2.17}$$

$$\Delta=y_b^2. \tag{2.18}$$

(2.18) follows from the continuity of e^λ at $r=R$. Further, if it be known that the ratio $\rho c^2/3p$ is not less than a certain value β^{-1}, then

$$rw'+3w\geq 3\beta^{-1}P,$$

and therefore, because of (2.15),

$$P\leq\beta w. \tag{2.19}$$

3. Limitations on Δ When $w'\leq 0$

(a) In this and the following three sections it will be supposed that the density does not increase outwards. This assumption is physically reasonable in the present context, for a physical realization of such a fluid sphere under highly relativistic conditions will essentially involve nuclear matter, and it is difficult to visualize

[2] Where convenient the constants k and c will hereafter be taken as unity. They are easily restored by inspection whenever necessary.

any circumstances under which this matter would undergo a modification (in the sense of phase change) such that the total energy density might increase with decreasing pressure. (Furthermore, a fluid sphere with ρ increasing outwards would presumably be highly unstable.) In view of (2.15), Eq. (2.14) now yields the inequality

$$\zeta_{,\xi\xi}\leq 0, \tag{3.1}$$

whence

$$(\zeta_{,\xi})_c\geq\zeta_{,\xi}\geq(\zeta_{,\xi})_b. \tag{3.2}$$

Now

$$(\zeta_{,\xi})_b=y_b\zeta_b'/2R=M/2R^3=\tfrac{1}{2}w_b, \tag{3.3}$$

in view of the continuity of ζ and ζ' at $r=R$, keeping in mind that the Schwarzschild exterior solution gives $\zeta=(1-2M/r)^{\frac{1}{2}}$ for $r\geq R$. (3.2) may be integrated from $r=0$ to $r=R$, so that

$$\Delta^{\frac{1}{2}}-\zeta_c\geq w_b\int_0^R rdr/y.$$

In view of (2.15), $w\geq w_b$ and therefore

$$y^2\leq 1-2w_bx. \tag{3.4}$$

Hence

$$\Delta^{\frac{1}{2}}-\zeta_c\geq\tfrac{1}{2}(1-y_b)=\tfrac{1}{2}(1-\Delta^{\frac{1}{2}}). \tag{3.5}$$

Since one must have $\zeta_c\geq 0$ it then follows that

$$\Delta\geq\tfrac{1}{9}. \tag{3.6}$$

Incidentally it is not difficult to see that equality holds only if ρ is constant and $\zeta_c=0$. *The value $\frac{1}{9}$ is therefore an absolute lower limit for all static fluid spheres whose density does not increase outwards.*

(b) An improved inequality may be obtained if it is assumed that the restriction (2.19) is valid for some value of β. With (3.3) the outer members of (3.2) give

$$(\zeta_{,\xi})\geq\tfrac{1}{2}w_b, \tag{3.7}$$

while (3.5) yields

$$\zeta_c\leq\tfrac{1}{2}(3\Delta^{\frac{1}{2}}-1). \tag{3.8}$$

Applying (2.9) and (2.19) at the center, together with (3.7) and (3.8), one obtains

$$(\beta+1)w_c\geq P_c+w_c=2(y\zeta_{,\xi}/\zeta)_c\geq 2(3\Delta^{\frac{1}{2}}-1)^{-1}w_b, \tag{3.9}$$

and therefore

$$\Delta\geq\frac{1}{9}\left(1+\frac{2\delta}{\beta+1}\right)^2. \tag{3.10}$$

In future the term "regular sphere" will designate a sphere whose central density is finite, and whose density does not increase outwards. It is generally held that the trace T of the energy-momentum tensor cannot be negative.[3] In the present notation this means that the value unity is to be assigned to β. Then (3.10) shows in particular that any *regular sphere with non-negative T has*

$$\Delta\geq\tfrac{1}{9}(1+\delta)^2. \tag{3.11}$$

For $\beta\rightarrow 0$ the inequality (3.10) is virtually empty for one is then dealing with a classical distribution for which $\Delta\rightarrow 1$ in any case.

