PHYSICAL REVIEW VOLUME 116, NUMBER 4 NOVEMBER 15, 1959

Direct Pair Production by Muons*

BYRON P. ROE† AND SEINOSUKE OZAKI‡
Laboratory for Nuclear Studies, Cornell University, Ithaca, New York

(Received April 3, 1959)

A measurement has been made of the direct pair production cross section for muon primaries between about 8 and 120 Bev. The muons were selected by a magnetic spectrometer and interactions were observed in a multiplate expansion cloud chamber. Results indicate that in the transferred-energy range above about 200 Mev the direct pair cross section is somewhat less than that predicted by the Murota-Ueda-Tanaka theory. This result does not, however, necessarily indicate any breakdown of fundamental electromagnetic theory. Possible theoretical inadequacies are discussed in the text.

INTRODUCTION

DIRECT pair production is the creation of an electron-positron pair by a charged particle as it passes through the electric field of a nucleus. Many measurements have been made of the direct pair cross section with electron primaries[1–15]—the so-called trident process. Some of the results of these experiments are plotted in Fig. 1. The trident experiments usually have been performed with nuclear emulsions and are beset by two serious difficulties.

Firstly, if cosmic-ray electrons are used, the energy of the primary electron, usually one of the members of a pair, must be determined from a measurement of relative scattering, a measurement of the opening angle of the pair, or from the dipole effect. All of these methods are subject to considerable error. Some investigators[10–13] have overcome this problem in the energy range below 1 Bev by using electrons obtained from accelerators.

Secondly, the problem of distinguishing real tridents from pseudo-tridents is a difficult one. Direct pair production is a second-order process, i.e., proportional to $(1/137)^2$. Therefore, for electron primaries it is of the same order as the combined probability (in targets of thickness on the order of 0.2 radiation length) of producing a bremsstrahlung photon followed by conversion of the produced photon. If the conversion occurs close to the track of the primary, this pseudo-trident event cannot be distinguished from a real trident. The proportion of pseudo-tridents increases with increasing primary energy. The evaluation of the exact pseudo-trident correction in emulsions is difficult since it involves exact knowledge of the resolution obtainable in examining the emulsions, the angular distribution of the bremsstrahlung photons, and the multiple scattering of the primary track.

A comparison of the pseudo-trident correction for electron primaries and muon primaries with the same Lorentz factor indicates that the pseudo-trident correction is smaller for muons by a factor of roughly $(m_e/m_\mu)^2$ for large E/mc^2. Thus a considerable advantage is obtained by the use of muon primaries. A disadvantage is also incurred, in that one must go to much higher energies for muons than for electrons to measure the same direct pair cross section, since the cross section is essentially a function of E/mc^2.

Avan and Avan[8,16] have exposed nuclear emulsions at several depths underground and have examined direct pair production by muons in the emulsions. Good agreement with theoretical predictions was obtained on the basis of 30 events at 300 meters of water

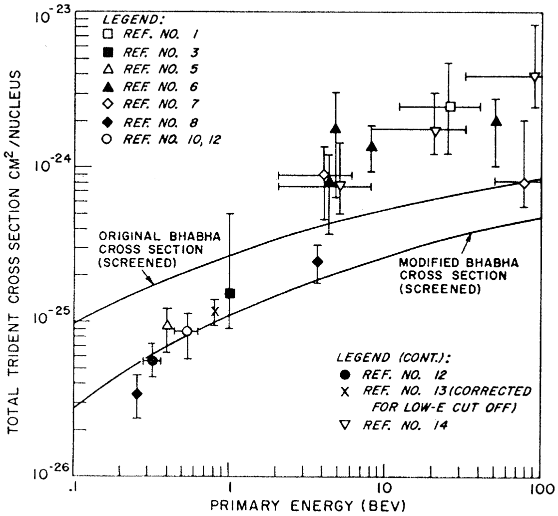

FIG. 1. Measurements of the total trident cross section for electron primaries.

---

* Research supported by the National Science Foundation.

† Now at University of Michigan, Ann Arbor, Michigan.

‡ Fulbright Fellow, 1955–58, on leave from the Institute of Polytechnics, Osaka City University, Osaka, Japan.

[1] E. Lohrmann, Nuovo cimento **3**, 820 (1956).

[2] J. E. Naugle and P. S. Freier, Phys. Rev. **92**, 1086 (1953).

