Meson Corrections to the Hyperfine Structure in Hydrogen

C. K. IDDINGS* AND P. M. PLATZMAN†
California Institute of Technology, Pasadena, California
(Received March 30, 1959)

The authors have previously shown that the form factor of the proton, measured by Hofstadter, implies an energy shift in the hyperfine structure of atomic hydrogen. Uncertainties in this method of applying the Hofstadter data, due to the effects of virtual mesons, are investigated in an approximate way and found to be small, on the order of one to two percent of the total correction. Under certain assumptions about the form of the amplitudes for various photoproduction processes, we conclude that the value of the fine-structure constant implied by the hyperfine structure (hfs) measurement of Kusch disagrees with the value obtained from the Lamb shift by 34±21 ppm.

IN a recent paper,[1] we have shown that the inclusion of a Hofstadter form factor for the proton removes a rather arbitrary cutoff present in previous calculations and produces a small additional correction to the hfs of atomic hydrogen. The effect is best discussed when combined with the so-called "recoil corrections" of Newcomb and Salpeter.[2] The combined shift was found to be -35 ± 3 ppm. However, the experimental limits set on this correction (comparison of hfs measurements and the Lamb shift)[3] are -1 ± 18 ppm. Our conclusion[1] was that meson effects might account for this difference. We have estimated these corrections and find that they are the order of 1 ppm.

The structure corrections to the hfs may be regarded as due to two diagrams, Figs. 1(a) and 1(b) in reference 1. The energy shift [Eqs. (4) and (9) in reference 1] is given in terms of $\mathrm{Tr}[M_{\mu\nu}(k,k)(1+\gamma_t)i\gamma_x\gamma_y]$. $M_{\mu\nu}$ is, within a constant multiplicative factor, the forward Compton scattering amplitude for virtual photons incident on protons. In the language of dispersion theory, our previous estimate included only the single-nucleon intermediate state (Fig. 1) with a Hofstadter form factor at the vertex. Although our previous work was based upon perturbation theory, the same result may indeed be derived by the use of a dispersion relation and a single-nucleon intermediate state. Since the bare proton and the single-meson, single-nucleon intermediate states seem to characterize the Compton process up to a few hundred Mev (for real photons),[4] we shall base our estimate of meson effects entirely on the process shown in Fig. 2.

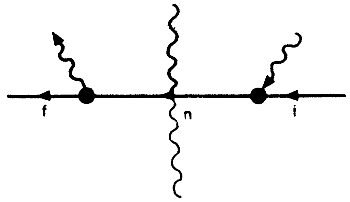

FIG. 1. Compton scattering. Single nucleon intermediate state.

The application of unitarity to the photoproduction amplitude yields the imaginary part of the Compton amplitude,

$$M_{fi}+M_{fi}^{\dagger}=-\sum_n M_{fn}^{\dagger}M_{ni}\delta^4(p_n-p_i), \qquad (1)$$

where

$$S_{fi}=\delta_{fi}+M_{fi}\delta^4(p_f-p_i).$$

For the case of real photons the values of M_{ni} and M_{fn} are experimentally well determined by the photoproduction cross sections. For virtual photons there are fewer experimental results,[5] and of course the square of the mass of the photon k_μ^2 is less than zero ($k_\mu^2=\omega^2-\mathbf{k}^2$). For real photons there are low-energy theorems which permit the determination of the number of subtractions and the values of these constants for certain amplitudes.

We assume that the virtual photoproduction amplitude for π mesons is pure $M1(\frac{3}{2},\frac{3}{2})$ in the center-of-mass (c.m.) system, although longitudinal quadrupole can also lead to the $(\frac{3}{2},\frac{3}{2})$ state. In addition, we shall take the energy dependence of the cross section to be a delta function at the well-determined c.m. resonance energy, $W_0=2.1M$. If only transverse photons of polarization $\mathbf{e}$ and momentum $\mathbf{k}$ are considered, the amplitude for $M1(\frac{3}{2})$ production of a meson of momentum $\mathbf{q}$ is, in the c.m. system,

$$M_{ni}=F_4(k_\mu^2)N(k_\mu^2,W)[2\hat{k}\times\mathbf{e}\cdot\hat{q}+i\boldsymbol{\sigma}\cdot(\hat{k}\times\mathbf{e})\times\hat{q}], \quad (2)$$

where $F_4(k_\mu^2)$ is the Hofstadter form factor, $\Lambda^4/(k_\mu^2-\Lambda^2)^2$, where $\Lambda=0.91M$. We take N proportional to $\delta(W-W_0)$ and by a comparison with the resonance fit of Gell-Mann and Watson,[6] the expression

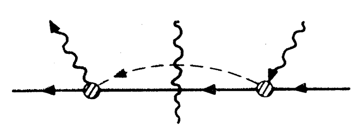

FIG. 2. Compton scattering. Single meson—single nucleon intermediate state.

