Collective Excitations in the Theory of Superconductivity*

G. RICKAYZEN†
Department of Physics, University of Illinois, Urbana, Illinois
(Received March 5, 1959)

The complex dielectric constant of a superconductor and the Meissner effect are derived in a manner which is gauge invariant, from the theory of superconductivity due to Bardeen, Cooper, and Schrieffer. The collective excitations are important in maintaining gauge invariance; the longitudinal collective excitations ensure that a static vector potential produces no longitudinal current and the transverse collective excitations contribute to the Meissner current an amount which depends on the angular properties of the two-body interaction. This contribution is estimated to be small. An earlier calculation of ultrasonic absorption in superconductors is justified. The whole investigation is based upon the generalized random-phase approximation introduced by Anderson and applies whether or not the Coulomb interaction between the electrons is taken into account. The equations of motion are linearized in such a way that the exchange terms are automatically screened if the Coulomb interaction is, in fact, taken into account. The region of applicability of most of the results is limited by the approximations to temperatures at or near absolute zero.

1. INTRODUCTION

MANY properties of a superconductor, notably the thermodynamic properties, can be understood in terms of independent quasi-particle excitations of the system. However, the force between one electron and another which brings about the superconducting transition also ensures that the system possess certain collective excitations. These collective excitations are essential for a complete understanding of certain properties of the superconducting system, particularly its interaction with external electric and magnetic fields. In this paper we wish to stress the collective aspects of the theory of superconductivity of Bardeen, Cooper, and Schrieffer.[1] The main contributions of the present paper are a completely gauge-covariant calculation of the Meissner effect and of the complex dielectric constant of a superconductor at absolute zero. These calculations take into account both the longitudinal and transverse collective excitations of the system.

Using the random-phase approximation, Anderson[2] and Bogoliubov, Tolmachov, and Shirkov[3] studied the existence and frequencies of the longitudinal collective excitations. They have found that the plasma frequency and the collective coordinates are practically unchanged in the transition to the superconducting state, in the long-wavelength limit. They have pointed out the existence of transverse and more complicated oscillations and BTS have attempted to calculate their frequencies. Anderson has shown that when the longitudinal collective modes are taken into account then, at least to order $(\epsilon_0/\hbar\omega)^2$, the longitudinal sum rules are satisfied.

Already in a normal metal the collective aspects of the interacting electron system are important. As is well known, if an external charge interacts with the electrons of the metal, all the electrons are perturbed in such a way that each electron is acted upon by the field of the external charge together with the perturbed fields of all the other electrons. The result is that every electron is perturbed by a screened field. (The screened field can be calculated by a self-consistent Hartree method[4] or equivalently by a canonical transformation.[5,6]) If the external charge density is a wave of long wavelength the screening is practically complete.

Another way of looking at the screening is to note that a part of the charge density of the electrons is a plasmon variable and that a low-frequency external field will not excite the plasmon states. We should like to develop this viewpoint using some of the ideas of Lipkin.[7] The system of electrons possesses longitudinal collective modes of wave vector $\mathbf{k}$ with coordinates Q_k, conjugate momenta P_k, and frequency ω_k. In the long-wavelength limit, Q_k is $(4\pi e^2)^{\frac{1}{2}}\sum_i \exp(i\mathbf{k}\cdot\mathbf{x}_i/2)(\mathbf{k}\cdot\mathbf{p}_i/nk)\exp(i\mathbf{k}\cdot\mathbf{x}_i/2)$ and P_k is $i(4\pi e^2/k^2)^{\frac{1}{2}}\sum_i \exp(i\mathbf{k}\cdot\mathbf{x}_i)$, where the $\mathbf{x}_i$ are the coordinates of the electrons and $\mathbf{p}_i$ are their momenta. There will be other operators, functions of $\mathbf{x}_i$ and $\mathbf{p}_i$, which, added to P_k, Q_k, will form a complete set. What these other variables are we leave aside for the moment. Now if an external field (say that associated with an incoming phonon) of wave vector $\mathbf{k}$, frequency Ω, and amplitude r_k acts on the system, there is an extra interaction term in the Hamiltonian,

$$M_k^2 r_k \rho_{-k}\exp(i\Omega t)+\text{c.c.}, \quad \rho_{-k}=\sum_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i),$$

where M_k^2 is the strength of the interaction. The component of the charge density ρ_{-k} can be expanded in terms of P_k, Q_k, and the other variables of the complete

* This work was supported in part by the Office of Ordnance Research, U. S. Army.
† Present address: Nuclear Physics Research Laboratory, University of Liverpool, England.
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[5] S. Nakajima, *Proceedings of the International Conference on Theoretical Physics, Kyoto and Tokyo, September, 1953* (Science Council of Japan, Tokyo, 1954).
[6] J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).
[7] H. J. Lipkin, Phys. Rev. Letters **2**, 159 (1959).

set; it then has the structure

$$\rho_{-k}=\alpha_k P_k+\rho_{-k,s},$$

where

$$-i\hbar\alpha_k=[\rho_{-k},Q_k]_-, \quad =-\hbar(4\pi e^2/k^2)^{-\frac{1}{2}},$$

and $\rho_{-k,s}$ is the residual screened charge density which vanishes in the long-wavelength limit. ($\rho_{-k,s}$ is of order g_k^2 relative to P_k, where g_k is the electron-plasmon coupling constant.) The first term of ρ_{-k} can lead to real transitions only if energies $\hbar\omega_k$ are involved. In general, however, $\omega_k \gg \Omega$, so that the only part of ρ_{-k} that can cause real transitions is $\rho_{-k,s}$, that is, the screened part.

One can eliminate the collective part of ρ_{-k} from the Hamiltonian by performing a canonical transformation with the unitary operator

$$\exp[ir_k e^{i\Omega t}M_k^2\alpha_k Q_{-k}/\hbar-\text{c.c.}];$$

as $k\rightarrow 0$ this approaches

$$\exp[ir_k e^{i\Omega t}\textstyle\sum_i \exp(-\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}_i) \times \mathbf{p}_i \exp(-\frac{1}{2}i\mathbf{k}\cdot\mathbf{x}_i)\cdot\mathbf{k}/nk^2-\text{c.c.}]$$

which is of the form found by Pines and Schrieffer[8] and can be interpreted as giving rise to a "dipolar backflow." The backflow is calculated in more detail at the end of Sec. 4.

In a metal the screening and collective coordinates are also affected by the extra phonon interaction between electrons which leads to the superconducting state. However, the contribution to the screening comes from all electrons within the Fermi sphere, whereas in the superconducting transition only those electrons within a small energy range kT_c of the Fermi surface are involved. Thus it is to be expected that in the superconducting state the screening will be practically the same as in the normal state. This is confirmed by the calculations of Anderson and BTS and by that of Sec. IV which is also based upon the generalized random phase approximation.

One problem that requires the introduction (explicit or implicit) of the collective modes for its solution is that of the Meissner effect. BCS have calculated by perturbation theory the current density produced by a static transverse vector potential. They made no attempt to derive the current density produced by a static longitudinal vector potential although it was earlier pointed out by Bardeen[9] that it would be necessary to take account of the longitudinal collective modes to do this. A static longitudinal vector potential contributes neither to the electric field nor to the magnetic field; it should, therefore, have no physical effects and should not give rise to a current. Because the BCS method, if applied without modification to the longitudinal vector potential, would give a nonzero (not even small) current, doubt has been cast on their calculation of the transverse current.[10] Anderson's verification[11] of the sum rules shows that the longitudinal current density is of order $(\epsilon_0/\hbar\omega)^2$, but as he does not introduce the wave-functions explicitly his proof is not a test of the method of BCS for calculating the transverse current.

Pines and Schrieffer,[8] by exploiting the smallness of the electron-plasmon coupling constant, g, have shown that the longitudinal current density is small. Their proof introduces the wave functions for the collective states and depends on the fact that the collective coordinates are practically unchanged in the superconducting transition. As we have seen, in the normal state the long-wavelength components of charge and longitudinal paramagnetic current densities are just collective coordinates (P_k and Q_k, respectively) to order g^2 and this is still true in the superconducting state. As Pines and Schrieffer show, the collective part of the longitudinal paramagnetic current density just cancels the diamagnetic current density and the total longitudinal current density is at the most of order g^4. An objection[10] has been raised against the argument of Pines and Schrieffer on the grounds that they use the controversial subsidiary condition of Bohm and Pines,[12] but in the way we have put the argument the subsidiary condition does not arise. In Sec. VI of this paper the corrections to the collective coordinates are taken into account and from the equations of motion obtained by Anderson within the generalized random-phase approximation, the transverse and longitudinal currents are calculated simultaneously. Because the longitudinal current is found to be zero it is believed that the calculation of the transverse current is to be trusted.

Because of the properties of the interaction which leads to the superconducting transition there exist, in a superconductor, transverse collective modes. (The effect of these excitations on the Meissner current has already been reported in a preliminary letter.[13]) In the presence of a static transverse vector potential they do not contribute to the current in the London limit (where the penetration depth is much greater than the coherence distance); only the single-particle excitations have to be considered and the calculation is exactly that of BCS. In the Pippard limit (where the penetration depth is much less than coherence distance) there is a contribution from the transverse modes and this depends on the angular properties of the two-particle interaction. We have assumed a simple angular dependence for the interaction to estimate the order of magnitude of the correction and we conclude that it is, in fact, small.

A simple model for a superconductor that is commonly used[3,14,15] is a gas of Fermi particles which do not interact with each other through the Coulomb interaction, but only through the phonons. For this model the

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long-wavelength longitudinal excitations are not charge or current density fluctuations and it is not so easy to see their part in the Meissner effect and the screening. The method given below applies to both this case and a model which is more realistic and the results are formally the same. Of course if the parameters involved in the two models were calculated from first principles they would be different. If the treatment of Bogoliubov is used and the Meissner effect calculated according to the method used previously by the author,[15] we believe that to obtain the correct result for the effect of the longitudinal vector potential, it is necessary to sum an infinite set of graphs. This same set of graphs should lead to a correction to the transverse current of the kind calculated in Sec. VI.

If one wishes to compute quantities of order g^2, for example the screening of a time-dependent external field, then the single-particle excitations of the system must be examined more closely and one must calculate $\rho_{-k,s}$. In the first calculations of the interaction of a time-dependent longitudinal external field with the superconducting electrons[1,16] it was assumed that one can use the normal form for the interaction with the same screening as in the normal state. For instance, it was implicitly assumed in BCS that the interaction between an acoustic wave and the superconducting electrons is of the form

$$\sum_Q (4\pi e^2/Q^2) r_{-Q} e^{-i\Omega t} \rho_Q f(Q) + \text{c.c.}, \qquad (1.1)$$

where $r_{-Q}\exp(i\Omega t)$ is the charge fluctuation associated with the acoustic wave and $f(Q)$ is the same screening factor as in the normal state. However, if the theory is gauge invariant one should be able to describe the interaction by either a vector potential or a scalar potential. Thus an equally good choice for the interaction would appear to be

$$\sum_Q (4\pi e^2/Q^2) r_{-Q} e^{-i\Omega t} [\mathbf{Q}\cdot\mathbf{j}(Q)] f(Q)/\Omega + \text{c.c.}, \qquad (1.2)$$

which is also the interaction in the normal state. If this interaction is actually used, then a result for the attenuation is obtained very different from that of BCS. This discrepancy arises because the equivalence of Eqs. (1.1) and (1.2) depends on the equation for the conservation of charge,

$$(E_n - E_m)\langle n|\rho|m\rangle = \hbar\langle n|\mathbf{Q}\cdot\mathbf{j}|m\rangle,$$

and this equation is not satisfied by the wave functions of BCS [not even to within $(\epsilon_0/\hbar\omega)^2$]. There is no general principle to show which result is correct. Therefore in order to ensure that the result be independent of gauge, it is necessary that the screening be calculated from improved wave functions, in a way that is gauge invariant. This calculation is performed in Sec. III, and it is concluded that for cases of practical importance $\langle n|\rho_Q|m\rangle$ can be replaced by $\langle n|\rho_Q|m\rangle_{\text{BCS}} f(Q)$ so that one can use the scalar potential screened as in the normal state. The interaction with a vector potential has to be modified so as to produce agreement.