[3] Hooper, King, and Morrish, Phil. Mag. **42**, 304 (1951).

[4] M. M. Block and D. T. King, Phys. Rev. **95**, 171 (1954).

[5] Block, King, and Wada, Phys. Rev. **96**, 1627 (1954).

[6] M. Koshiba and M. F. Kaplon, Phys. Rev. **100**, 327 (1955).

[7] J. E. Naugle and P. S. Freier, Phys. Rev. **104**, 804 (1956).

[8] L. Avan, Ph.D. thesis, L'Université de Caen, 1956 (unpublished).

[9] M. Koshiba and M. F. Kaplon, Phys. Rev. **97**, 193 (1955).

[10] F. J. Loeffler, Phys. Rev. **108**, 1058 (1957).

[11] M. Camac, Phys. Rev. **88**, 745 (1952).

[12] S. L. Leonard, Bull. Am. Phys. Soc., Ser. II, **1**, 167 (1956).

[13] G. Lutjens, Z. Naturforsch. **13A**, 510 (1958).

[14] R. Weill *et al.*, Nuovo cimento **6**, 1430 (1957).

[15] W. H. Barkas *et al.*, Phys. Rev. **86**, 59 (1952).

[16] M. Avan and L. Avan, Nuovo cimento **6**, 1590 (1957).

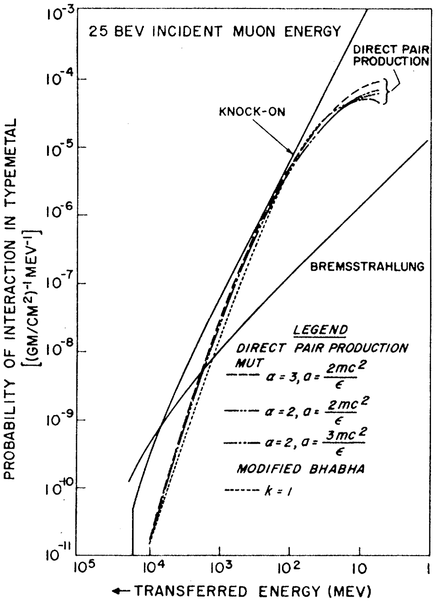

FIG. 2. Theoretical cross sections *vs* energy transfer for electromagnetic processes of muons of 25 Bev. For the case of pair production, "Modified Bhabha" represents the calculations of Bhabha[18,19] as evaluated by Block, King, and Wada.[5] MUT refers to the calculations of Murota, Ueda, and Tanaka[22] for two values of each of the parameters α and a.

equivalent (mwe) and 31 events at 580 mwe. The mean energies of the two groups of muons were 60 and 120 Bev, respectively; the mean energy transfers in the measured processes were 20 and 50 Mev. The present experiment concerns similar meson energies but is restricted to higher values of the transferred energy, namely values above 200 Mev.

THEORIES

In the present experiment showers caused by three electromagnetic processes are observed (nuclear showers being rejected for the present purposes). These are the knock-on process, the bremsstrahlung process, and the direct pair process. The first two are believed to be well accounted for by theory, and the formulas describing them are summarized in Rossi's book.[17]

The direct pair production cross section has been computed by several authors,[18–22] the most recent paper being by Murota, Ueda, and Tanaka[22] (hereafter denoted by MUT).

The various theories are now in fair agreement. The original Bhabha theory was in considerable disagreement with all others; but Block, King, and Wada[5] (hereafter denoted by BKW) showed that this was due to the neglect of certain lower-order terms. BKW took these terms into account and obtained good agreement with the Racah theory.[20] The MUT theory gives a slightly higher cross section than the modified Bhabha theory for a given primary energy in some regions of transferred energy. All of these theories have been computed under Born approximation and neglect both nuclear recoil and the finite size of the nucleus. Also, most of the theories have two arbitrary parameters known to be equal to 1 only in order of magnitude (although arbitrarily set equal to 1 in some theories). One of these parameters arises from approximations about the amount of transverse momentum transfer, and the other from uncertainties in the lower energy limit of the produced electron and positron, since all of the theories are accurate only for highly relativistic produced particles.