(c) A somewhat different inequality will appear later [see Sec. 6(b)] as an incidental result. It is

$$\Delta\geq j\delta, \tag{3.12}$$

with

$$j^{\frac{1}{2}}=(1+\beta/3)/(1+\beta). \tag{3.13}$$

(3.12) is very weak when δ is sufficiently small [see, however, Sec. 6(b)]. On the other hand, it is stronger than (3.10) whenever

$$1>\delta>\tfrac{1}{4}(\beta+1)^2. \tag{3.14}$$

4. Impossibility of Certain Classical Distributions

The basic equations of Newtonian gravitational theory arise from Einstein's equations in first approximation; and it is characteristic of this approximation that the pressure does not enter into Poisson's equation for the gravitational potential. When contemplating distributions in which p becomes comparable with ρ at the center, one might nevertheless be tempted into believing that one could perhaps deal with the problem by means of some set of equations, similar to the classical equations (but supplemented by terms involving the pressure), for instance when the mass of the sphere is sufficiently small. In this section it will be shown that this is not possible.

Equation (2.12) may be written

$$(y\zeta_{,x})_{,x}/(y\zeta_{,x})=\tfrac{1}{2}w_{,x}\zeta/y^2\zeta_{,x}=w_{,x}/(P+w), \tag{4.1}$$

because of (2.9). For a regular sphere, since $P\geq 0$ and $w_{,x}\leq 0$,

$$w_{,x}/(P+w)\geq w_{,x}/w,$$

whence it follows that $y\zeta_{,x}/w$ is an increasing function of x. Hence, comparing its central with its boundary value, one has

$$(\zeta_{,x})_c\leq\tfrac{1}{2}w_c. \tag{4.2}$$

Now if it be given that $p_c=\frac{1}{3}\beta\rho_cc^2$, i.e., $P_c=\beta w_c$, it follows from (2.9) and (4.2) that

$$(\beta+1)w_c\zeta_c=2(\zeta_{,x})_c\leq w_c,$$

and therefore

$$\zeta_c\leq 1/(\beta+1).$$

If the conditions at the center of the sphere are such that $\beta=1$ (sufficiently nearly) they will be called "radiation-like." Then one has the result that any regular sphere whose central condition is radiation-like has

$$(g_{44})_c\leq\tfrac{1}{4}. \tag{4.3}$$

It follows that with any such sphere one must neces-

[3] E.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison Wesley Press, Cambridge, 1951), Chap. 4, p. 89.

sarily operate with the field equations in the highly nonlinear domain, and the equation cannot resemble those of Newton's theory. Incidentally the condition that w_c be finite is not necessary for the validity of (4.3).

5. Minimal Theorem for Pressure

A well-known theorem[4] of classical astrophysics states that the central pressure of a regular sphere satisfies the inequality

$$p_c \geq 3kM/8\pi R^4. \qquad (5.1)$$

Equation (3.9) shows that

$$P_c \geq (M/R^3)[2/(3\Delta^{\frac{1}{2}}-1)-1/\delta]. \qquad (5.2)$$

When $\delta=1$, (5.2) indeed goes over into (5.1) as $M/R \to 0$. However, for any given Δ there are values of δ for which (5.2) becomes empty. It is clearly desirable to have an equality which does not exhibit this feature. For this purpose consider the quantity

$$\psi = y^{-2n}(P+\xi)/(P+\eta). \qquad (5.3)$$

Then, using (2.11), one has

$$y^2(P+\xi)(P+\eta)\psi'/r\psi = -(\eta-\xi)(P+w)(P+rw'+3w) + 2n(rw'+2w)(P+\xi)(P+\eta). \qquad (5.4)$$

The right-hand member of (5.4) will not be positive for any value of r if the factors multiplying the various powers of P are separately nonpositive. Taking into account that $w' \leq 0$, and $rw'+3w \geq 0$, this requires

$$4nw_c \leq (\eta-\xi), \quad n(\eta+\xi) \leq (\eta-\xi), \quad 4n\xi\eta \leq 3(\eta-\xi)w_b.$$