* Rand Fellow.
† Howard Hughes Fellow.
[1] C. K. Iddings and P. M. Platzman, Phys. Rev. **113**, 192 (1959).
[2] W. A. Newcomb and E. E. Salpeter, Phys. Rev. **97**, 1146 (1955).
[3] Triebwaser, Dayhoff, and Lamb, Phys. Rev. **89**, 106 (1953).
[4] Jon Mathews, doctoral thesis, California Institute of Technology, 1957 (unpublished).
[5] W. K. H. Panofsky and E. A. Allton, Phys. Rev. **110**, 1155 (1958).
[6] M. Gell-Mann and K. M. Watson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, California, 1954), Vol. 4, pp. 219–270.

is normalized for zero-mass photons. Since

$$\sigma_{\frac{3}{2}}=(2\pi)^2\frac{E_1}{W}\sum_f\sum_{\text{Av }i}\delta^4(p_f-p_i)|M_{fi}|^2, \quad (3)$$

and since[6]

$$\sigma(\gamma+p\rightarrow\pi^0+p)=\frac{2}{3}\sigma_{\frac{3}{2}}=\frac{2\pi}{\mathbf{k}^2}\frac{\Gamma_\gamma\Gamma}{(W-W_0)^2+\frac{1}{4}\Gamma^2}, \quad (4)$$

by taking

$$\frac{\frac{1}{2}\Gamma}{(W-W_0)^2+\frac{1}{4}\Gamma^2}=\pi\delta(W-W_0), \quad (5)$$

we find that

$$|N|^2=\frac{W^2}{qq_0E_1E_2}\frac{3}{8\pi}\frac{\Gamma_\gamma}{\mathbf{k}^2}\delta(W-W_0), \quad (6)$$

where E_1 and E_2 are the initial and final energies, respectively, of the nucleon and q_0 is the energy of the meson.

For a fixed c.m. energy, the quantities E_1 and ω may be considered analytic functions of k_μ^2; i.e., $E_1=(W^2+M^2-k_\mu^2)/2W$, $\omega=(W^2-M^2+k_\mu^2)/2W$. For values of $k_\mu^2\geq 0.1M^2$, the quantities $|\mathbf{k}|$ and ω make no physical sense. Thus a dependence of $|N|^2$ on k_μ^2 via the kinematical quantities E_1, $|\mathbf{k}|$, and $|\omega|$ has no clear meaning. We neglect this dependence.

We may write $M_{fi}^{\text{c.m.}}$ [if we assume only $M1(\frac{3}{2},\frac{3}{2})$ intermediate states] as

$$M_{fi}^{\text{c.m.}}=h_1^{\text{c.m.}}(k_\mu^2,W)i\mathbf{e}\cdot\mathbf{e}'+h_2^{\text{c.m.}}(k_\mu^2W)\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}. \quad (7)$$

The application of unitarity (1) to (2) gives

$$\text{Im}h_2^{\text{c.m.}}=\frac{3}{4}\left(\frac{W}{E_1}\right)\frac{\Gamma_\gamma}{\omega_{\text{c.m.}}^2}\delta(W-W_0). \quad (8)$$

In order to apply a dispersion relation to (8), we must transform to the laboratory system; this we do by a simple relativistic generalization of the M matrix and evaluation in the laboratory frame:

$$\text{Im}h_2^{\text{lab}}=h_2^{\text{c.m.}}\frac{E_1\omega_{\text{c.m.}}}{M\omega_{\text{lab}}}=f(k_\mu^2,W)h_2^{\text{c.m.}}. \quad (9)$$

Since h_1 does not contribute to the hfs, we neglect it. Again we neglect the dependence of $f(k_\mu^2,\omega)$ on k_μ^2 because of the uncertain nature of the kinematical quantities which give rise to it. A more sophisticated treatment would, of course, include these effects. In the laboratory system we do include the k_μ^2 dependence of $\delta(\omega-\omega_{\text{res}})$, and we include (in one estimate) a Hofstadter form factor at each vertex. Recent experimental results[5] for small k_μ^2, and a more careful dispersion treatment[7] indicate that our photoproduction amplitude is at least qualitatively correct.

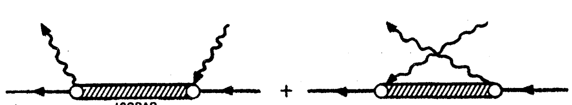

FIG. 3. Isobar approximation to the process shown in Fig. 2.