The basis of this paper is the set of linear equations of motion determined by Anderson[2]; he used the generalized random-phase approximation and neglected the exchange terms. We solve these equations in Sec. III and obtain wave functions for which the equation governing the conservation of charge is satisfied. We then find a simple approximation to the matrix elements. In Sec. IV we add the effect of an external time-dependent charge fluctuation to the equations of motion and from the solution determine the generalized dielectric constant which depends on the frequency and wave vector of the external field. A form for the interaction with the external field is given, which can be used with the wave functions of BCS. From this interaction it is shown in Sec. V that corrections to the calculation of ultrasonic attenuation by BCS are negligible. It is also shown that the exact wave functions incorporate a backflow around an external charge. In Sec. VI, the effect of an external static vector potential is added to the equations of motion and from the solution the Meissner effect is calculated in an arbitrary gauge.

As the results of this paper are based upon the generalized random-phase approximation (RPA), their region of applicability is limited. The essence of the RPA is that products of a pair of single-particle operators are treated as bosons. The approximation takes account of the transition of a pair of quasiparticles out of the Fermi sea or into it but neglects the scattering of a particle outside the sea. This approximation should be valid as long as the number of single-particle excitations outside the sea is small, a condition which restricts the discussion to the region of temperature near absolute zero. The result for the real part of the dielectric constant is as accurate as all but the most recent calculations of the dielectric constant of a free-electron gas. If we were to include the exchange terms we should even obtain Hubbard's result[17] but we shall not enlarge on this here.

One defect of the present analysis is that the two-body interaction is arbitrarily chosen so that the Hamiltonian is gauge invariant and still leads to a superconducting transition. (The two-body interaction used by BCS is such that their Hamiltonian is not strictly gauge invariant.) Most of the results are not sensitive to the potential used and can be applied directly to the model of BCS. The physical principles underlying the mathematics are generally so clear that it is easy to see how the results can be applied. Those corrections which are sensitive to the form of the interaction cannot be trusted quantitatively except perhaps so far as the order of magnitude. In any case our model provides an example against which calculations of the electromagnetic properties of superconductors can be tested. To

[16] Bardeen, Tewordt, and Rickayzen, Phys. Rev. **113**, 982 (1959).

[17] J. Hubbard, Proc. Roy. Soc. (London) **A240**, 539 (1957).

improve upon our method it would be necessary to go back to the original gauge-covariant Hamiltonian from which the electron-phonon interaction has not been eliminated.

An unsatisfactory feature of Anderson's approach is the assumption that in the terms of the equations of motion that lead to the superconductlng transition, the two-body interactions are screened. This point has been discussed by Anderson, who suggests that these equations are the second step in a self-consistent calculation. It is clear from the work of BCS and BTS that the two-body interaction in this term is in fact screened, but it would be more satisfactory to have a justification of Anderson's equations of motion which is basic. In Appendix A we attempt to provide this justification. We are led to a system of nonlinear equations which can be approximated by Anderson's equations in just the way he suggests. Although we do not take into account the exchange terms, it is possible to do this by the method of Appendix A and to obtain a set of linear equations with the exchange terms properly screened.

2. NOTATION

In this section the notation to be used is summarized. The operators which create electrons in the states of momentum $\mathbf{k}$ and spin σ are denoted by $c_{k,\sigma}$. The Bloch energies of the electrons in the normal state are ϵ_k, measured from the Fermi level, ϵ_F. In general ϵ_k will be assumed to be $(\hbar^2k^2/2m)-\epsilon_F$. The velocity at the Fermi surface is v_0. We define

$$\rho_k{}^Q=c_{k+Q\uparrow}{}^*c_{k\uparrow},\quad \bar{\rho}_k{}^Q=c_{-k\downarrow}{}^*c_{-k-Q\downarrow},$$
$$b_k{}^Q=c_{-k-Q\downarrow}c_{k\uparrow},\quad \bar{b}_k{}^Q=c_{k+Q\uparrow}{}^*c_{-k\downarrow}{}^*.$$

The operators b_k, ρ_k corresponding to $\mathbf{Q}=0$ are always taken as c-numbers, their expectation values in the BCS ground state,

$$b_k=\langle c_{-k\downarrow}c_{k\uparrow}\rangle_0=b_k{}^*,$$
$$\rho_k=\langle c_{k\uparrow}{}^*c_{k\uparrow}\rangle_0=\bar{\rho}_k=n_k.$$

The wave functions are normalized to unit volume. The Hamiltonian is

$$H=H_K+H_V+H_C,$$

where

$$H_K=\sum_{k,\sigma}\epsilon_kc_{k,\sigma}{}^*c_{k,\sigma},$$

$$H_V=\tfrac{1}{2}\sum_{k,k',q}\sum_{\sigma,\sigma'}v(\mathbf{k},\mathbf{k}')c_{k'\sigma'}{}^*c_{-k'+q\sigma}{}^*c_{-k+q\sigma}c_{k\sigma'},$$

$$H_C=\sum_{k,k',q}\sum_{\sigma,\sigma'}2\pi e^2|\mathbf{k}-\mathbf{k}'|^{-2}c_{k'\sigma'}{}^*c_{-k'+q\sigma}{}^*c_{-k+q\sigma}c_{k\sigma'}.$$

For the theory to be gauge invariant, $v(\mathbf{k},\mathbf{k}')$ is taken to be a function of $(\mathbf{k}-\mathbf{k}')$ only, $v(\mathbf{k}-\mathbf{k}')$. The direct interaction is denoted by

$$V_D(\mathbf{Q})=4\pi e^2Q^{-2}+v(\mathbf{Q}),$$

and the interaction which leads to the superconducting transition is denoted by

$$V(\mathbf{k},\mathbf{k}')=[v(\mathbf{k},\mathbf{k}')+4\pi e^2|\mathbf{k}-\mathbf{k}'|^{-2}]\times(\text{screening factor}).$$

The potential $V(\mathbf{k},\mathbf{k}')$ is predominantly negative when $\mathbf{k}$ and $\mathbf{k}'$ are near the Fermi surface. If the method of Appendix A is followed, the screening factor is zero for $|\mathbf{k}-\mathbf{k}'|<Q_{\max}$ and unity otherwise. The wave number $Q_{\max}$ is the cutoff of Sawada *et al.*[18]

The energy gap parameter, analogous to ϵ_0 of BCS, is I_K where

$$I_K=-\sum_k V(\mathbf{K},\mathbf{k})b_k=-\sum_k V(\mathbf{K},\mathbf{k})b_k{}^*.$$

Near the Fermi surface it is a constant which we shall denote by ϵ_0. The energy of a single quasi-particle excitation is

$$E_k=(\epsilon_k{}^2+I_k{}^2)^{\frac{1}{2}}.$$

The energy of a pair of particles with momenta $-\mathbf{k}$ and $\mathbf{k}+\mathbf{Q}$ is

$$\nu_k(\mathbf{Q})=E_k+E_{k+Q}.$$

The coherence distance is $\xi_0=\hbar v_0/\pi\epsilon_0$. The ground state is such that

$$b_k=b_k{}^*=u_kv_k>0,\quad n_k=v_k{}^2,$$

where

$$u_k{}^2=\tfrac{1}{2}(1+\epsilon_k/E_k),\quad v_k{}^2=\tfrac{1}{2}(1-\epsilon_k/E_k).$$

We shall use the operators,

$$\gamma_{k0}=u_kc_{k\uparrow}-v_kc_{-k\downarrow}{}^*,$$
$$\gamma_{k1}=u_kc_{-k\downarrow}+v_kc_{k\uparrow}{}^*,$$

the four coherence factors,

$$l(\mathbf{k},\mathbf{Q})=u_ku_{k+Q}+v_kv_{k+Q},$$
$$m(\mathbf{k},\mathbf{Q})=u_kv_{k+Q}+v_ku_{k+Q},$$
$$n(\mathbf{k},\mathbf{Q})=u_ku_{k+Q}-v_kv_{k+Q},$$
$$p(\mathbf{k},\mathbf{Q})=u_kv_{k+Q}-v_ku_{k+Q},$$

and the three collective variables

$$\rho(\mathbf{Q})=\sum_k(\rho_k{}^Q+\bar{\rho}_k{}^Q)$$
$$=\sum_k[m(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+\gamma_{k+Q1}\gamma_{k0})$$
$$+n(\mathbf{k},\mathbf{Q})(\gamma_{k1}{}^*\gamma_{k+Q1}+\gamma_{k+Q0}{}^*\gamma_{k0})],$$
$$B_K(\mathbf{Q})=\sum_k V(\mathbf{K},\mathbf{k})(b_k{}^Q+\bar{b}_k{}^Q)$$
$$=\sum_k V(\mathbf{K},\mathbf{k})[n(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+\gamma_{k+Q1}\gamma_{k0})$$
$$-m(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k0}+\gamma_{k1}{}^*\gamma_{k+Q1})],$$
$$A_K(\mathbf{Q})=\sum_k V(\mathbf{K},\mathbf{k})(b_k{}^Q-\bar{b}_k{}^Q)$$
$$=-\sum_k V(\mathbf{K},\mathbf{k})[l(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q1}\gamma_{k0})$$
$$+p(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k+Q1})].$$

[18] Sawada, Brueckner, Fukuda, and Brout, Phys. Rev. **108**, 507 (1957).

3. SOLUTION OF THE EQUATIONS OF MOTION

In this section we are concerned with the solution of the equations of motion derived from the Hamiltonian H, and establishing the equation of conservation of charge,

$$(E_n-E_0)(\Psi_n,\rho(\mathbf{Q})\Psi_0)=\hbar(\Psi_n,\mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0). \quad (3.1)$$

Having obtained matrix elements for which the equation of conservation holds, we find approximate expressions for them which are valid in the usual physical situations for which they are required.

The equations of motion that are the basis for this work are the linear equations derived by Anderson. They were obtained from the full equations of motion for $\rho_k{}^Q$, $\bar{\rho}_k{}^Q$, $b_k{}^Q$, $\bar{b}_k{}^Q$, by replacing those products of pairs of operators which have nonzero expectation values in the ground state by those expectation values, i.e., $c_{k\sigma}{}^*c_{k\sigma}$, $c_{-k\downarrow}c_{k\uparrow}$ are replaced by the c-numbers n_k, b_k, respectively. All remaining terms containing the product of four operators are neglected. In those terms in which b_k appears the potential is screened. Anderson gave reasons for this screening but since this procedure is not in the spirit of his approximations we provide a justification for his equations in Appendix A. The resultant linearized equations of motion are

$$[H,\rho_K{}^Q]=(\epsilon_{K+Q}-\epsilon_K)\rho_K{}^Q-V_D(\mathbf{Q})\rho(\mathbf{Q})(n_{K+Q}-n_K)-I_{K+Q}b_K{}^Q+I_K\bar{b}_K{}^Q+b_K\textstyle\sum_k V\bar{b}_k{}^Q-b_{K+Q}\sum_k Vb_k{}^Q, \quad (3.2a)$$

$$[H,\bar{\rho}_K{}^Q]=(\epsilon_K-\epsilon_{K+Q})\bar{\rho}_K{}^Q+V_D(\mathbf{Q})\rho(\mathbf{Q})(n_{K+Q}-n_K)-I_Kb_K{}^Q+I_{K+Q}\bar{b}_K{}^Q+b_{K+Q}\textstyle\sum_k V\bar{b}_k{}^Q-b_K\sum_k Vb_k{}^Q, \quad (3.2b)$$

$$[H,b_K{}^Q]=-(\epsilon_K+\epsilon_{K+Q})b_K{}^Q-V_D(\mathbf{Q})\rho(\mathbf{Q})\times(b_K+b_{K+Q})-I_K\bar{\rho}_K{}^Q-I_{K+Q}\rho_K{}^Q-(1-n_K-n_{K+Q})\textstyle\sum_k Vb_k{}^Q, \quad (3.2c)$$

$$[H,\bar{b}_K{}^Q]=(\epsilon_K+\epsilon_{K+Q})\bar{b}_K{}^Q+V_D(\mathbf{Q})\rho(\mathbf{Q})\times(b_K+b_{K+Q})+I_{K+Q}\bar{\rho}_K{}^Q+I_K\rho_K{}^Q+(1-n_K-n_{K+Q})\textstyle\sum_k V\bar{b}_k{}^Q. \quad (3.2d)$$

As the mathematical detail may obscure the essentially simple steps involved, it will be useful to outline the procedure beforehand. We first find those linear combinations $\mu_k{}^*(\mathbf{Q})$, $\mu_k(\mathbf{Q})$, of the operators ρ, $\bar{\rho}$, b, $\bar{b}$, which are the normalized normal modes of the equations of motion (3.2), and then we define the ground state Ψ_0 of the problem by

$$\mu_k(\mathbf{Q})\Psi_0=0,$$

where the $\mu_k(\mathbf{Q})$ are the destruction operators. Then the unscreened charge density and current density are written in terms of the normal modes, and the matrix elements

$$(\Psi_0\mu_k{}^*(\mathbf{Q}),\rho(\mathbf{Q})\Psi_0), \quad (\Psi_0\mu_k{}^*(\mathbf{Q}),\mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0)$$

are calculated. It is shown explicitly that Eq. (3.1) governing the conservation of charge is satisfied. As already pointed out, Eq. (3.1) implies that the calculation of transition probabilities is independent of gauge. In the succeeding sections the effects of external electromagnetic fields are added to the equations of motion. These produce changes in the operators, $\mu_k(\mathbf{Q})$, from which the complex dielectric constant and the Meissner effect can be calculated.