Adopting equality in each case

$$n = \tfrac{1}{2}(4-3\delta)^{\frac{1}{2}}, \quad \eta = 2w_c(1+n), \quad \xi = 2w_c(1-n). \qquad (5.5)$$

With these values of the parameters involved, ψ is then a nonincreasing function of r. Accordingly, comparison of its central and boundary values then gives the required result:

$$P_c \geq 3w_b(1-\Delta^n)/2[(1+n)\Delta^n-(1-n)]. \qquad (5.6)$$

This is an appropriate generalization of (5.1), and when $M/R \to 0$, (5.6) reduces to (5.1), i.e.,

$$P_c \geq 3M^2/2R^4, \qquad (5.7)$$

independently of the value of δ. (5.6), like (4.3), has invariant significance, for $\Delta = (g_{44})_b$, and g_{44} behaves as a scalar under transformations of the space coordinates alone.

(5.6) incidentally implies yet another inequality governing the minimum value of Δ, but this does not seem to be of any further interest.

[4] A. S. Eddington, *The Internal Constitution of the Stars* (Cambridge University Press, Cambridge, 1926), Chap. 4, p. 91.

6. Proper Energy and Potential Energy

(a) M, that is to say the field-producing mass of the sphere as measured in terms of the motion of a test particle at infinity, is at the same time the total energy of the sphere including the gravitational field energy,[5] i.e.,

$$M = \int (T_4{}^4 - T_1{}^1 - T_2{}^2 - T_3{}^3)(-g)^{\frac{1}{2}} d^{(3)}x. \qquad (6.1)$$

The proper energy M_0 on the other hand is defined as the integral of the energy density ρ over the elements of proper volume. Thus

$$M = \int_0^R d\mu(r), \quad M_0 = \int_0^R y^{-1} d\mu(r). \qquad (6.2)$$

The negative gravitational potential energy Ω is then

$$\Omega = M_0 - M > 0. \qquad (6.3)$$

A peculiar difficulty enters into any attempt at estimating maximum values of M_0 on the basis of (6.2) which arises from the fact that, for some r, y may be an *increasing* function of r. In other words, although according to what has already been proved, $y_b \geq \frac{1}{3}$, it is by no means known that $y(r) \geq \frac{1}{3}$ for all r. Indeed, if one considers for instance a sphere whose boundary density is zero, then, as one goes inward from the boundary, $y(r)$ will *decrease* as long as $\rho < \frac{1}{3}\bar{\rho}$. It is difficult to see whether regions in which $y(r) < y_b$ will profoundly affect the value of the integral for M_0 when Δ is of the order of magnitude $\frac{1}{9}$. The following analysis yields the best limitation upon M_0 I have been able to develop, despite a persistent effort to improve upon it. (See also Sec. 9.)

(b) The constants ξ and η in (5.3) turned out to be proportional to w_c, the constants of proportionality depending on δ. This suggests the replacement of ξ and η by the functions ξw and ηw, respectively, where ξ and η are positive numerical factors, $\eta > \xi$. Proceeding as before one soon finds that there do not exist any values of ξ, η, and n such that one can conclude ψ' to be certainly not positive for any r, unless some *a priori* restrictions on the form of w are imposed. Consider therefore now the function

$$\psi = w^s y^{-2n}(P+\xi w)/(P+\eta w), \qquad (6.4)$$

where ξ, η, s, and n are all positive numbers, $\eta > \xi$. By inspection of the factors multiplying the various powers of P in the derivative of ψ', keeping in mind the regularity of the sphere, one then finds in the manner of Sec. 5 that $\psi' \leq 0$ provided

$$4n \leq \eta-\xi, \quad s \geq (\eta-\xi)/(\eta+\xi),$$
$$n \leq (\eta-\xi)/(\eta+\xi), \quad \tfrac{4}{3}n\xi\eta \leq \eta-\xi.$$

[5] See reference 2, Chap. 11, p. 323.