In regions of low transferred energy the direct pair process can be pictured in terms of the semiclassical Weizsäcker-Williams model. The incoming particle is surrounded by a cloud of virtual photons, one of which converts, forming an electron-positron pair. This picture is valid only if the transferred energy $\epsilon = \epsilon_+ + \epsilon_- \ll (m_e/M)E$, where M and E are the mass and energy, respectively, of the primary particle. Figures 2 and 3 show electromagnetic cross sections of interest in the present experiment, plotted as functions of transferred energy for two values of the primary muon energy.

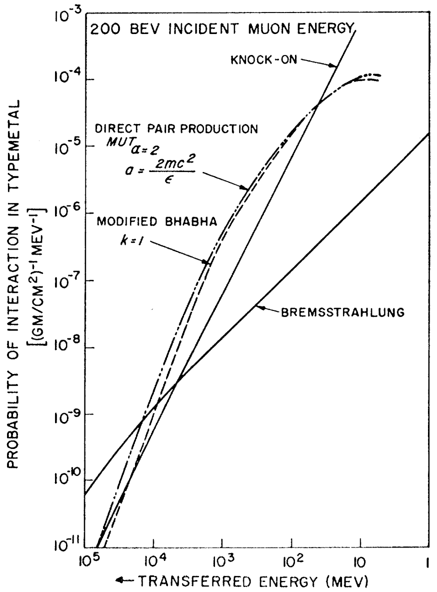

FIG. 3. Theoretical cross sections *vs* energy transfer for electromagnetic processes of muons of 200 Bev. Note that for transferred energies in the range 0.1 to 10 Bev, direct electron pair creation is expected to be the dominant process.

---

[17] B. Rossi, *High-Energy Particles* (Prentice-Hall, New York, 1952).

[18] H. J. Bhabha, Proc. Roy. Soc. (London) **A152**, 559 (1935).

[19] H. J. Bhabha, Proc. Cambridge Phil. Soc. **31**, 394 (1935).

[20] G. Racah, Nuovo cimento **14**, 93 (1937).

[21] Nishina, Tomonaga, and Kobayasi, Sci. Papers Inst. Phys. Chem. Research (Tokyo) **27**, 137 (1935).

[22] Murota, Ueda, and Tanaka, Progr. Theoret. Phys. (Kyoto) **16**, 482 (1956).

EXPERIMENTAL ARRANGEMENT

The present experiment was performed with the aid of the Cornell magnetic spectrometer. This consists of a magnet producing a 13-kilogauss field within a gap of dimensions $8\frac{1}{4}$ in.×22 in.×45 in., and trays of Geiger counters and hodoscope counters. This is the same spectrometer described by Pine[23] and Pine, Davisson, and Greisen[24] except that the fine momentum determination is now accomplished by means of three trays of Conversi hodoscope neon tubes instead of three cloud chambers, in order to obtain better operating efficiency, and the distance between the Geiger counter trays has been changed. Therefore only a brief description will be given of most of the spectrometer.

Geiger counter trays with axes parallel to the magnetic field perform the rough momentum selection. Counters with axes perpendicular to the direction of the magnetic field are used to help guard against scattering in the gap. By insisting on a straight path through both sets of Geiger trays, muons above about 8 Bev are selected. The median energy of this group of muons is about 25 Bev since the aperture is very small for energies of 8 to 15 Bev.

The fine momentum selection is accomplished by means of Conversi hodoscope tube chambers. These chambers are photographed whenever the rough sagitta requirements are satisfied. A maximum measurable momentum of about 120 Bev/c can be obtained by use of these chambers. The chambers are filled with a total of about 3000 small sealed Pyrex tubes, 5 mm in outer diameter, which are viewed endwise.[25–27] The lengths of the tubes are about 8 in. in the top and bottom chambers and 6 in. in the center chamber. Each chamber contains six rows of tubes.

These tubes have been baked out for several hours and then filled with neon plus 0.2% argon to 2.3 atmospheres pressure. 99.6% pure neon and argon were used.

When a particle passes through one of these tubes, some ions are formed in the gas, and if a field of about 7 kilovolts per centimeter is applied across the tubes, the tubes glow momentarily, as small neon lamps. The voltage must be applied within about 2 microseconds after the passage of the particle, or there is a severe loss in the efficiency of the tubes. However, if the tubes are lit once, many ions are formed and a second pulse arriving even some milliseconds later again lights the tubes with high efficiency.