It is actually easier to work with the operators γ_k of Bogoliubov[19] and Valatin.[20] In terms of these operators it is found (after considerable algebraic manipulation) that

$$[H,\gamma_{k+Q0}{}^*\gamma_{k1}{}^*]=(E_k+E_{k+Q})\gamma_{k+Q0}{}^*\gamma_{k1}{}^*+V_D(\mathbf{Q})m(\mathbf{k},\mathbf{Q})\rho(\mathbf{Q})+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})\times B_k(\mathbf{Q})-\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q}), \quad (3.3a)$$

$$[H,\gamma_{k+Q1}\gamma_{k0}]=-(E_k+E_{k+Q})\gamma_{k+Q1}\gamma_{k0}-V_D(\mathbf{Q})m(\mathbf{k},\mathbf{Q})\rho(\mathbf{Q})-\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})\times B_k(\mathbf{Q})-\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q}), \quad (3.3b)$$

$$[H,\gamma_{k+Q0}{}^*\gamma_{k0}]=(E_{k+Q}-E_k)\gamma_{k+Q0}{}^*\gamma_{k0},$$

$$[H,\gamma_{k1}{}^*\gamma_{k+Q1}]=-(E_{k+Q}-E_k)\gamma_{k1}{}^*\gamma_{k+Q1}.$$

Evidently, half of the normal modes are given by the operators $\gamma_{k+Q0}{}^*\gamma_{k0}$, $\gamma_{k1}{}^*\gamma_{k+Q1}$, which have eigenvalues $(E_{k+Q}-E_k)$ and (E_k-E_{k+Q}), respectively. These are just the modes called unphysical by Anderson. Since all the physical states Ψ satisfy

$$\gamma_{k+Q0}{}^*\gamma_{k0}\Psi=\gamma_{k1}{}^*\gamma_{k+Q1}\Psi=0,$$

these operators may be taken to be zero throughout the remainder of this section.

In the absence of the collective coordinates $\rho(\mathbf{Q})$, $A_K(\mathbf{Q})$, $B_K(\mathbf{Q})$, the other normal modes are $\gamma_{k+Q0}{}^*\gamma_{k1}{}^*$ and $\gamma_{k+Q1}\gamma_{k0}$ which oscillate with frequencies (E_k+E_{k+Q}) and $-(E_k+E_{k+Q})$, respectively. In the presence of the collective coordinates there will be modes with frequencies $\pm(E_k+E_{k+Q})$ and also collective modes with frequencies outside the range of the $\pm(E_k+E_{k+Q})$. We shall let $\mu_i(\mathbf{Q})$ denote any collective coordinates. We shall use $\mu_k(\mathbf{Q})^*$ to denote a mode of frequency (E_k+E_{k+Q}) and $\mu_{-k}(-\mathbf{Q})$ to denote the mode of frequency $-(E_k+E_{k+Q})$. The operator $\mu_k{}^*(\mathbf{Q})$ adds energy to the system and is, therefore, a creation operator; $\mu_k(-\mathbf{Q})$ subtracts energy from the system and is, therefore, a destruction operator which must satisfy $\mu\Psi_0=0$. The notation is consistent because the equations of motion are invariant under time reversal. The products $\gamma_{k+Q0}{}^*\gamma_{k1}{}^*$, $\gamma_{k+Q1}\gamma_{k1}$, and the collective variables $\rho(\mathbf{Q})$, $A_k(\mathbf{Q})$, $B_k(\mathbf{Q})$, can be expanded in terms of the μ's and

[19] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958).
[20] J. G. Valatin, Nuovo cimento **7**, 843 (1958).

written as

$$\gamma_{k+Q0}{}^*\gamma_{k1}{}^*=\sum_{k'}\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})\mu_{k'}{}^*(\mathbf{Q})+\sum_{k'}\alpha_2(\mathbf{k},\mathbf{k}',\mathbf{Q})\mu_{-k'}(-\mathbf{Q})+\sum_i\alpha_i(\mathbf{k},\mathbf{Q})\mu_i(\mathbf{Q}), \quad (3.4a)$$

$$\gamma_{k+Q1}\gamma_{k0}=\sum_{k'}\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})\mu_{k'}{}^*(\mathbf{Q})+\sum_{k'}\beta_2(\mathbf{k},\mathbf{k}',\mathbf{Q})\mu_{-k'}(-\mathbf{Q})+\sum_i\beta_i(\mathbf{k},\mathbf{Q})\mu_i(\mathbf{Q}), \quad (3.4b)$$

$$\rho(\mathbf{Q})=\sum_{k'}M_{k'}\mu_{k'}{}^*(\mathbf{Q})+\text{terms in } \mu_{-k'}(-\mathbf{Q}) \text{ and } \mu_i,$$

$$A_k(\mathbf{Q})=\sum_{k'}L_{kk'}\mu_{k'}{}^*(\mathbf{Q})+\text{terms in } \mu_{-k'}(-\mathbf{Q}) \text{ and } \mu_i,$$

$$B_k(\mathbf{Q})=\sum_{k'}N_{kk'}\mu_{k'}{}^*(\mathbf{Q})+\text{terms in } \mu_{-k'}(-\mathbf{Q}) \text{ and } \mu_i.$$

If these expressions are substituted into the equations of motion, the α's and β's must be chosen so that the coefficients of all the $\mu_k{}^*(\mathbf{Q})$ are separately zero. Hence

$$-[\nu_k(\mathbf{Q})-\nu_{k'}(\mathbf{Q})]\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})=\Phi_{k,k'}+\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'},$$

$$[\nu_k(\mathbf{Q})+\nu_{k'}(\mathbf{Q})]\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})=-\Phi_{k,k'}+\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'},$$

where

$$\Phi_{kk'}=V_D m(\mathbf{k},\mathbf{Q})M_{k'}+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})N_{kk'}.$$

A set of orthogonal solutions of the equations are given by

$$\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})=\delta_{k,k'}+\frac{\Phi_{kk'}+\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}}{\nu_{k'}-\nu_k+i\epsilon}, \quad (3.5a)$$

$$\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})=-\frac{\Phi_{kk'}-\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}}{\nu_k+\nu_{k'}}. \quad (3.5b)$$

In the Appendix it is shown that these solutions are normalized. From the definitions of ρ, A, and B and Eqs. (3.3) for $\gamma^*\gamma^*$ and $\gamma\gamma$ it is found that

$$M_{k'}=\sum_k m(\mathbf{k},\mathbf{Q})[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})+\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})], \quad (3.6a)$$

$$L_{kk'}=\sum_{k''}V(\mathbf{k},\mathbf{k}'')l(\mathbf{k}'',\mathbf{Q})\times[\alpha_1(\mathbf{k}'',\mathbf{k}',\mathbf{Q})-\beta_1(\mathbf{k}'',\mathbf{k}',\mathbf{Q})], \quad (3.6b)$$

$$N_{kk'}=\sum_{k''}V(\mathbf{k},\mathbf{k}'')n(\mathbf{k}'',\mathbf{Q})\times[\alpha_1(\mathbf{k}'',\mathbf{k}',\mathbf{Q})+\beta_1(\mathbf{k}'',\mathbf{k}',\mathbf{Q})]. \quad (3.6c)$$

When the α's and β's, as given by Eqs. (3.5), are substituted in these equations we have a set of three linear simultaneous integral equations for the functions $M_{k'}$, $L_{kk'}$, $N_{kk'}$. The situation is not as bad as it seems, for in most cases of interest $M_{k'}$ is much greater than $L_{kk'}$ and $N_{kk'}$, and it is necessary to solve only a simple algebraic equation.

It is at once apparent from the definition of M_k and the fact that the operators are normalized, that

$$(\Psi_0\mu_k{}^*(\mathbf{Q}),\rho(\mathbf{Q})\Psi_0)=M_k(\mathbf{Q}).$$

In the normal state $M_k(\mathbf{Q})$ is the function $[\phi_+(\omega_Q)]^{-1}$ of Brout.[21] For the continuity equation it is necessary to calculate

$$(\Psi_0\mu_k{}^*(\mathbf{Q}),\ \mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0),$$

where

$$\mathbf{j}(\mathbf{Q})=(\hbar/2m)\sum_k(2\mathbf{k}+\mathbf{Q})(\rho_k{}^Q-\bar{\rho}_k{}^Q).$$

[Note that

$$\mathbf{j}(\mathbf{r})=\sum_Q\mathbf{j}(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{r})].$$

Now

$$\hbar\mathbf{Q}\cdot\mathbf{j}(Q)=\sum_k(\epsilon_{k+Q}-\epsilon_k)(\rho_k{}^Q-\bar{\rho}_k{}^Q)$$
$$=\sum_k(\epsilon_{k+Q}-\epsilon_k)[l(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k+Q1})-p(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q1}\gamma_{k0})].$$

Hence

$$\hbar(\Psi_0\mu_k{}^*(Q),\ \mathbf{Q}\cdot\mathbf{j}(Q)\Psi_0)=-\sum_{k'}(\epsilon_{k'+Q}-\epsilon_{k'})p(\mathbf{k}',\mathbf{Q})[\alpha_1(\mathbf{k}',\mathbf{k},\mathbf{Q})-\beta_1(\mathbf{k}',\mathbf{k},\mathbf{Q})].$$

We now use the identity

$$-p(\mathbf{k},\mathbf{Q})(\epsilon_{k+Q}-\epsilon_k)\equiv\nu_k(\mathbf{Q})m(\mathbf{k},\mathbf{Q})-(I_k+I_{k+Q})l(\mathbf{k},\mathbf{Q}) \quad (3.7)$$

to obtain

$$\hbar(\Psi_0\mu_k{}^*(\mathbf{Q}),\ \mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0)$$
$$=\sum_{k'}\nu_{k'}(\mathbf{Q})m(\mathbf{k}',\mathbf{Q})\left[\delta_{k'',k}+\frac{\Phi_{k'k}2\nu_k(\mathbf{Q})}{\nu_k{}^2-\nu_{k'}{}^2}+\frac{l(k',Q)L_{k'k}\nu_{k'}}{\nu_k{}^2-\nu_{k'}{}^2}\right]-\sum_{k'}(I_{k'}+I_{k'+Q})l(\mathbf{k}',\mathbf{Q})\times[\alpha_1(\mathbf{k}',\mathbf{k},\mathbf{Q})-\beta_1(\mathbf{k}',\mathbf{k},\mathbf{Q})].$$

Now

$$\sum_{k'}\frac{\nu_{k'}(Q)^2m(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})L_{k'k}}{\nu_k{}^2-\nu_{k'}{}^2}$$
$$=\sum_{k'}\frac{\nu_k{}^2m(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})L_{k'k}}{\nu_k{}^2-\nu_{k'}{}^2}-\sum_{k'}m(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})L_{kk'},$$

and from Eq. (3.6b)

$$\sum_{k'}m(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})L_{k'k}$$
$$=\sum_{k',k''}(u_{k'}v_{k'}+u_{k'+Q}v_{k'+Q})V(\mathbf{k}',\mathbf{k}'')l(\mathbf{k}'',\mathbf{Q})\times[\alpha_1(\mathbf{k}'',\mathbf{k},\mathbf{Q})-\beta_1(\mathbf{k}'',\mathbf{k},\mathbf{Q})]$$
$$=-\sum_{k''}(I_{k''}+I_{k''+Q})l(\mathbf{k}'',\mathbf{Q})\times[\alpha_1(\mathbf{k}'',\mathbf{k},\mathbf{Q})-\beta_1(\mathbf{k}'',\mathbf{k},\mathbf{Q})].$$