Again adopting equality in each case, there comes

$$s=n=\tfrac{1}{2},\quad \xi=1,\quad \eta=3. \tag{6.5}$$

ψ is then a nonincreasing function of r; thus

$$\frac{P_c+w_c}{P_c+3w_c}w_c^{\frac{1}{2}}\geq\frac{P+w}{P+3w}\frac{w^{\frac{1}{2}}}{y}\geq\frac{1}{3}\frac{w_b^{\frac{1}{2}}}{\Delta^{\frac{1}{2}}}. \tag{6.6}$$

The outer members of (6.6) immediately give the inequality (3.12). [The following remarks concerning the latter may be of some interest. In the case of certain distributions the weakness of (3.12) seems very surprising. Consider for example the sphere which has

$$\begin{aligned} w&=\text{const}=a, \quad (r\leq r_1);\\ w&=\text{const}=b, \quad (r_1<r\leq R), \end{aligned} \tag{6.7}$$

where $r_1\ll R$, and $a\gg b$. Then one would certainly expect the limiting value of Δ to be close to j, yet δ can be made arbitrarily small. Now actually it follows from (6.6) that

$$\Delta\geq jw_b y^2/w, \quad (\text{all } r). \tag{6.8}$$

Hence in place of (3.12) one may write

$$\Delta\geq jw_b(y^2/w)_{\max}. \tag{6.9}$$

In certain cases y^2/w will have a maximum at $r>0$ rather than at $r=0$; and in the case (6.7), for instance, one can indeed show that this maximum will have a value arbitrarily close to $1/w_b$ if a is sufficiently large compared with b and r_1 sufficiently small.]

The first two members of (6.6) now yield, for all r,

$$y^2/w\geq j/w_c,$$

whence

$$y\geq(1+\tau^2r^2/R^2)^{-\frac{1}{2}}, \tag{6.10}$$

with

$$\tau^2=2M/j\delta R, \tag{6.11}$$

so that a definite lower limit for y is thereby established.

(c) As a next step the integral (6.2) for M_0 will be given another form which will in any case be needed later on. One has

$$\begin{aligned} M_0&=\int_0^R(r^3w'+3r^2w)y^{-1}dr\\ &=-\Delta^{\frac{1}{2}}R+\int_0^R(y+r^2w/y)dr, \end{aligned}$$

on integrating by parts, and this may be written in the required form,

$$M_0=-\Delta^{\frac{1}{2}}R+\tfrac{1}{2}\int_0^R(y+y^{-1})dr. \tag{6.12}$$

The integrand of the integral on the right is a monotonically decreasing function of y when $y<1$. Therefore one may apply (6.10), *viz.*,

$$M_0\leq-\Delta^{\frac{1}{2}}R+(R/2\tau)\int_0^\tau[(1+t^2)^{\frac{1}{2}}+(1+t^2)^{-\frac{1}{2}}]dt,$$

or

$$M_0/R\leq-\Delta^{\frac{1}{2}}+\tfrac{1}{4}[(1+\tau^2)^{\frac{1}{2}}+3\,\text{ar}\sinh\tau/\tau]. \tag{6.13}$$

This, then, is the best upper limit for M_0 which I have been able to devise. An upper limit for the potential energy is, of course, implied. It may be of interest to examine its classical limit. In that case one may put $\beta=0$, $j=1$, while M/R is very small compared with unity. Disregarding the case in which δ is small enough to cause τ, nevertheless, not to be small compared with unity, (6.13) gives asymptotically

$$\Omega\leq\tfrac{1}{2}(1+1/5\delta^2)(M^2/R)+O(M^3/R^2). \tag{6.14}$$

For the uniform sphere, in fact, $\Omega=3M^2/5R$, in conformity with (6.14). If one considers Emden polytropes as examples of nonuniform distributions, (6.14) is somewhat weak[6]; a purely numerical crude approximation of Ω of the form

$$\Omega_n=\tfrac{1}{2}(1+1/5\delta_n{}^s)(M^2/R)$$

for a polytrope of index n will, in fact, have s only of the order of $\frac{1}{2}$.