Therefore a fast coincidence is taken of the six trays of Geiger counters. This is fed into a Thyratron giving a high-voltage pulse at the hodoscope chambers 1.5 microseconds after the time of passage of the particle. Meanwhile, the signals from the individual Geiger tubes are fed into an electronic circuit which computes the coarse sagitta of the muon track. If the requirement of zero deflection (within the limits of Geiger tube resolution) is satisfied, a solenoid is actuated to open the camera shutter, and a delayed sequence of pulses is applied to the neon tubes as soon as the shutter is open.

The total light output of a tube is not proportional to the duration of the applied pulse. Instead the ion cascade causes a shielding action to occur which eventually extinguishes the tube. Therefore it is necessary to excite the tubes with a burst of pulses while the shutter is open. The burst used is a series of 22 pulses 50 microseconds long and spaced at 3-millisecond intervals. The size of the pulses is reduced by about 20% after the first pulse in the burst in order to reduce the number of "flashers" (tubes that light even though no particle has passed through them).

Flashers can be caused by impurities in the tubes, but they can also be caused by light from one tube entering a second tube, since the photons from the neon can ionize the argon. Therefore it is necessary to cover the wall of each tube with black tape.

It may be noted that although the above characteristics of the hodoscope tubes agree with the characteristics found by Conversi *et al.*,[25,26] they disagree with the properties observed by Gardener *et al.*,[27] who obtain very good performance from soda glass tubes which were not baked out but merely flamed. Gardener *et al.* claim to have far fewer flashers, and are able to obtain high efficiency even if the tubes are pulsed many microseconds after the passage of the particle. Their tubes are, however, of larger diameter than the tubes used here.

For the present experiment a multiplate expansion cloud chamber with a sensitive volume of 12 in.×12 in. ×22 in. was placed under the spectrometer. The chamber contained nine typemetal plates (essentially Pb) each 1.06 radiation lengths thick. The chamber was expanded whenever a single particle traversed the spectrometer with a momentum sufficiently high that it was not deflected by the magnetic field within the resolution of the Geiger counter trays in the spectrometer.

The experimental run lasted from September, 1957 to August, 1958, except for some times of shutdown for servicing the equipment.

RESULTS

Six thousand and forty-six cloud chamber tracks were examined, of which 4492 were obtained with the spectrometer in a vertical position and 1554 with the spectrometer at 68° to the vertical. For 1125 of the tracks obtained with the spectrometer in vertical position, a 1.5-in. Pb block was placed directly over the chamber.

---

[23] J. Pine, Ph.D. thesis, Cornell University, 1959 (unpublished).

[24] Pine, Davisson, and Greisen, Nuovo cimento (to be published).

[25] M. Conversi and A. Gozzini, Nuovo cimento **2**, 189 (1955).

[26] G. Barsanti *et al.*, *Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956), Vol. II, p. 56.

[27] M. Gardener *et al.*, Proc. Phys. Soc. (London) **B70**, 687 (1957).

TABLE I. Predictions and observations of the numbers of electromagnetic showers produced by mesons. $\bar{N}$ is a measure of the energy of the shower as explained in the text.

Sagitta (mm)	Median energy (Bev)		$\bar{N}$ = 4–5	6–20	21–100
0–2	83	No. observed showers	6	13	1
		No. K.O.+bremsstrahlung showers predicted	5	3	0.8
		No. D.P. showers predicted	11	15	3.0
2–8	25	No. observed showers	21	23	6
		No. K.O.+bremsstrahlung showers predicted	17	15	2.8
		No. D.P. showers predicted	18	16	1.4
8–15	12	No. observed showers	11	8	1
		No. K.O.+bremsstrahlung showers predicted	8.4	6.9	1.03
		No. D.P. showers predicted	2.6	1.2	.073
All sagitta	22	No. observed showers	73	74	12
		No. K.O.+bremsstrahlung showers predicted	46	40	7.3
		No. D.P. showers predicted	45	46	6.6

The sagitta spectrum of the incoming mesons was determined on the basis of about 2000 hodoscope pictures. These results and their interpretation will be reported in a future work. Having obtained this distribution, it was considered necessary to scan further only the 451 hodoscope pictures for which the corresponding cloud chamber pictures contained showers. In approximately $\frac{1}{3}$ of these cases the hodoscope pictures were unusable because of malfunctions of that part of the apparatus.