Therefore

$$\hbar(\Psi_0\mu_k{}^*(Q),\ \mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0)$$
$$=\nu_k(\mathbf{Q})\sum_{k'}m(\mathbf{k}',\mathbf{Q})\left[\delta_{k,k'}+\frac{2\Phi_{k'k}\nu_{k'}}{\nu_k{}^2-\nu_{k'}{}^2}+\frac{l(\mathbf{k}',\mathbf{Q})L_{k'k}\nu_k}{\nu_k{}^2-\nu_{k'}{}^2}\right]$$
$$=\nu_k(\mathbf{Q})\sum m(\mathbf{k}',\mathbf{Q})[\alpha_1(\mathbf{k}',\mathbf{k},\mathbf{Q})+\beta_1(\mathbf{k}',\mathbf{k},\mathbf{Q})]$$
$$=\nu_k(\mathbf{Q})M_k(\mathbf{Q}),$$

[21] R. Brout, Phys. Rev. **108**, 515 (1957).

and

$$\hbar(\Psi_0\mu_k^*(\mathbf{Q}),\ \mathbf{Q}\cdot\mathbf{j}(\mathbf{Q})\Psi_0) = \nu_k(\mathbf{Q})(\Psi_0\mu_k(\mathbf{Q})^*,\rho(\mathbf{Q})\Psi_0), \quad (3.1)$$

and the wave functions are such that our results will be gauge invariant.

Now that the matrix elements satisfy the equation of continuity, it is possible to approximate them in a consistent manner. We shall compute M_k. In general the states $\mathbf{k}$ and $\mathbf{k}+\mathbf{Q}$ are close to the Fermi surface, much closer than $\hbar\omega$, the average spread in energy allowed by $V(\mathbf{k},\mathbf{k}')$, which in turn is much less than the Fermi energy. This means that $A_k(\mathbf{Q})$, $B_k(\mathbf{Q})$, $L_{kk'}$, $N_{kk'}$ are all approximately independent of $\mathbf{k}$ and that the states $\mathbf{k}'$, $\mathbf{k}'+\mathbf{Q}$ in the integrals involving L and N will lie within an energy $\hbar\omega$ of the Fermi surface. As the interaction with a *longitudinal* field is being calculated, all these variables will be independent of angle, too.[3] In this case the set of three integral equations reduces to a set of three linear algebraic equations which are

$$M_k(\mathbf{Q}) = m(\mathbf{k},\mathbf{Q}) + \sum_{k'} m(\mathbf{k}',\mathbf{Q})\Phi_{k'k}\frac{2\nu_{k'}}{\nu_k^2-\nu_{k'}^2} + \sum_{k'} m(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})\frac{\nu_k L_k}{\nu_k^2-\nu_{k'}^2},$$

$$L_k(\mathbf{Q}) = Vl(\mathbf{k},\mathbf{Q}) + \sum_{k'} Vl(\mathbf{k}',\mathbf{Q})\Phi_{k'k}\frac{2\nu_k}{\nu_k^2-\nu_{k'}^2} + \sum_{k'} Vl^2(\mathbf{k}',\mathbf{Q})\frac{\nu_{k'}L_k}{\nu_k^2-\nu_{k'}^2},$$

$$N_k(\mathbf{Q}) = Vn(\mathbf{k},\mathbf{Q}) + \sum_{k'} Vn(\mathbf{k}',\mathbf{Q})\Phi_{k'k}\frac{2\nu_{k'}}{\nu_k^2-\nu_{k'}^2} + \sum_{k'} Vn(\mathbf{k}',\mathbf{Q})l(\mathbf{k}',\mathbf{Q})\frac{\nu_k L_k}{\nu_k^2-\nu_{k'}^2}.$$

Normally $M_k(\mathbf{Q})$ is required to obtain the probability of absorption of a wave of energy $\nu_k(\mathbf{Q})$, for which $\hbar v_0 Q \gg \nu_k$ (e.g., acoustic wave). Since the formalism above applies only for absorption when the system is initially in the ground state, the condition $\nu_k \geq 2\epsilon_0$ must hold. Therefore $\hbar v_0 Q \gg 2\epsilon_0$. In this case it is easy to show that to order $(\nu_k/\hbar v_0 Q)^2$ and $(\epsilon_0/\hbar\omega)^2$, L_k and N_k can be neglected. Then

$$M_k(\mathbf{Q}) = m(\mathbf{k},\mathbf{Q}) \Big/ \left(2V_D(\mathbf{Q})\sum_{k'}\frac{m^2(\mathbf{k}',\mathbf{Q})\nu_{k'}}{\nu_{k'}^2-\nu_k^2}+1\right) = \langle 0|\rho(\mathbf{Q})|\mathbf{k},\mathbf{k}+\mathbf{Q}\rangle_{\mathrm{BCS}} \Big/ \left(2V_D(\mathbf{Q})\sum_{k'}\frac{m^2(\mathbf{k}',\mathbf{Q})\nu_{k'}}{\nu_{k'}^2-\nu_k^2}+1\right).$$

To the same order of accuracy the denominator of this expression can be replaced by its value in the normal state. This shows that to calculate the probability of absorption of the system at the absolute zero of temperature one can use the interaction term in the Hamiltonian,

$$H_1 = \sum_{k,Q} V_D(\mathbf{Q}) r_Q e^{i\Omega t} c_{k,\sigma}^* c_{k-Q,\sigma} + \text{complex conj.},$$

where, if we choose r_Q to be the external charge fluctuation screened as in the normal state, we need consider only the single-particle wave functions of BCS. In the next section we approach the problem from a slightly different point of view and see how to generalize this result so that it applies at all temperatures and takes into account the corrections to the screening from the superconducting transition.

4. DIELECTRIC CONSTANT OF A SUPERCONDUCTOR

The procedure developed by Nozières and Pines[22] for the normal metal is adopted; a time-dependent longitudinal external field (unscreened) is allowed to act on the system and the polarization it induces is calculated and related to the complex dielectric constant. Let us suppose there is an oscillating test charge of wave vector $\mathbf{Q}$ and frequency Ω acting on the system. Its charge density is

$$r_Q \exp[-i(\Omega t - \mathbf{Q}\cdot\mathbf{r})] + \text{c.c.}$$

The interaction of this test charge with the system adds the term H_1 to the Hamiltonian where

$$H_1 = V_D(\mathbf{Q})[\rho(-\mathbf{Q}) r_Q e^{-i\Omega t} + \text{c.c.}] e^{\eta t}.$$

[In this section the contribution of the phonons to $V_D(\mathbf{Q})$ is neglected so that $V_D(\mathbf{Q})$ is just the Coulomb interaction. The term neglected is only of the order of the electron mass divided by the mass of an ion.] The infinitesimally small quantity η is introduced to ensure that the test charge is switched on adiabatically; in the mathematics, η indicates which contour to choose for the integrals that arise.

The interaction H_1 leads to extra terms in the equations of motion so that these equations become

$$[H,\gamma_{k+Q0}^*\gamma_{k1}^*] = \nu_k(\mathbf{Q})\gamma_{k+Q0}^*\gamma_{k1}^* + V_D m(\mathbf{k},\mathbf{Q}) \times [\rho(\mathbf{Q}) + r_Q e^{-i\Omega t+\eta t} + \text{c.c.}] + \tfrac{1}{2}n(\mathbf{k},\mathbf{Q}) \times B_k(\mathbf{Q}) - \tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q}), \quad (4.1a)$$

$$[H,\gamma_{k+Q1}\gamma_{k0}] = -\nu_k(\mathbf{Q})\gamma_{k+Q1}\gamma_{k0} - V_D m(\mathbf{k},\mathbf{Q}) \times [\rho(\mathbf{Q}) + r_Q e^{-i\Omega t+\eta t} + \text{c.c.}] - \tfrac{1}{2}n(\mathbf{k},\mathbf{Q}) \times B_k(\mathbf{Q}) - \tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q}), \quad (4.1b)$$

$$[H,\gamma_{k+Q0}^*\gamma_{k0}] = (E_{k+Q}-E_k)\gamma_{k+Q0}^*\gamma_{k0},$$

$$[H,\gamma_{k1}^*\gamma_{k+Q1}] = (E_k-E_{k+Q})\gamma_{k1}^*\gamma_{k+Q1}.$$

[22] P. Nozières and D. Pines, Nuovo cimento **9**, 470 (1958).

As shown by Nozières and Pines,[22] only the part of $\langle\rho(\mathbf{Q})\rangle$ that varies as $\exp(-i\Omega t)$ is required. Then $[\epsilon(\mathbf{Q},\Omega)^{-1}-1]$ is the ratio of this to $r_Q\exp(-i\Omega t)$. Now the effect of the extra terms in the equations of motion is to change the normal coordinates by adding c-numbers, and since $\rho(\mathbf{Q})$ is linear in the old normal coordinates it too is increased only by a c-number. The expectation value of $\rho(\mathbf{Q})$ in the ground state will be just this c-number. Hence it is necessary to solve the equations of motion treating the operators as c-numbers, remembering that the commutators on the left-hand sides are to be replaced by time derivatives. Thus as it is necessary to treat only the part of the test charge that varies as $e^{-i\Omega t+\eta t}$, we find

$$\begin{aligned}\rho(\mathbf{Q})=&\sum_k[V_D(\rho(\mathbf{Q})+r_Qe^{-i\Omega t+\eta t})m(\mathbf{k},\mathbf{Q})\\&+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})B_k(\mathbf{Q})]m(\mathbf{k},\mathbf{Q})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k(Q)}-\frac{1}{-\hbar\Omega-i\eta+\nu_k(Q)}\right]\\&-\tfrac{1}{2}\sum_k l(\mathbf{k},\mathbf{Q})A_k(Q)m(\mathbf{k},\mathbf{Q})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k}+\frac{1}{-\hbar\Omega-i\eta+\nu_k}\right],\end{aligned}\tag{4.2a}$$

$$\begin{aligned}B_K(\mathbf{Q})=&\sum_k[V_D(\rho(\mathbf{Q})+r_Qe^{-i\Omega t})m(\mathbf{k},\mathbf{Q})\\&+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})B_k(Q)]V(\mathbf{K},\mathbf{k})n(\mathbf{k},\mathbf{Q})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k}-\frac{1}{-\hbar\Omega-i\eta+\nu_k}\right]\\&-\tfrac{1}{2}\sum_k l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})V(\mathbf{K},\mathbf{k})n(\mathbf{k},\mathbf{Q})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k}+\frac{1}{-\hbar\Omega-i\eta+\nu_k}\right],\end{aligned}\tag{4.2b}$$

$$\begin{aligned}A_K(\mathbf{Q})=&-\sum_k[V_D(\rho(\mathbf{Q})+r_Qe^{-i\Omega t})m(\mathbf{k},\mathbf{Q})\\&+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})B_k(\mathbf{Q})]l(\mathbf{k},\mathbf{Q})V(\mathbf{K},\mathbf{k})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k}+\frac{1}{-\hbar\Omega-i\eta+\nu_k}\right]\\&+\tfrac{1}{2}\sum_k l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})V(\mathbf{K},\mathbf{k})l(\mathbf{k},\mathbf{Q})\\&\times\left[\frac{1}{-\hbar\Omega-i\eta-\nu_k}-\frac{1}{-\hbar\Omega-i\eta+\nu_k}\right].\end{aligned}\tag{4.2c}$$

From these three integral equations, $\rho(\mathbf{Q})$ and hence the complex dielectric constant can be determined. It is not difficult to show that the imaginary part of the dielectric constant leads to a result for the absorption which is in agreement with that calculated from the matrix elements of the previous section.