7. Explicit Example

It may be of interest to compare some of the general results obtained with exact data relating to an explicit solution of the field equations. As a matter of convenience one may aim at a solution which is expressible entirely in terms of elementary algebraic functions and which is free from singularities at the origin. Now it is easily confirmed that with the choice

$$2w=a/(1+cx), \quad (a,c=\text{const}>0), \tag{7.1}$$

for example, Eq. (2.12) becomes a hypergeometric equation:

$$t(t-1)\zeta_{,tt}+\tfrac{1}{2}\zeta_{,t}-\tfrac{1}{4}[a/(a-c)]\zeta=0, \tag{7.2}$$

with $t=(a-c)(1+cx)/a$. By considering its general solution one may then choose the ratio a/c to be such that one or other or both of the independent solutions of (7.2) become elementary functions. One such value is $\frac{3}{2}$, and this will now be adopted. Then it turns out that, with $cx=\xi$,

$$\zeta=A(1+\xi)^{\frac{3}{2}}+B(5+2\xi)(2-\xi)^{\frac{1}{2}}. \tag{7.3}$$

If $cR^2=\xi_1$ it follows directly from (7.1) that

$$\delta=1/(1+\xi_1),\quad \Delta=(1-\xi_1/2)/(1+\xi_1)=\tfrac{1}{2}(3\delta-1). \tag{7.4}$$

One sees incidentally that certainly $\delta>11/27$ for any such physically admissible solution. From (2.9) and

[6] See reference 3, Chap. 4, p. 89.

TABLE I. Comparison of parameters of special solution with general limits.

β	ξ_1	Δ	Δ^*	δ	ζ_c	$(\beta+1)^{-1}$
∞	1.2826	0.1574	0.1111	0.4381	0	0
3	1.0000	0.2500	0.1736	0.5000	0.1768	0.2500
1	0.6335	0.4183	0.2888	0.6122	0.4302	0.5000
0.3	0.3059	0.6496	0.5271	0.7658	0.7573	0.7629
$\to 0$	$\sim(8/9)\beta$	$\sim 1-\frac{4}{3}\beta$	$\sim 1-(68/27)\beta$	$\sim 1-(8/9)\beta$	$\sim 1-\beta$	$\sim 1-\beta$

(7.3),

$$\zeta P=(9c/4)(1+\xi)^{-1}[A(1-\xi)(1+\xi)^{\frac{1}{2}} -B(1+2\xi)(2-\xi)^{\frac{1}{2}}]. \quad (7.5)$$

The right-hand member of (7.5) must vanish at $\xi=\xi_1$, and the ratio A/B is thus known in terms of ξ_1. The requirement (2.19) then implies

$$(3-\beta)A\leq\sqrt{2}(5\beta+3)B. \quad (7.6)$$

Inserting here the known ratio A/B, there comes

$$8\beta(2\beta+3)-(19\beta^2+6\beta+27)\xi_1 -6(3\beta^2+2\beta+3)(\xi_1^2-\xi_1^3)\geq 0. \quad (7.7)$$

The largest and smallest numerical values respectively of ξ_1 and Δ as computed from (7.4) and (7.7) are given in the second and third columns of Table I for a few selected values of β. Δ^* is the value of the right-hand member of (3.10), calculated with the values of δ given in the fifth column of the table, which are those of the exact solutions. Actually, owing to the fact that w involves only a single parameter one may in this instance express in (3.10) δ in terms of Δ [see Eq. (7.4)], and so obtain an inequality which may be applied without *any* reference to the exact solution. The results are, of course, somewhat worse, e.g., when $\beta=1$, Δ^* turns out to be 0.2500. Finally, in the last two columns of Table I, the actual value of ζ_c is compared with $1/(\beta+1)$, (see Sec. 4).