The application of dispersion relations in the laboratory frame then results in a Compton amplitude which behaves as if the usual Feynman diagram were replaced by one with an intermediate "isobar" of total mass equal to the resonance energy of the $(\frac{3}{2},\frac{3}{2})$ state (Fig. 3). (It is to achieve this behavior that the dependence of ω on k_μ^2 in the argument of the δ-function was retained.) We have calculated the single-meson correction to the Compton amplitude by the use of dispersion relations with a single subtraction and for the case of dispersion relations with no subtractions. For one subtraction we have

$$\text{Re}[\omega h_2(\omega,k_\mu^2)]=\omega C+\frac{2\omega^3}{\pi}\int\frac{\text{Im}[\omega' h_2(\omega',k_\mu^2)]}{\omega'^2(\omega'^2-\omega^2)}d\omega'. \quad (10)$$

The arbitrary constant in the single-subtraction case is put equal to zero, its value in the Born approximation. We assume no meson effects at low energies. Thus the correction to the Compton amplitude due to meson

TABLE I. Calculated energy shifts.[a]

Subtractions	Hofstadter form factor	No Hofstadter form factor
None		$|2|$
One	$<|2.6|$	$|\frac{1}{2}|$

[a] The units are $(\alpha m/M\pi\mu_p)$hfs.

absorptive effects has no arbitrary constants in it. For real photons the low-energy theorem guarantees this. For $k_\mu^2\neq 0$, we do not know this. We now use these additional contributions to the Compton amplitude to give corrections to the hfs according to the method mentioned above.[1] Since all the integrals are convergent, a Hofstadter form factor is not required. In the case of a single subtraction we have included a Hofstadter form factor to ascertain its effect. In all cases the shift is of the order of 1 to 2 ppm. Results are given in Table I. These results are negligible compared to the energy shift for a single-proton intermediate state[1] of -76 ± 6 in these same units where the uncertainty comes from the measurements of the Hofstadter form factor. Although our treatment of photoproduction by virtual photons is crude, we feel it should reveal the main features of the process, especially insofar as it is dominated by $(\frac{3}{2},\frac{3}{2})$ resonance.

The following summarizes our conclusions on the validity of the whole theoretical calculation. The energy shift calculated in reference 1, formula (11), is the most important term. The result obtained for the energy shift, from this term, does not depend appreciably on the approximate analytic form we have chosen

[7] Fubini, Nambu, and Wataghin, Phys. Rev. 111, 329 (1958).

for the Hofstadter form factor $[F_4(k_\mu^2)]$, except in so far as it is experimentally determined. We need assume only that the singularities of the true structure factor lie as shown in Fig. 4. For then, instead of evaluating the integral for ΔE_{3s-1s}, Eq. (11) in reference 1, by integrating along the real ω axis, we can rotate the path of integration through 90 degrees in the complex ω plane so as to integrate along $i\omega$. When this is done, the resulting metric is positive definite; the structure factor is needed now only for negative k_μ^2, and over 90% of the contribution to the integral comes from values of $-k_\mu^2$ in the range zero to $0.6M^2$ where $F_4(k_\mu^2)$ is directly measured.

The other question which needs to be answered is what to do about diagrams with higher-mass intermediate states. There are two possibilities. If we postulate that all the amplitudes for photoproduction by a virtual photon contain a factor $F_4(k_\mu^2)$, then we are justified in neglecting these higher-mass diagrams. The same rotation will be possible, and the corrections for these diagrams will come from low-mass, low-energy photons. The contributions in this range are then correctly given by the low-mass intermediate states. Even the contribution from the $(\frac{3}{2},\frac{3}{2})$ resonance in the photoproduction amplitude is small compared with that of the bare proton. Higher states are expected to be further damped. On the other hand, virtual photons may not act through $F_4(k_\mu^2)$ in all processes. In this paper, we have calculated the case of a single-meson, single-nucleon intermediate state, without using $F_4(k_\mu^2)$ and found that even so, these corrections change our answer by only (approximately) one percent. We cannot, however, rule out the *unlikely* possibility that higher-mass diagrams do contribute a significant amount to the energy shift.

The structure and recoil corrections $P_{\rm exp}=[+1-(1.4\pm18)\times10^{-6}]$ are obtained using the value of α determined from the measurements of fine structure in deuterium.[3] $P_{\rm calc}$, on the other hand, equals $[1-(35\pm3)\times10^{-6}]$, where the ±3 is due to the experimental uncertainties in the Hofstadter form factor. There is an apparent discrepancy of 34 ± 21 ppm between the two separate determinations. Some of this difference may be attributed to various small terms neglected in the calculation and in the theoretical formula for deuterium splitting.

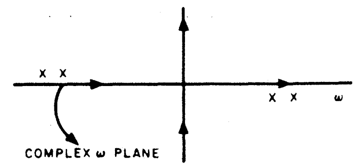

FIG. 4. Assumed location of singularities in true structure factor.

The possible sources of the difference might be the following: (1) recoil corrections in deuterium; (2) uncertainty in making corrections for the polarizability of the oil sample (used in determining the proton anomaly ratio), (3) meson effects, as estimated in this paper; (4) higher-order corrections ($\alpha^3 \ln\alpha$). Because all these effects have been estimated only approximately and because they may not all add in the same direction, we feel that they could at the most account for about 10 ppm of the difference. Although the experiments and calculations have been performed very carefully, the apparent discrepancy is just at the limit of both, and therefore we do not know how seriously to view it.

ACKNOWLEDGMENTS

We would like to thank Professor M. Gell-Mann for suggesting the method of attack in this problem. We are also indebted to Professor R. P. Feynman for his continued encouragement and many helpful comments and criticisms.