It is apparent from the structure of the equations of motion (4.1) that if one wishes to calculate the probability of a transition caused by an external charge fluctuation $\rho(Q)e^{-i\Omega t}$ ($\Omega\ll v_0Q$) one can take as the interaction term in the Hamiltonian

$$\begin{aligned}H'=&V_D(\rho(\mathbf{Q})+r_Qe^{-i\Omega t})\rho_{0p}(-\mathbf{Q})\\&+\tfrac{1}{2}\textstyle\sum_k B_k(\mathbf{Q})(b_k{}^{-Q}+\bar{b}_k{}^{-Q})\\&-\tfrac{1}{2}\textstyle\sum_k A_k(\mathbf{Q})(b_k{}^{-Q}-\bar{b}_k{}^{-Q}),\end{aligned}\tag{4.3}$$

where $\rho(\mathbf{Q})$, $B_k(\mathbf{Q})$, and $A_k(\mathbf{Q})$ are the solutions of Eqs. (4.2) and are c-numbers. ρ, B, and A are all proportional to $r_Qe^{-i\Omega t}$ and the constants of proportionality need be determined once for all interactions. The result (4.3) has been proved only for transitions into and out of the ground state. We guess that it is correct for all transitions and that $\rho(\mathbf{Q})$ is hardly altered by a change of temperature. For temperatures $kT\ll\epsilon_0$, $A(\mathbf{Q})$ and $B(\mathbf{Q})$ will also be unchanged. For many problems $A(\mathbf{Q})$ and $B(\mathbf{Q})$ can be neglected; for these problems the interaction is H_1 even up to T_c. As an example of the use of Eq. (4.3), we shall show in Sec. V that the corrections to ultrasonic attenuation as calculated by BCS are of order $(u/v_0)^2$, where u is the phase velocity of the sound wave. Notice that if the corrections are important then, because the terms involving A and B are not single-particle operators, the interaction cannot be described as a screened charge acting on each excitation.

As a check on the formula for $\epsilon(\mathbf{Q},\Omega)$ we shall investigate its behavior as $\Omega\to\infty$. According to Nozières and Pines[22] we should obtain

$$\epsilon(Q,\Omega)-1\to-\omega_p{}^2/\Omega^2\quad\text{as}\quad\Omega\to\infty.\tag{4.4}$$

If we write

$$\begin{aligned}A_k(\mathbf{Q})&=V_D(\mathbf{Q})\alpha_k(\mathbf{Q})[\rho(\mathbf{Q})+r_Qe^{-i\Omega t}],\\B_k(\mathbf{Q})&=V_D(\mathbf{Q})\beta_k(\mathbf{Q})[\rho(\mathbf{Q})+r_Qe^{-i\Omega t}],\end{aligned}$$

then

$$\begin{aligned}\epsilon-1=&\frac{-\rho(\mathbf{Q})}{\rho(\mathbf{Q})+r_Qe^{-i\Omega t}}\\=&-V_D\sum_k\left\{[2m(\mathbf{k},\mathbf{Q})+n(\mathbf{k},\mathbf{Q})\beta_k(\mathbf{Q})]\right.\\&\left.\times\frac{m(\mathbf{k},\mathbf{Q})\nu_k}{\hbar^2\Omega^2-\nu_k{}^2}+\frac{l(\mathbf{k},\mathbf{Q})m(\mathbf{k},\mathbf{Q})\alpha_k(\mathbf{Q})\hbar\Omega}{\hbar^2\Omega^2-\nu_k{}^2}\right\}\to\\&-(V_D/\Omega^2)\sum_k\{[2m(\mathbf{k},\mathbf{Q})+n(\mathbf{k},\mathbf{Q})\beta_k(\mathbf{Q})]\nu_k\\&+l(\mathbf{k},\mathbf{Q})\alpha(\mathbf{k},\mathbf{Q})\hbar\Omega\}m(\mathbf{k},\mathbf{Q}).\end{aligned}$$

As $\Omega\to\infty$ both β and α are proportional to Ω^{-1}. There-

fore β can be neglected and

$$\alpha_K(\mathbf{Q})=(1/\hbar\Omega^2)\sum_k V(\mathbf{K},\mathbf{k})2m(\mathbf{k},\mathbf{Q})l(\mathbf{k},\mathbf{Q})\Omega$$
$$=-(2/\hbar\Omega)(I_K+I_{K+Q}).$$

Therefore

$$\epsilon-1=-[V_D(\mathbf{Q})/\hbar^2\Omega^2]\sum_k m(\mathbf{k},\mathbf{Q})\times[2m(\mathbf{k},\mathbf{Q})\nu_k(\mathbf{Q})-2(I_k+I_{k+Q})l(\mathbf{k},\mathbf{Q})]$$
$$=[2V_D(\mathbf{Q})/\hbar^2\Omega^2]\sum_k m(\mathbf{k},\mathbf{Q})p(\mathbf{k},\mathbf{Q})(\epsilon_{k+Q}-\epsilon_k)$$
$$=[2V_D(\mathbf{Q})/\hbar^2\Omega^2]\sum(u_k^2v_{k+Q}^2-v_k^2u_{k+Q}^2)(\epsilon_{k+Q}-\epsilon_k)$$
$$=[2V_D(\mathbf{Q})/\hbar^2\Omega^2]\sum(v_{k+Q}^2-v_k^2)(\epsilon_{k+Q}-\epsilon_k)$$
$$=-[4V_D(\mathbf{Q})/\hbar^2\Omega^2]\sum v_k^2(\hbar^2Q^2/2m)$$
$$=-[V_D(\mathbf{Q})/\Omega^2](Q^2/m)N$$
$$=-(\omega_p^2/\Omega^2).$$

Had we neglected α we should have found that $(\epsilon-1)$ behaves like Q^{-2}. The proof of Eq. (4.4) is implicitly a proof of the sum rules

$$\sum_k \nu_k(\mathbf{Q})M_k^2(\mathbf{Q})=N\hbar^2Q^2/2m. \tag{4.5}$$

Notice that as Q tends to zero the energy $\nu_k(\mathbf{Q})$ remains finite but the matrix element $M_k(\mathbf{Q})$ is proportional to Q. In the normal metal it is the matrix element that remains finite while the energy is proportional to Q^2. One can see from this discussion that $\epsilon(\mathbf{Q},\Omega)$ is significantly different from its value in the normal metal only if $\hbar\Omega\sim\hbar v_0Q\lesssim\epsilon_0$.

Before leaving the subject of the dielectric constant we shall consider the connection of this work with the ideas of backflow.[8] An external charge fluctuation $r_Qe^{-i\Omega t}$ causes a charge fluctuation $\rho(\mathbf{Q})$ in the superconducting system. Therefore there is a current flow given by the density $\Omega\mathbf{Q}\rho(\mathbf{Q})Q^{-2}$. If an external point charge moving with velocity $\mathbf{V}$ interacts with the system the external charge density is

$$\delta(\mathbf{r}-\mathbf{V}t)=\sum_Q \exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{V}t)],$$

and the induced current flow is

$$\mathbf{j}(\mathbf{r})=\sum_Q(\mathbf{Q}\cdot\mathbf{V})\mathbf{Q}\rho(\mathbf{Q})Q^{-2}\exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{V}t)]/r_Q\exp(-i\Omega t)$$
$$=\sum_Q(\mathbf{Q}\cdot\mathbf{V})\mathbf{Q}\left(\frac{1}{\epsilon(\mathbf{Q},\mathbf{Q}\cdot\mathbf{V})}-1\right)Q^{-2}\exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{V}t)]$$
$$=-\nabla(\mathbf{V}\cdot\nabla)\sum_Q(1/\epsilon-1)Q^{-2}\exp[i\mathbf{Q}\cdot(\mathbf{r}-\mathbf{V}t)].$$

When $Q\to 0$, $\epsilon(\mathbf{Q},\mathbf{Q}\cdot\mathbf{V})^{-1}\to 0$. Therefore, at large distances from the moving charge the flow is just that due to a dipole of strength[23] $(-\mathbf{V}/4\pi)$ as it is in the normal metal. If $V\ll v_0$ or $V\gg v_0$, the flow everywhere is as in the normal metal. However, if $V\sim v_0$ then the flow in the superconductor is different from that in the normal metal at distances less than the coherence distance from the external charge.

[23] This backflow is analogous to the backflow around a foreign atom in liquid helium. The strength is the same, see R. P. Feynman and M. Cohen, Phys. Rev. **102**, 1189 (1956).

5. ULTRASONIC ABSORPTION

The interaction (4.3) will be used to show that the correction to the ultrasonic absorption calculated by BCS is (at least at the lowest temperatures) of order $(u/v_0)^2$, where u is the velocity of sound in the metal. Near absolute zero, $kT<\epsilon_0(T)$, $\hbar\Omega\ll\hbar v_0Q\ll\epsilon_0$ for the acoustic waves of interest. If the integrands of Eqs. (4.2) are expanded in powers of (Ω/v_0Q) and $(\hbar v_0Q/\epsilon_0)$, it is found that

$$A=-(40/17)V_D[r_Qe^{-i\Omega t}+\rho(\mathbf{Q})]\epsilon_0\Omega(\hbar v_0Q)^{-2}$$
$$=\alpha_QV_D[r_Qe^{-i\Omega t}+\rho(Q)]=\alpha_QV_Dr_Qe^{-i\Omega t}\epsilon^{-1},$$
$$B=\tfrac{4}{5}V_D[r_Qe^{-i\Omega t}+\rho(\mathbf{Q})]\hbar v_0Q/\epsilon_0$$
$$=\beta_QV_D[r_Qe^{-i\Omega t}+\rho(Q)]=\beta_QV_Dr_Qe^{-i\Omega t}\epsilon^{-1}.$$

The correction to $\rho(\mathbf{Q})$ is of order $(\Omega/\hbar v_0Q)^2$ and $(\hbar v_0Q/\epsilon_0)^2$. As only the order of magnitude of the correction is being estimated, these corrections to $\rho(\mathbf{Q})$ can be ignored. (BCS have already ignored corrections of this order of magnitude.) Hence, the absorption is proportional to

$$\int d^3k\,\delta(E_k-E_{k+Q}+\hbar\Omega_Q)(f_{k+Q}-f_k)\{[n(\mathbf{k},\mathbf{Q})+\beta_Qm(\mathbf{k},\mathbf{Q})+\alpha_Qp(\mathbf{k},\mathbf{Q})]^2+[n(\mathbf{k},\mathbf{Q})+\beta_Qm(\mathbf{k},\mathbf{Q})-\alpha_Qp(\mathbf{k},\mathbf{Q})]^2\}$$
$$=2\int d^3k\,\delta(E_k-E_{k+Q}+\hbar\Omega_Q)(f_{k+Q}-f_k)\times\left\{\frac{\epsilon^2}{E^2}+\frac{4}{5}\frac{\epsilon(\hbar v_0Q)}{E^2}+\frac{16}{25}\left(\frac{\hbar v_0Q}{E}\right)^2+\frac{5}{4}\frac{\Omega^2}{(\hbar v_0Q)^2}\frac{\epsilon_0^4}{E^4}\right\}.$$

The first term in the curly brackets gives the result of BCS. The corrections are evidently no bigger than terms already neglected. Since the term involving $\rho(\mathbf{Q})$ is the only important one, this result will be valid for temperatures up to T_c.

6. MEISSNER EFFECT

We shall calculate the current within the superconductor due to an external static magnetic field. The connection between this current and the existence of a Meissner effect and the calculation of the penetration depth has been discussed sufficiently elsewhere[24] for it to be omitted here. If the static magnetic field is described by the vector potential $\mathbf{a}(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{r})$, the extra perturbing term in the Hamiltonian is H_1, given

[24] J. Bardeen, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 19. This review article contains further references on this topic.

by

$$H_1=-\alpha\sum_k \mathbf{a}(\mathbf{Q})\cdot(2\mathbf{k}+\mathbf{Q})(\rho_k{}^Q-\tilde{\rho}_k{}^Q)^*$$
$$=-\alpha\sum_k \mathbf{a}(\mathbf{Q})\cdot(2\mathbf{k}+\mathbf{Q})[l(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k0}-\gamma_{k1}{}^*\gamma_{k+Q1})$$
$$-p(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q1}\gamma_{k0})]^*, \quad (6.1)$$
$$\alpha=e\hbar/2mc.$$

A straightforward way of calculating the paramagnetic part of the current to first order in $\mathbf{a}(\mathbf{Q})$ would be to use first-order perturbation theory and obtain

$$\mathbf{j}_p(\mathbf{Q})=\sum_n \frac{\langle 0|j_{p,0p}|n\rangle\langle n|H_1|0\rangle}{E_0-E_n}+\text{c.c.},$$

where the states $|n\rangle$ are the states $\mu^*|0\rangle$ of Sec. III. To obtain the result explicitly (i.e., without invoking sum rules) it would be necessary to obtain all the solutions μ^*. In order to avoid the excessive computation involved we use the following quicker method which involves no extra assumptions.