8. Positive Density Gradient

(a) The spheres hitherto considered have all been subject to the condition $w'\leq 0$. The question arises as to what effects the relaxation of this condition might have. As mentioned in Sec. 3(a), spheres which have w' not restricted to be nonpositive are not likely to be of physical importance. Accordingly I shall content myself with a brief discussion—analogous to that of Sec. 3—of spheres which have $w'\geq 0$ throughout. To begin with, one would expect the restriction $\Delta\geq\frac{1}{9}$ to be no longer necessarily valid. Thus Wyman[7] has given a detailed discussion of the Volkoff sphere, in which the density is constant but a singularity of e^{λ} is allowed at the origin. Such spheres can certainly have $\Delta<\frac{1}{9}$. Now the solution of the problem in which one has a small core of constant density surrounded by a large mantle of sufficiently large constant density will approximate Wyman's solution closely within the mantle. Accordingly one may expect Δ to be less than $\frac{1}{9}$ in suitable cases, and this feature will not be the result of the existence of a singularity at the origin.

It turns out, however, that one can still derive for spheres having $w'\geq 0$ a result analogous to (3.6). Equation (I.5) gives at once

$$(\zeta/y)'\geq 0, \quad (8.1)$$

which on taking boundary values into account gives

$$\zeta\leq y. \quad (8.2)$$

Equation (2.12) therefore gives rise to

$$(y\zeta_{,x})_{,x}=\tfrac{1}{2}w_{,x}\zeta/y\leq\tfrac{1}{2}w_{,x}. \quad (8.3)$$

Accordingly

$$y\zeta_{,x}-\tfrac{1}{2}w\geq(y\zeta_{,x})_b-\tfrac{1}{2}w_b=0, \quad (8.4)$$

by (3.3). Now, by hypothesis, $w\geq w_c$ everywhere, so that

$$y^2\leq 1-2r^2w_c,$$

and therefore, in view of (8.4),

$$[\zeta+\tfrac{1}{2}(1-2r^2w_c)^{\frac{1}{2}}]'\geq 0. \quad (8.5)$$

Comparing central and boundary values, this gives

$$\Delta^{\frac{1}{2}}+\tfrac{1}{2}(1-2R^2w_c)^{\frac{1}{2}}\geq\zeta_c+\tfrac{1}{2}\geq\tfrac{1}{2},$$

which may be written

$$\Delta^{\frac{1}{2}}\geq\tfrac{1}{2}\{1-[1-(1-\Delta)/\delta]\}^{\frac{1}{2}}.$$

Solving this inequality for Δ, one finally obtains a lower bound for Δ in the form

$$\Delta\geq(4\delta-1)^{-2}. \quad (8.6)$$

(b) When $\delta=\infty$ as a consequence of w_c being zero, (8.6) gives only $\Delta\geq 0$, which is empty. This defect is the result of the weakness of the inequality which leads from (8.4) to (8.5). Returning to (8.4) therefore one has

$$\Delta^{\frac{1}{2}}\geq\int_0^R rwy^{-1}dr, \quad (8.7)$$

and the integral on the right is certainly positive. However, it is scarcely worthwhile to pursue the somewhat unphysical situations of this section any further, except for the remark that since $\zeta'\geq 0$ (if only $P+w$ is not negative), $\Delta>0$ *always* unless $\zeta=0$ throughout the sphere; but such a solution is physically meaningless.

[7] M. Wyman, Phys. Rev. 75, 1930 (1949).

9. The Physical Radius

(a) A result such as $M \leq 4R/9$ [derived from (3.6)] does not possess invariant character of any sort, as has already been pointed out. To give it invariant significance one may introduce in place of R the physical radius R^*,

$$R^* = \int_0^R dr/y. \tag{9.1}$$

Under the conditions of Sec. 3, using (3.4), one easily finds

$$M \lesssim 0.3404 R^*, \tag{9.2}$$

which indeed represents an *invariant* limitation upon the mass of any static fluid sphere with $w' \leq 0$ of given (physical) radius. Now if one agrees to deal with R^* in place of R, then various invariant results can be established in an almost trivial manner. Thus from $0 \leq e^{-\lambda} \leq 1$ it follows that

$$M \leq \tfrac{1}{2}R < \tfrac{1}{2}R^*, \tag{9.3}$$

giving an upper bound for M invariantly for *any* kind of fluid sphere (if only $w \geq 0$). Again, upper bounds of M_0 or Ω were considered in Sec. 6 under the assumption that $w' \leq 0$. Since only the crudest inequalities are now contemplated one may write, from (6.12),

$$M_0 < \int_0^R dr/y, \tag{9.4}$$

since $y + 1/y \leq 2/y$ when $y \leq 1$. But the integral on the right of (9.4) is just R^*. Consequently *the proper energy and the* (*negative*) *potential energy of any static fluid sphere are finite and less than the physical radius.*