The numbers of showers obtained in various intervals of primary energy and transferred energy are shown in Table I. $\bar{N}$ is defined as the sum of the number of electrons in the gap between Pb plates at which the shower is at maximum development and the numbers of electrons in the gaps directly above and below this one. Showers with $\bar{N}\leqq 3$ are neglected in this table because experimental biases and theoretical difficulties make these low-energy data very difficult to interpret.

Define N_1 to be the number of electrons below the Pb plate in which the shower originates. The values of N_1 (where they could be clearly distinguished) are shown in Table II, for showers with $4\leqq\bar{N}\leqq 10$. These observations are useful in distinguishing between the knock-on process and direct pair production, since the most probable value of N_1 is 1 for knock-on events and 2 for direct pair events.

MEASUREMENT OF THE ENERGY OF ELECTRON SHOWERS IN THE CLOUD CHAMBER

Unfortunately the current shower theories are not adequate in the energy regions of interest here, from about 200 Mev to over 1 Bev in Pb. Therefore extrapolations were made from the results of Wilson's Monte-Carlo calculations[28,29] using asymptotic theories as guides.[30,31] Wilson's calculations have a 6-Mev low-energy cutoff, which corresponds roughly to neglecting all tracks which travel at an angle of greater than 60° to the direction of the primary particle. Such tracks were therefore ignored in the determinations of the transferred energy parameter $\bar{N}$.

TABLE II. Observed numbers of showers, subdivided according to energy of primary muon, energy in shower (indicated by $\bar{N}$), and value of N_1, which is the number of secondary particles in the first compartment under the plate in which the shower originates. This tabulation includes only those pictures in which it can clearly be distinguished whether N_1 is one or more than one.

Median energy (Bev)	Sagitta range (mm)	N_1	$\bar{N}=4$	$\bar{N}=5$	$\bar{N}=6$–10
83	0–2	1	3	0	2
		>1	0	0	3
25	2–8	1	9	6	4
		>1	2	4	10
12	8–15	1	1	2	2
		>1	4	2	3
22	Unmeasured sagitta	1	10	4	10
		>1	5	5	7

Results summed over all sagitta ranges

	$\bar{N}=4$		$\bar{N}=5$		$\bar{N}=6$–10	
N_1	Observed	Predicted	Observed	Predicted	Observed	Predicted
1	23	16	12	10	18	18
>1	11	17	11	13	23	23

Results summed over all $\bar{N}$, $4\leqq\bar{N}\leqq 10$

	$s=0$–2 mm		$s=2$–8 mm		$s=8$–15 mm		All sagitta	
N_1	Obs.	Pred.	Obs.	Pred.	Obs.	Pred.	Obs.	Pred.
1	5	2	19	16	5	9	53	44
>1	3	6	16	19	9	5	45	54

Numerical calculations made from the Wilson data indicated no serious difference between the values of $\bar{N}$ (or its variance) obtained for direct pair production and for knock-on processes involving the same total transferred energy. However, S_m, the position of the shower maximum, was different for the two processes. The results obtained are

$$\bar{N}=0.043E/(\ln E-2.03)^{\frac{1}{2}},$$
$$\sigma^2(\bar{N})=\tfrac{1}{2}\bar{N}=0.0215E/(\ln E-2.03)^{\frac{1}{2}},$$
$$S_m\text{ (direct pair)}=\ln E-3.45,$$
$$S_m\text{ (knock-on)}=\ln E-2.45,$$

where E is in Mev, S_m in radiation lengths.

Thom[32] is currently using a multiplate cloud chamber to examine showers produced by electrons from the Cornell synchrotron. Preliminary results with mean primary energy of 950 Mev indicate $\bar{N}=17\pm 1$ while the extrapolation used in our work predicts 18. The statistical accuracy of the values of $\bar{N}$ derived from Wilson's Monte Carlo data at 300 Mev is also about 5%. Therefore our values of $\bar{N}$ are believed accurate to 10% within the energy range of present interest. The pre-

---

[28] R. R. Wilson, Phys. Rev. **79**, 204 (1950).
[29] R. R. Wilson (private communication, 1957).
[30] S. Z. Belenky, C. R. Acad. Sci. U.S.S.R. **30**, 608 (1941).
[31] I. Tamm and S. Z. Belenky, Phys. Rev. **70**, 660 (1946).
[32] H. Thom (private communication, 1958).

liminary value obtained by Thom for $\sigma^2(\bar{N})$ is, however, almost double that predicted by the above formula.