The equations of motion when H_1 is added to the Hamiltonian are

$$[H,\gamma_{k+Q0}{}^*\gamma_{k1}{}^*]$$
$$=\nu_k(\mathbf{Q})\gamma_{'+Q0}{}^*\gamma_{k1}{}^*+V_D(\mathbf{Q})\rho(\mathbf{Q})m(\mathbf{k},\mathbf{Q})$$
$$+\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})B_k(\mathbf{Q})-\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})$$
$$+\alpha p(\mathbf{k},\mathbf{Q})\mathbf{a}(\mathbf{Q})\cdot(2\mathbf{k}+\mathbf{Q}), \quad (6.2a)$$
$$[H,\gamma_{k+Q1}\gamma_{k0}]$$
$$=-\nu_k(\mathbf{Q})\gamma_{k+Q1}\gamma_{k0}-V_D(\mathbf{Q})\rho(\mathbf{Q})m(\mathbf{k},\mathbf{Q})$$
$$-\tfrac{1}{2}n(\mathbf{k},\mathbf{Q})B_k(\mathbf{Q})-\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})$$
$$+\alpha p(\mathbf{k},\mathbf{Q})\mathbf{a}(\mathbf{Q})\cdot(2\mathbf{k}+\mathbf{Q}). \quad (6.2b)$$

[These and the following equations still apply if the Coulomb term is omitted. We then have to omit the Coulomb contributions to $V_D(\mathbf{Q})$ and $V(\mathbf{k}',\mathbf{k})$.] As in Sec. IV, we look only for the steady-state solution of these equations. Because the external field is static the left-hand side of the equations is zero. Subtracting the two equations, one quickly finds

$$\rho(\mathbf{Q})=B_k(\mathbf{Q})=0.$$

From the sum of the two equations

$$-\nu_k(\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q1}\gamma_{k0})$$
$$=-l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})+2\alpha p(\mathbf{k},\mathbf{Q})\mathbf{a}\cdot(2\mathbf{k}+\mathbf{Q}).$$

It follows from this equation and the definition of $A_K(\mathbf{Q})$,

$$A_K(\mathbf{Q})=-\sum_k V(\mathbf{K},\mathbf{k})l(\mathbf{k},\mathbf{Q})(\gamma_{k+Q0}{}^*\gamma_{k1}{}^*-\gamma_{k+Q1}\gamma_{k0}),$$

that

$$A_K(\mathbf{Q})=-\sum_k V(\mathbf{K},\mathbf{k})[l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})$$
$$-2\alpha p(\mathbf{k},\mathbf{Q})\mathbf{a}\cdot(2\mathbf{k}+\mathbf{Q})]\nu_k{}^{-1}l(\mathbf{k},\mathbf{Q}). \quad (6.3)$$

The paramagnetic part of the current density $\mathbf{j}(\mathbf{Q})$ is given by

$$(2m/e\hbar)\mathbf{j}_p(\mathbf{Q})$$
$$=\sum_k(2\mathbf{k}+\mathbf{Q})(\rho_k{}^Q-\tilde{\rho}_k{}^Q)$$
$$=\sum_k(2\mathbf{k}+\mathbf{Q})[-l(\mathbf{k},\mathbf{Q})A_k(\mathbf{Q})$$
$$+2\alpha p(\mathbf{k},\mathbf{Q})\mathbf{a}\cdot(2\mathbf{k}+\mathbf{Q})]p(\mathbf{k},\mathbf{Q})\nu_k{}^{-1}. \quad (6.4)$$

The second term on the right is the one that is calculated from the BCS wave functions alone. As the result is linear in $\mathbf{a}(\mathbf{Q})$, one can calculate the effects of longitudinal and transverse fields separately. This we proceed to do.

(1) Longitudinal Field

As pointed out in the Introduction, a longitudinal static vector potential cannot give rise to any current. It will now be shown that the effect of including the collective term $A_k(\mathbf{Q})$ is to ensure that this result is satisfied. If $\mathbf{a}(\mathbf{Q})$ is a longitudinal potential, one can write

$$\mathbf{a}(\mathbf{Q})=\mathbf{Q}(\hbar^2/2m)\phi(\mathbf{Q}).$$

Then the solution of the integral equation for $A_k(\mathbf{Q})$ is

$$A_k(\mathbf{Q})=2(I_k+I_{k+Q})\alpha\phi(\mathbf{Q}). \quad (6.5)$$

This can be checked directly. If the formula (6.5) is substituted into the right-hand side of Eq. (6.3), that side becomes [when Eq. (3.7) is used]

$$-2\alpha\phi\sum_k V(\mathbf{K},\mathbf{k})[l(\mathbf{k},\mathbf{Q})(I_k+I_{k+Q})$$
$$-(\epsilon_{k+Q}-\epsilon_k)p(\mathbf{k},\mathbf{Q})]l(\mathbf{k},\mathbf{Q})\nu_k{}^{-1}$$
$$=-2\alpha\phi\sum_k V(\mathbf{K},\mathbf{k})m(\mathbf{k},\mathbf{Q})l(\mathbf{k},\mathbf{Q})$$
$$=2\alpha\phi(I_K+I_{K+Q})$$
$$=A_K(\mathbf{Q}).$$

Therefore the two sides of Eq. (6.3) are equal. If one substitutes for $A_k(\mathbf{Q})$ in the current density it is found that

$$(2m/e\hbar)\mathbf{j}_p(\mathbf{Q})=+2\alpha\phi\sum_k(2\mathbf{k}+\mathbf{Q})[(\epsilon_{k+Q}-\epsilon_k)p(\mathbf{k},\mathbf{Q})$$
$$-(I_k+I_{k+Q})l(\mathbf{k},\mathbf{Q})]p(\mathbf{k},\mathbf{Q})\nu_k{}^{-1}$$
$$=-2\alpha\phi\sum_k(2\mathbf{k}+\mathbf{Q})m(\mathbf{k},\mathbf{Q})p(\mathbf{k},\mathbf{Q})$$
$$=-2\alpha\phi\sum_k(2\mathbf{k}+\mathbf{Q})(u_k{}^2v_{k+Q}{}^2-v_k{}^2u_{k+Q}{}^2)$$
$$=-2\alpha\phi\sum(2\mathbf{k}+\mathbf{Q})(v_{k+Q}{}^2-v_k{}^2)$$
$$=2\alpha\phi 2\sum(2\mathbf{k}+\mathbf{Q})v_k{}^2$$
$$=2\alpha\phi\mathbf{Q}N$$
$$=(4m/\hbar^2)\alpha N\mathbf{a}(\mathbf{Q}),$$
$$\mathbf{j}_p(\mathbf{Q})=(ne^2/mc)\mathbf{a}(\mathbf{Q}).$$

Hence the paramagnetic current density just cancels the diamagnetic current density, the total longitudinal current being zero.

(2) Transverse Field

If $\mathbf{a}$ is transverse and $V(\mathbf{K},\mathbf{k})$ is independent of angle, then

$$\mathbf{a}\cdot\sum_k V(\mathbf{K},\mathbf{k})(2\mathbf{k}+\mathbf{Q})l(\mathbf{k},\mathbf{Q})p(\mathbf{k},\mathbf{Q})\nu_k^{-1}$$

is zero because the sum must be proportional to $\mathbf{Q}$, and $\mathbf{a}\cdot\mathbf{Q}=0$. In this case $A_k(\mathbf{Q})$ is zero and there are no corrections to the result of BCS. In general, however, $V(\mathbf{K},\mathbf{k})$ is a function of angle and there will be a contribution from the transverse excitations. In the London limit, $Q\to 0$, $p(\mathbf{k},\mathbf{Q})\to 0$ and as there is no singularity in the solution one finds no contribution to $\mathbf{j}_p$ in the limit. Thus the London equation is obtained as $Q\to 0$. We have tried to estimate the order of magnitude of the correction in the Pippard limit, $Q\xi_0\gg 1$, by treating a specific example. The dependence of V on $|k|$ and $|K|$ is not important provided we cut off the integrals appropriately. Because the phonon interaction tends to zero as the angle, θ, between $\mathbf{k}$ and $\mathbf{K}$ tends to zero we have chosen a $V(\mathbf{K},\mathbf{k})$ that possesses this property. The simplest potential that gives a nonzero contribution is

$$V(\mathbf{K},\mathbf{k})=-\tfrac{3}{4}V(1-\cos\theta)^2. \tag{6.6}$$

Then

$$-\langle V(\mathbf{K},\mathbf{k})\rangle_{\text{Av}}=V$$

is the same parameter as used by BCS. With this form for $V(\mathbf{K},\mathbf{k})$, it is found that

$$\sum_k V(\mathbf{K},\mathbf{k})\mathbf{a}\cdot(2\mathbf{k}+\mathbf{Q})\nu_k^{-1}p(\mathbf{k},\mathbf{Q})l(\mathbf{k},\mathbf{Q})$$
$$=-\tfrac{3}{4}V\sum_k(\mathbf{a}\cdot 2\mathbf{k})(1-\cos\theta)^2\nu_k^{-1}\epsilon_0(\epsilon-\epsilon')/2EE',\qquad \epsilon'=\epsilon_{k+Q}.$$

If the x axis is chosen along $\mathbf{Q}$ and the z axis along $\mathbf{a}$, the sum is (neglecting terms of order Q/k_0)

$$\frac{3Va}{2}\sum_k k_z\left[-\frac{k_zK_z}{kK}+\frac{k_zK_z}{k^2K^2}(k_xK_x+k_yK_y)\right]\times\frac{\epsilon_0\hbar^2k_xQ_x/m}{E_kE_{k+Q}(E+E')}.$$

If we substitute $(-k_x-Q_x)$ for k_x we find that the integrand, apart from the term in square brackets, is an odd function of k_x (to order Q/k_0). Hence the sum is

$$\frac{3Va}{2}\frac{K_xK_z}{K^2}\frac{\hbar^2}{m}Q\sum_k\frac{k_x^2k_z^2}{k^2}\frac{\epsilon_0}{E_kE_{k+Q}(E_k+E_{k+Q})}$$
$$=\frac{3}{2}\frac{V(\mathbf{a}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{K})}{K^2Q}\sum_k\frac{k_x\frac{1}{2}(k^2-k_x^2)}{k^2}\frac{\epsilon_0(\epsilon'-\epsilon)}{EE'(E+E')}$$
$$=\tfrac{3}{4}N(0)V\frac{(\mathbf{a}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{K})}{K^2Q}k_0\int d\epsilon\times\int_{-1}^{1}\frac{d\mu}{2}\mu(1-\mu^2)\frac{\epsilon_0(\epsilon'-\epsilon)}{EE'(E+E')}$$
$$=\tfrac{3}{8}N(0)Vk_0\frac{(\mathbf{a}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{K})}{K^2Q}J(Q)\quad\text{(say)}.$$

This result suggests that we try

$$A_K(Q)=\frac{(\mathbf{a}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{K})}{K^2}\bar{A}(Q).$$