PART II

10. Energy Radiated by Contracting Spheres

This final section deals with a certain question which has sometimes been asked,[8] although its character lies somewhat outside the framework of the rest of the paper. The problem may be stated in the following idealized form. An amount of gas consisting of N particles each of mass m is initially completely dispersed.[9] It subsequently contracts symmetrically to form a star, i.e., a fluid sphere, and in the course of the contraction radiation is emitted.[10] W is the total amount of emitted radiation received by an observer O located outside B. Then the question is: can W ever exceed $Nm = W_0$, say?[11] Though elementary, it nevertheless deserves a definitive answer since doubts evidently exist whether the latter might not perhaps turn out to be in the affirmative.

One may first inquire as to the way in which O might, in principle, measure the amount of radiation received from S. For this purpose he would surround S by an opaque spherical shell A which would act as an absorber. (One may take A to be a perfect absorber, for to suppose otherwise would only introduce quite irrelevant complications, and a similar remark applies to the assumption which will now be made, namely, that the initial mass of A is zero. Also it is convenient to take the inner radius of A to be just b, while its outer radius will be denoted by a.) Then at any arbitrary time, t, O measures the energy M_{0A} [see Eq. (6.2)] of A, and he will understand this to be the energy he has received from S up to time t. Now in virtue of Birkhoff's theorem,[12] the total mass M (defined as usual in terms of the motion of a test particle "at infinity") of the whole system, consisting of S, A and any radiation between them, is constant in time. (There is no radiation *outside* A: all sources of the gravitational field were, by hypothesis, absorbed by A.) Therefore $M = W_0$, M/b having been supposed negligible. In terms of a metric of the usual type (see Appendix II),

$$M = 4\pi \int_0^a T_4{}^4 r^2 dr. \tag{10.1}$$

Within A, again using (II.2),

$$1 \geq e^{-\lambda} = 1 - 8\pi r^{-1} \int_0^r T_4{}^4 r^2 dr \geq 1 - 2M/b,$$

keeping (10.1) in mind. Since M/b is negligible it follows that e^{λ} is effectively unity within A. Therefore

$$M_{0A} = 4\pi \int_b^a T_4{}^4 r^2 dr.$$

Consequently

$$M_{0A} = W_0 - 4\pi \int_0^b T_4{}^4 r^2 dr. \tag{10.2}$$

Since the integral on the right cannot be negative, it follows that *the total radiation received by the observer O from the star S can never exceed the total energy which its constituent matter possessed when in a state of complete dispersion.* It will be seen that this result has been arrived at in an entirely elementary manner, and no use has had to be made of any differential conservation theorems.

Finally it may be remarked that any expectation of an affirmative answer to the question treated above

[8] E. E. Salpeter (private discussion).

[9] The particles may be taken as initially at rest. Furthermore the gas will be imagnined initially to be confined uniformly to a sufficiently large but finite spherical region B of radius b: the term "complete dispersion" is therefore to be interpreted as meaning that the initial density is to be smaller than any preassigned positive limit.

[10] *Any* kind of radiation, electromagnetic, material, etc., is contemplated. Even energy transfer through gravitational waves is not excluded in principle.

[11] Units are so chosen that $k = c = 1$.