DISCUSSION OF RESULTS

Table I also gives theoretical predictions for the number of events expected in each interval of primary energy and transferred energy. These results were obtained from the theoretical cross sections by folding in the energy spectrum of the primary muons, the energy resolution of the apparatus, and the variance in $\bar{N}$. The calculations also included a factor representing the change of efficiency of the cloud chamber for showers of varying sizes. The criterion used was that the shower had to be visible in the cloud chamber for one gap beyond its maximum development unless the shower were very large, in which case the criterion used was that the shower had to be visible in at least five gaps. Assuming perfectly accurate electromagnetic theories, the above calculations are believed to be accurate within ±25% except in the 0–2 mm sagitta range in which they are believed accurate within ±30%. Most of this error is caused by the uncertainty in the relation between $\bar{N}$ and E. The experimental bias in observing tracks and events is believed less than ±10%.

Table II gives rough theoretical predictions for the distribution of N_1 values, based on Wilson's Monte Carlo charts.[28,29] The probabilities of various N_1 values did not appear to be strong functions of energy and the average probabilities, used in calculating the theoretical entries in Table II, were as follows:

N_1	Knock-on	Direct pair
1	80%	15%
>1	20%	85%

Previous experiments summarized in a recent article by Fowler and Wolfendale[33] have obtained good agreement with theory for the knock-on and bremsstrahlung cross sections with muon primaries. The present results also indicate good agreement with theoretical predictions in the 8–15 mm sagitta range in which direct pair production is not important. Therefore it can probably be assumed that the knock-on and bremsstrahlung cross sections are in agreement with theory throughout the energy range of the present experiment, and the experiment can be regarded as a test of direct pair production.

The results in Tables I and II indicate that the direct pair cross section is somewhat smaller than expected theoretically. Define f to be the ratio, assumed constant, of the experimental to theoretical direct pair cross section. From application of the method of maximum likelihood to Table I, one obtains a value $f=0.48\pm0.11$, and from Table II one finds $f=0.63\pm0.17$. The statistics are too poor to allow any statement to be made about the energy dependence of f, except that it seems as if f does not increase with increasing primary energy. Since the two determinations of f both use the same events, it is not permissible to combine them statistically and so reduce the total error. It is, however, of interest to note that the two determinations agree within their statistical errors. The determination of f by means of the N_1 data does not have the large uncertainty caused by inaccuracies in the $\bar{N}$ *vs* E relation. If the tentative results of Thom's shower experiment are considered perfectly accurate, the value of f obtained from Table I would be increased by 0.1 or 0.2 and would give even better agreement between the values obtained for f from Table I and Table II.

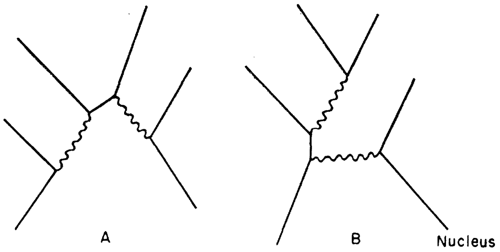

FIG. 4. Feynman diagrams of principal interactions leading to electron pair production.

The present results do not necessarily indicate any failure of basic electromagnetic theory. For one thing the MUT theory used here gives somewhat higher predictions in the region of present interest than does the BKW modification of Bhabha's theory. Furthermore, though the value of the arbitrary parameter entering from the low-energy cutoff causes only a small effect for the present experiment (see Fig. 1), the value of the other arbitrary parameter, k in the Bhabha theory and $\alpha/2$ in the MUT theory, can make some difference. The predictions given in Tables I and II were calculated assuming $\alpha/2=1$.

If the use of the Born approximation for direct pair production introduces the same inaccuracies as in ordinary pair production, then the present theories probably overestimate the direct pair cross section in Pb by about 10% and the correction to the theories should be proportional to Z^2.