In that case

$$\sum_k V(\mathbf{K},\mathbf{k})A_k(Q)l(\mathbf{k},\mathbf{Q})^2\nu_k(\mathbf{Q})^{-1}$$
$$=-\tfrac{3}{4}V\bar{A}(Q)\sum_k\left(1-\frac{\mathbf{k}\cdot\mathbf{K}}{kK}\right)^2\times\frac{(\mathbf{a}\cdot\mathbf{k})(\mathbf{Q}\cdot\mathbf{k})}{k^2}\frac{1}{2}\left(1+\frac{\epsilon\epsilon'+\epsilon_0^2}{EE'}\right)\nu_k(\mathbf{Q})^{-1}$$
$$=-\tfrac{3}{4}V\bar{A}(Q)\sum_k\frac{2k_x^2K_xk_z^2K_z}{k^4K^2}aQ\tfrac{1}{2}\left(1+\frac{\epsilon\epsilon'+\epsilon_0^2}{EE'}\right)\nu_k(Q)^{-1}$$
$$=-\tfrac{3}{4}N(0)V\bar{A}(Q)\frac{(a\cdot K)(Q\cdot K)}{K^2}\times\int_0^{\hbar\omega}d\epsilon\int\frac{d\mu}{2}\frac{\mu^2(1-\mu^2)}{E+E'}\frac{1}{2}\left(1+\frac{\epsilon\epsilon'+\epsilon_0^2}{EE'}\right).$$

After some calculation it is found that if $\hbar v_0Q<\hbar\omega$ and $\epsilon_0\ll\hbar v_0Q$, the sum is

$$-(3/20)N(0)VA_K(\mathbf{Q})\ln(2\hbar\omega/\hbar v_0Q)\cong-(3/20)A_K(\mathbf{Q})[1-N(0)V\ln(\hbar v_0Q/\epsilon_0)].$$

Then the equation for $A_k(\mathbf{Q})$ is

$$\left[1-\frac{3}{20}+\frac{3}{20}N(0)V\ln(\hbar v_0Q/\epsilon_0)\right]A_K(Q)=\tfrac{3}{4}\alpha N(0)Vk_0\frac{(\mathbf{a}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{K})}{K^2Q}J(Q).$$

As we are only estimating the order of magnitude of the correction, we shall keep only the first term of the square brackets. Hence

$$\bar{A}(Q)=\tfrac{3}{4}(\alpha/q)N(0)Vk_0J(Q).$$

The correction to the current density is given by

$$\left(\frac{2m}{e\hbar}\right)j_z(Q)=-\frac{3}{4}\frac{\alpha}{q}N(0)Vk_0J(Q)\times\sum_k k_z\frac{\epsilon_0(\epsilon-\epsilon')}{EE'(E+E')}\frac{aQk_xk_z}{k^2}$$
$$=\tfrac{3}{4}\alpha N(0)^2Vk_0J(Q)a\times\int d\epsilon\int_{-1}^{1}\frac{d\mu}{2}\mu(1-\mu^2)\frac{\epsilon_0(\epsilon'-\epsilon)}{2EE'(E+E')}$$
$$=\tfrac{3}{16}a\alpha N(0)^2Vk_0^2J(Q)^2.$$

$J(Q)$ has been evaluated and is

$$-\frac{16\epsilon_0}{3\hbar v_0 Q}[\ln(\hbar v_0 Q/\epsilon_0)-\tfrac{4}{3}].$$

Therefore, the total transverse current density is

$$j_\perp(Q)=\frac{-3c}{16Q\xi_0\lambda_L^2(0)}\left\{1-\frac{16}{\pi^3 Q\xi_0}\ln(\pi Q\xi_0)-\frac{8N(0)V}{3\pi^3 Q\xi_0}[\ln(\pi Q\xi_0)-\tfrac{4}{3}]^2\right\}a_\perp(Q).$$

The first two terms are those given by BCS while the third is the new correction. As the formula is valid only for $Q\xi_0>1$, it is reasonable to test the correction using $\pi Q\xi_0=10$. If we also choose $N(0)V=0.3$, the ratio of the third term to the second is 0.02, which suggests that the correction is small. It *is* possible to choose a potential that makes the correction large by making the potential vary considerably with angle and change sign. For example, if one chooses

$$V(\mathbf{K},\mathbf{k})=V_1-\tfrac{3}{4}V_2(1-\cos\theta)^2,\quad V_1-V_2<0,$$

the correction is enhanced by the factor $V_2/(V_1-V_2)$ which can be made as large as one pleases by making (V_1-V_2) sufficiently small. But, although $V(\mathbf{K},\mathbf{k})$ may oscillate widely over small angles because of the contribution of the umklapp processes, we expect that on the average it will not vary widely enough over 180° to make the correction large. As the correction is sensitive to the dependence of the potential on angle the argument is not conclusive.

One can see the connection with the work of Pines and Schrieffer[8] in the following way. The operator which creates a plasmon is

$$\mu_{\rm pl}^*(\mathbf{Q})=\sum_k[\alpha(\mathbf{k},\mathbf{Q})\gamma_{k+Q1}^*\gamma_{k0}^*-\beta(\mathbf{k},\mathbf{Q})\gamma_{k+Q0}\gamma_{k1}],$$

where

$$\alpha(\mathbf{k},\mathbf{Q})=[\Phi_k+\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_k]/(\hbar\omega_{\rm pl}-\nu_k),$$
$$\beta(\mathbf{k},\mathbf{Q})=-[\Phi_k-\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_k]/(\hbar\omega_{\rm pl}+\nu_k),$$

and $\omega_{\rm pl}$ is the plasma frequency. Φ_k and L_k satisfy Eqs. (3.6) without the inhomogeneous terms and with $\nu_{k'}$ replaced by $\hbar\omega_{\rm pl}$. Hence

$$\mu_{\rm pl}^*(\mathbf{Q})-\mu_{\rm pl}(-\mathbf{Q})=\sum_k[\alpha(\mathbf{k},\mathbf{Q})+\beta(\mathbf{k},\mathbf{Q})]\times[\gamma_{k+Q1}^*\gamma_{k0}^*-\gamma_{k+Q0}\gamma_{k1}].$$

If terms of second order in the electron-plasmon coupling constant are neglected,

$$\begin{aligned}\alpha(\mathbf{k},\mathbf{Q})+\beta(\mathbf{k},\mathbf{Q})&=[2\Phi_k\nu_k+\hbar\omega_{\rm pl}l(\mathbf{k},\mathbf{Q})L_k](\hbar\omega_{\rm pl})^{-2}\\&=2M_{\rm pl}V_D(\mathbf{Q})[m_k\nu_k-(I_k+I_{k+Q})l(\mathbf{k},\mathbf{Q})](\hbar\omega_{\rm pl})^{-2}\\&=-2M_{\rm pl}V_D(\mathbf{Q})(\epsilon_{k+Q}-\epsilon_k)p(\mathbf{k},\mathbf{Q})(\hbar\omega_{\rm pl})^{-2}.\end{aligned}$$

Hence

$$\mu_{\rm pl}^*(\mathbf{Q})-\mu_{\rm pl}(-\mathbf{Q})=-\frac{2M_{\rm pl}V_D(\mathbf{Q})}{(\hbar\omega_{\rm pl})^2}\sum_k(\epsilon_{k+Q}-\epsilon_k)(\gamma_{k+Q1}^*\gamma_{k0}^*-\gamma_{k+Q0}\gamma_{k1}),$$

which is proportional to $\mathbf{j}_p(\mathbf{Q})$. $M_{\rm pl}$ has to be chosen so that the creation operators are properly normalized. Then the analysis follows that of Pines and Schrieffer.

Note added in proof.—Since this paper was submitted a number of papers and preprints have appeared on the theory of the Meissner effect. The reader is referred to K. Yosida [Prog. Theoret. Phys. (Kyoto) **21**, 731 (1959)], Blatt, Matsubara, and May [Prog. Theoret. Phys. (Kyoto) **21**, 745 (1959)], N. N. Bogoliubov (preprint) and Y. Nambu (preprint).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the advice of Professor J. Bardeen at all stages of this work and several stimulating discussions with Professor H. J. Lipkin and Professor D. Pines. The author is also grateful to Dr. P. W. Anderson for letting him see a copy of his paper prior to its publication.

APPENDIX A

In this Appendix, Anderson's linear equations of motion are derived and the screening of the exchange terms justified. It is apparent from a comparison of the treatments of Nakajima[5] and of Bardeen and Pines[6] of the electron-phonon interaction, that in order to obtain the interaction properly screened it is necessary to separate out the plasma degrees of freedom. Accordingly, let us try to separate out these degrees of freedom. We will suppose at first that the operators μ_Q^* which create plasma oscillations of wave vector $\mathbf{Q}$ in the superconductor are known. For these modes the RPA is certainly a good approximation; one can write

$$\mu_Q^*=\sum_k[\alpha(\mathbf{k},\mathbf{Q})\rho_k{}^Q+\beta(\mathbf{k},\mathbf{Q})\bar\rho_k{}^Q+\gamma(\mathbf{k},\mathbf{Q})b_k{}^Q+\phi(\mathbf{k},\mathbf{Q})\bar b_k{}^Q].$$

Ultimately the coefficients α, β, γ, ϕ will have to be determined.

The "intrinsic" Hamiltonian, $H_{\rm int}$, is introduced by

$$H_{\rm int}=H-\sum_{Q<Q_{\max}}\hbar\omega_Q\mu_Q^*\mu_Q;$$

$H_{\rm int}$ is a function of operators $\mu_k(\mathbf{Q})$ and $\mu_k(\mathbf{Q})^*$ which commute with μ_Q and μ_Q^* and which destroy and create states in which a pair of particles are excited. We can find these operators from the equations of motion derived from $H_{\rm int}$. Let us consider just one of these

equations, the one for $\rho_K{}^Q$. One finds

$$
\begin{aligned}
&[H_{\rm int},\rho_K{}^Q]\\
&=[H-\textstyle\sum \hbar\omega_{Q'}\mu_{Q'}{}^*\mu_{Q'},\ \rho_K{}^Q]\\
&=[H,\rho_K{}^Q]-\textstyle\sum_{Q'}\hbar\omega_{Q'}\mu_{Q'}{}^*[\mu_{Q'},\rho_K{}^Q]\\
&\qquad\qquad -\textstyle\sum_{Q'}\hbar\omega_{Q'}[\mu_{Q'}{}^*,\rho_K{}^Q]\mu_{Q'}\\
&=[H,\rho_K{}^Q]-\textstyle\sum_{Q'}\hbar\omega_{Q'}\mu_{Q'}{}^*\sum_{k'}\{\alpha(\mathbf{k}',\mathbf{Q}')\\
&\quad\times[c_{k'\uparrow}{}^*c_{k\uparrow}\delta_{k'+Q',\,k+Q}-c_{k+Q\uparrow}{}^*c_{k'+Q'\uparrow}\delta_{k',\,k}]\\
&\quad-\gamma(\mathbf{k}',\mathbf{Q}')[c_{k+Q\uparrow}{}^*c_{-k'-Q'\downarrow}{}^*\delta_{k',\,k}]+\phi(\mathbf{k}',\mathbf{Q}')\\
&\quad\times[c_{-k'\downarrow}c_{k\uparrow}\delta_{k'+Q',\,k+Q}]\}-\textstyle\sum_{Q'}\hbar\omega_{Q'}\sum_{k'}\{\alpha(\mathbf{k}',\mathbf{Q}')\\
&\quad\times[c_{k'+Q'\uparrow}{}^*c_{k\uparrow}\delta_{k',\,k+Q}-c_{k+Q\uparrow}{}^*c_{k'\uparrow}\delta_{k'+Q',\,k}]\\
&\quad+\gamma(\mathbf{k}',\mathbf{Q}')c_{-k'-Q'\downarrow}c_{k\uparrow}\delta_{k',\,k+Q}\\
&\qquad\qquad -\phi(\mathbf{k}',\mathbf{Q}')c_{k+Q\uparrow}{}^*c_{-k'\downarrow}{}^*\delta_{k'+Q',\,k}]\}\mu_{Q'}\\
&=[H,\rho_K{}^Q]-\textstyle\sum_{Q'}\hbar\omega_{Q'}\mu_{Q'}{}^*\{\alpha(\mathbf{k}+\mathbf{Q}-\mathbf{Q}',\mathbf{Q}')\\
&\quad\times[c_{k+Q-Q'\uparrow}{}^*c_{k\uparrow}-\alpha(\mathbf{k},\mathbf{Q}')c_{k+Q\uparrow}{}^*c_{k+Q'\uparrow}]\\
&\quad-\gamma(\mathbf{k},\mathbf{Q}')c_{k+Q\uparrow}{}^*c_{-k-Q'\downarrow}{}^*\\
&\quad+\phi(\mathbf{k}+\mathbf{Q}-\mathbf{Q}',\mathbf{Q}')c_{-k-Q+Q'\downarrow}c_{k\uparrow}\}\\
&\quad-\textstyle\sum_{Q'}\hbar\omega_{Q'}\{\alpha(\mathbf{k}+\mathbf{Q},\mathbf{Q}')c_{k+Q+Q'\uparrow}{}^*c_{k\uparrow}\\
&\quad-\alpha(\mathbf{k}-\mathbf{Q}',\mathbf{Q}')c_{k+Q\uparrow}{}^*c_{k-Q'\uparrow}\\
&\quad+\gamma(\mathbf{k}+\mathbf{Q},\mathbf{Q}')c_{-k-Q-Q'\downarrow}c_{k\uparrow}\\
&\qquad\qquad -\phi(\mathbf{k}-\mathbf{Q}',\mathbf{Q}')c_{k+Q\uparrow}{}^*c_{-k+Q'\downarrow}{}^*\}\mu_{Q'}.
\end{aligned}
$$