[12] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, Oxford, 1934), Chap. 7, p. 252.

presumably arises from the notion that energy could be radiated indefinitely as a consequence of the continued contraction of S. This would, however, involve a continual increase of Ω, so that to keep M positive—which it must be according to (10.1)—M_0 must also continue to increase. In Newtonian terms, the work done by the gravitational field to an ever growing extent is limited to merely increasing the "mechanical" energy of S. (Indeed, Birkhoff's theorem here says just this: namely, that in a spherically symmetric process without radiation, all the gravitational work done goes towards increasing the mechanical energy of S.) It should also be kept in mind that any radiation emitted by S will suffer an ever-increasing "red-shift," as seen by O, as the contraction proceeds. However, the simple argument presented above avoids any difficulties which might be encountered if one tried to answer the question by considering the details of the mechanism of contraction.

ACKNOWLEDGMENT

I wish to thank Dr. J. Robert Oppenheimer for the splendid hospitality, and support, of the Institute for Advanced Study, and also for reading the original manuscript and his valuable comments thereon.

APPENDIX I

Throughout this paper all formal developments are carried out in a particular coordinate system, *viz.*, that in which the metric takes the form

$$ds^2=c^2e^\nu dt^2-e^\lambda dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \qquad (I.1)$$

in which λ and ν are functions of r only. The energy-momentum tensor is diagonal with $T_1{}^1=T_2{}^2=T_3{}^3=-p$, $T_4{}^4=\rho c^2$, where p is the hydrostatic pressure and ρc^2 the energy density, which includes all forms of energy, with the exception, of course, of gravitational energy. p and ρ are scalar quantities and it is supposed that *neither can be negative.* The field equations are then

$$8\pi kc^{-4}r^2p=e^{-\lambda}(r\nu'+1)-1, \qquad (I.2)$$

$$8\pi kc^{-2}r^2\rho=e^{-\lambda}(r\lambda'-1)+1, \qquad (I.3)$$

$$2p'=-(p+\rho c^2)\nu'. \qquad (I.4)$$

k is Newton's constant, and primes indicate differentiation with respect to r. The cosmical constant has been taken as zero. If one adds (I.2) and (I.3) one has

$$e^{-\lambda}(\lambda'+\nu')=8\pi kc^{-4}r(\rho c^2+p). \qquad (I.5)$$

APPENDIX II

The spherical symmetry of a metric (which may be nonstatic), i.e., its invariance under the group of spatial rotations, requires that it be of the form

$$ds^2=e^\nu dt^2+2qdrdt-e^\lambda dr^2-r^2e^\mu(d\theta^2+\sin^2\theta d\phi^2), \qquad (II.1)$$

where λ, μ, ν, and q are functions of r and t. Since one has the freedom to replace r and t by new variables which are functions of r and t one can always arrange μ and q to be zero. Then of the resulting field equations (see reference 12, p. 251), the only one explicitly required here is that corresponding to (I.3), i.e.,

$$8\pi r^2T_4{}^4=e^{-\lambda}(r\lambda'-1)+1, \qquad (II.2)$$

from which Eq. (10.1) of the text follows at once if one takes into account that at $r=a$ the metric must go over into the Schwarzschild exterior solution.

PHYSICAL REVIEW VOLUME 116, NUMBER 4 NOVEMBER 15, 1959

Upper Bounds on Scattering Lengths for Static Potentials*

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(Received June 22, 1959)

It is shown that in the zero-energy scattering of a particle by a center of force, where no bound state exists, the Kohn variational principle provides an upper bound on the scattering length. A bound may also be obtained from Hulthén's method, although with the same form of trial function the Kohn result will be lower (and therefore better) than the one obtained from the Hulthén principle. The Rubinow formulation need not provide a bound; for those calculations which have been performed in this form, the results may be converted without any further calculations so that they correspond to the Kohn form, and therefore, under the circumstances considered, do give a bound. Analogous results hold for states of nonzero orbital angular momentum. Direct generalizations of the above results are valid for scattering by a compound system.

1. INTRODUCTION

VARIATIONAL methods have proved to be of great value in the theoretical analysis of the problem of the scattering of a particle by a center of force. However, the utility of variational techniques in the more complicated problem of scattering by a compound system is considerably impaired by the fact that it is more difficult, in the many-body problem, to

* The research reported in this article was done at the Institute of Mathematical Sciences, New York University, under the sponsorship of both the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, and the Office of Ordnance Research, U. S. Army.