From the above arguments it can also be seen that the present experiment is not in disagreement with that of Avan and Avan.[8,16] The latter experiment utilized nuclear emulsions and the Born approximation correction should not be important in emulsions. Furthermore, very few direct pairs with energy above 200 Mev were found in the emulsions, while almost all of the events utilized in the present experiment had energy greater than 200 Mev. Thus most of the events of Avan and Avan fall into the region in which the Weizsäcker-Williams method is applicable, while most of the events

---

[33] G. N. Fowler and A. W. Wolfendale, in *Elementary Particle and Cosmic-Ray Physics*, edited by J. G. Wilson and S. A. Wouthuysen (North Holland Publishing Company, Amsterdam, 1958), Chap. 3.

found in the present experiment fall into a region in which the Weizsäcker-Williams picture is not valid.

Furthermore, the present results are not necessarily in disagreement with those electron experiments in which the measured cross section has appeared to be larger than that predicted by the theory. The theory takes into account only the type of interaction represented by the Feynman diagram of Fig. 4(A); the process portrayed by the diagram of Fig. 4(B) is neglected. Process B is certainly of negligible importance for incident particles as heavy as muons, but may make a considerable contribution to the probability of direct pair production by electrons. For completeness, it should also be noted that no direct pair theory has yet included the effect of exchange on the electron cross section, which would probably reduce the theoretical value.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the help of Dr. K. Greisen who suggested the problem and whose continuous help and advice were invaluable. They also wish to acknowledge the help of W. Pak who assisted in keeping the apparatus running during the latter half of the run and who scanned almost half of the cloud chamber pictures. Mrs. M. Nielsen scanned most of the hodoscope pictures.

PHYSICAL REVIEW VOLUME 116, NUMBER 4 NOVEMBER 15, 1959

General Relativistic Fluid Spheres

H. A. BUCHDAHL*
Institute for Advanced Study, Princeton, New Jersey

(Received June 16, 1959)

In Part I of this paper certain well known results concerning the Schwarzschild interior solution are generalized to more general static fluid spheres in the form of inequalities comparing the boundary value of g_{44} with certain expressions involving only the mass concentration and the ratio of the central energy density to the central pressure. A minimal theorem appropriate to the relativistic domain is derived for the central pressure, corresponding to a well-known classical result. Inequalities involving the proper energy and the potential energy are also considered, as is the introduction of the physical radius in place of the coordinate radius. A singularity-free elementary algebraic solution of the field equations is presented and exact values obtained from it compared with the limits prescribed by some of the inequalities. In Part II an answer is given to the question whether the total amount of radiation emitted during the symmetrical gravitational contraction of an amount of matter whose initial energy, at complete dispersion, is W_0 can ever exceed W_0.

PART I

1. Introduction

VARIOUS questions concerning fluid spheres in static (thermodynamic) equilibrium in the context of the general relativity theory are treated not infrequently on the basis of special models, i.e., of special explicit solutions of the field equations, the best known of these probably being the so-called Schwarzschild interior solution.[1] In particular, it is known for this solution that—in terms of the usual coordinate system (see Appendix I)—the ratio of the total mass M to the (coordinate) radius R of the sphere cannot have a value greater than 4/9, or 5/18 if the trace of the energy-momentum tensor is postulated to be non-negative. In other words, although the quantity

$$\Delta = 1 - 2M/R \tag{1.1}$$

certainly must not be negative, this particular solution does not allow Δ to approach zero but prescribes the minimum value $\frac{1}{9}$. This limitation arises essentially from the condition that g_{44} must not be negative anywhere. It should be noted that the result $M \leq 4R/9$ depends upon the coordinates used. However, one may replace R by the physical radius R^* of the sphere, and then in this case one has $M \lesssim 0.3404R^*$, which represents an *invariant* limitation upon the mass of the sphere (invariantly defined in terms of the motion of a test particle "at infinity") if its radius be prescribed. Again, one may calculate quantities such as g_{44} at the center of that Schwarzschild sphere whose central pressure is just one third of the central energy density, and this turns out to be $\frac{1}{4}$, which shows clearly the strongly non-Galilean character of the metric; and so on.

Now, instead of having results of this kind available for a few special models it seems desirable to establish analogous general results for arbitrary static fluid spheres in the form of inequalities, the distributions being subject only to some limitations of a general kind. The present paper accordingly makes a simple attack on this problem. In Secs. 3 and 8 of Part I certain limitations on Δ are established, it being supposed alternatively that the density does not increase or does not decrease outwards; and the possibility is taken into

---

* On leave from the Physics Department, University of Tasmania, Hobart, Tasmania, Australia.

[1] R. C. Tolman, Phys. Rev. **55**, 364 (1939).