To linearize this equation, products of pairs are replaced by their expectation values in the ground state. The first term when linearized looks just like the right-hand side of Eq. (3.1a) but the potential appearing in it is unscreened. This term will be written as $[H,\rho_K{}^Q]_L$. Then (omitting the exchange terms to save space)

$$
\begin{aligned}
&[H_{\rm int},\rho_K{}^Q]\\
&=[H,\rho_K{}^Q]_L+\textstyle\sum_{Q'}\hbar\omega_{Q'}\{-\mu_{Q'}{}^*\langle[\mu_{Q'},\rho_K{}^Q]\rangle\\
&\quad+\langle[\mu_{Q'}{}^*,\rho_K{}^Q]\rangle\mu_{Q'}-\beta(\mathbf{k}',\mathbf{Q}')\alpha(\mathbf{k}+\mathbf{Q}-\mathbf{Q}',\mathbf{Q}')\\
&\quad\times[b_{k'}\delta_{k',\,k+Q-Q'}c_{-k'-Q'\downarrow}c_{k\uparrow}\\
&\quad-c_{-k'\downarrow}{}^*c_{k+Q-Q'\uparrow}{}^*\delta_{k'+Q',\,k}b_k]+\beta(\mathbf{k}',\mathbf{Q}')\alpha(\mathbf{k},\mathbf{Q}')\\
&\quad\times[b_{k'}\delta_{k',\,k+Q}c_{-k'-Q'\downarrow}c_{k+Q'\uparrow}-b_{k'+Q'}c_{-k'\downarrow}{}^*c_{k+Q\uparrow}{}^*\delta_{k,\,k'}]\\
&\quad-\alpha(\mathbf{k}+\mathbf{Q},\mathbf{Q}')\beta(\mathbf{k}',\mathbf{Q}')[-b_{k'+Q'}c_{k\uparrow}c_{-k'\downarrow}\delta_{k+Q,\,k'}\\
&\quad+b_kc_{k+Q+Q'\uparrow}{}^*c_{-k'-Q'\downarrow}{}^*\delta_{k,\,k'}]+\alpha(\mathbf{k}-\mathbf{Q}',\mathbf{Q}')\beta(\mathbf{k}',\mathbf{Q}')\\
&\quad\times[-b_{k+Q}\delta_{k+Q,\,k'+Q'}c_{k-Q'\uparrow}c_{-k'\downarrow}\\
&\qquad+b_{k-Q'}\delta_{k-Q',\,k'}c_{k+Q\uparrow}{}^*c_{-k'-Q'\downarrow}{}^*]+\text{terms in }\gamma\text{ and }\delta,
\end{aligned}
$$

where $\langle[\mu_{Q'},\rho_K{}^Q]\rangle$ is the expectation value of the commutator in the ground state, i.e.,

$$
\begin{aligned}
\langle[\mu_{Q'},\rho_K{}^Q]\rangle&=\delta_{Q,\,Q'}\{\alpha(\mathbf{k},\mathbf{Q}')[n_k-n_{k+Q}]\\
&\qquad -\gamma(\mathbf{k},\mathbf{Q}')b_{k+Q}+\phi(\mathbf{k},\mathbf{Q}')b_k\},
\end{aligned}
$$

and

$$
\begin{aligned}
\langle[\mu_{Q'}{}^*,\rho_K{}^Q]\rangle&=\delta_{Q,-Q'}\{\alpha(\mathbf{k}+\mathbf{Q},\mathbf{Q}')(n_k-n_{k+Q})\\
&\qquad +\gamma(\mathbf{k}+\mathbf{Q},\mathbf{Q}')b_k-\phi(\mathbf{k}+\mathbf{Q},\mathbf{Q}')b_{k+Q}\}.
\end{aligned}
$$

One now has a set of equations (not all independent) from which to determine the single-particle excitations. Since the coefficients α, β, γ, and ϕ are also unknown, these equations together with

$$[H_{\rm int},\mu_Q{}^*]=[H_{\rm int},\mu_Q]=0$$

determine these coefficients and also the cutoff on Q'. The equations are not linear in the coefficients.

In order to see the connection with the Eqs. (3.1) we first make what appears to be a reasonable approximation and replace α, β, γ, ϕ by the values we should obtain in the normal state (neglecting second-order terms in the electron-plasmon coupling constant), that is, we take

$$\gamma=\phi=0,\quad \beta(\mathbf{k},\mathbf{Q})=\alpha(\mathbf{k},\mathbf{Q})=[V(\mathbf{Q})/2\hbar\omega_Q]^{\frac{1}{2}},\quad \text{(A1)}$$

where $V(Q)$ is the sum of the Coulomb and phonon interactions. $V(Q)$ is unscreened, the phonon part corresponds to the interaction obtained by Nakajima, not that of Bardeen and Pines. This approximation can be made the first step of a self-consistent calculation. It follows that

$$
\begin{aligned}
&[H_{\rm int},\rho_k{}^Q]\\
&=[H,\rho_k{}^Q]_L-\textstyle\sum_{Q'}V(\mathbf{Q}')\\
&\quad\times[b_{k+Q-Q'}b_k{}^Q+b_k\bar{b}_{k-Q'}{}^Q-b_{k+Q}b_{k+Q'}{}^Q-b_{k+Q'}\bar{b}_k{}^Q]\\
&\qquad -\hbar\omega_Q\{\mu_Q{}^*\langle[\mu_Q,\rho_k{}^Q]\rangle+\langle[\mu_{-Q}{}^*,\rho_k{}^Q]\rangle\mu_Q\}\\
&=[H,\rho_k{}^Q]_A-\hbar\omega_Q\{\mu_Q{}^*\langle[\mu_Q,\rho_k{}^Q]\rangle\\
&\qquad\qquad +\langle[\mu_{-Q}{}^*,\rho_k{}^Q]\rangle\mu_{-Q}\},\quad \text{(A2)}
\end{aligned}
$$

where $[H,\rho_k{}^Q]_A$ stands symbolically for the commutator written down by Anderson. This equation and the corresponding equations for $\bar{\rho}$, b, and $\bar{b}$ are together equivalent to those of Anderson. This can be seen in the following way. The equations for the coefficients $\alpha(\mathbf{k},\mathbf{Q})$, β, γ, and ϕ are found from the equation for $\mu_Q{}^*$. From Eqs. (A2) and the corresponding equations for b, $\bar{b}$, and $\bar{\rho}$ one obtains

$$
\begin{aligned}
[H_{\rm int},\mu_Q{}^*]&=[H,\mu_Q{}^*]_A-\hbar\omega_Q\{\mu_Q{}^*\langle[\mu_Q,\mu_Q{}^*]\rangle\\
&\qquad +\langle[\mu_{-Q}{}^*,\mu_Q{}^*]\rangle\mu_Q\}.
\end{aligned}
$$

Since $\mu_Q{}^*$ commutes with $H_{\rm int}$,

$$[H,\mu_Q{}^*]_A=\hbar\omega_Q\mu_Q{}^*,$$

which is the equation one obtains from Anderson. For the single-particle excitations, $\mu_k{}^*(\mathbf{Q})$, one finds in the same way

$$[H,\mu_k{}^*(\mathbf{Q})]_A=[H_{\rm int},\mu_k{}^*(\mathbf{Q})]=[H,\mu_k{}^*(\mathbf{Q})],$$

since $\mu_k{}^*(\mathbf{Q})$ commutes with μ_Q and $\mu_Q{}^*$. This proves that Eqs. (A2) are equivalent to Eqs. (3.1). The reason the terms that lead to the superconducting transition appear screened is that as far as these terms are concerned $H_{\rm int}$ is

$$H-\sum_{Q'<Q_{\max}}\hbar\omega_{Q'}\mu_{Q'}{}^*\mu_{Q'}\approx H-\tfrac{1}{2}\sum_{Q'<Q_{\max}}V(\mathbf{Q}')\rho_{-Q'}\rho_{Q'}.$$

In this Hamiltonian the two-body interaction is screened. For the same reason, had the exchange terms been kept we should have found that these, too, are screened.

It would seem possible to generalize the equations by supposing all the operators μ_i^* to be known linear combinations of ρ_k^Q, $\bar{\rho}_k^Q$, b_k^Q, and $\bar{b}_k^Q$. Then one would obtain a set of nonlinear integral equations for the coefficients by making the equations

$$[H-\sum_{j\neq i}\hbar\omega_j\mu_j^*\mu_j,\,\mu_i^*]=\hbar\omega_i\mu_i^*$$

linear in the operators. Of course, Eqs. (A1) would only be a first approximation to the plasmon operators.

APPENDIX B

In this appendix it will be shown that the operators, μ_k^* defined by Eqs. (3.4) and (3.5) form an orthonormal set. The plan is to show that the operators $X_k^*(\mathbf{Q})$, defined by

$$X_k^*(\mathbf{Q})=\sum_{k'}[\alpha_1^*(\mathbf{k}',\mathbf{k},\mathbf{Q})\gamma_{k'+Q1}^*\gamma_{k'0}^*-\beta_1^*(\mathbf{k}',\mathbf{k},\mathbf{Q})\gamma_{k'+Q0}\gamma_{k'1}],$$

form an orthonormal set. It then will follow that the coefficients of $X_k^*(\mathbf{Q})$ in the expansions of $\gamma_{k+Q1}^*\gamma_{k0}^*$ and $\gamma_{k+Q0}\gamma_{k1}$ in terms of the X's are, respectively, $\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})$ and $\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})$. By direct substitution it can be seen that $X_k^*(\mathbf{Q})$ satisfies the equations of motion with eigenvalue $\nu_k(\mathbf{Q})$. Hence $X_k(\mathbf{Q})$ can be identified with $\mu_k(\mathbf{Q})$ and the result will be proved. Now

$$\begin{aligned}
&[X_{k''}(\mathbf{Q}),X_{k'}(\mathbf{Q})^*]\\
&=\sum_k[\alpha_1(\mathbf{k},\mathbf{k}'',\mathbf{Q})\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*-\beta_1(\mathbf{k},\mathbf{k}'',\mathbf{Q})\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*]\\
&=\delta_{k'k''}+\frac{\Phi_{k''k'}^*+\frac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}^*}{\nu_{k'}-\nu_{k''}-i\epsilon}\\
&\quad+\frac{\Phi_{k'k''}+\frac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}}{\nu_{k''}-\nu_{k'}+i\epsilon}\\
&\quad+\sum_k\frac{(\Phi_{kk'}^*+\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}^*)(\Phi_{kk''}+\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''})}{(\nu_k-\nu_{k''}-i\epsilon)(\nu_k-\nu_{k'}+i\epsilon)}\\
&\quad-\sum_k\frac{(\Phi_{kk'}^*-\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}^*)(\Phi_{kk''}-\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''})}{(\nu_k+\nu_{k''})(\nu_k+\nu_{k'})}\\
&=\Big\{\tfrac{1}{2}\delta_{k'k''}+\frac{1}{\nu_{k'}-\nu_{k''}-i\epsilon}\Big[\Phi_{k''k'}^*+\tfrac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}^*\\
&\quad-\sum_k\Big(\frac{[\Phi_{kk'}^*+\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}^*][\Phi_{kk''}+\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''}]}{(\nu_k-\nu_{k'}+i\epsilon)}\\
&\quad+\frac{[\Phi_{kk'}^*-\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}^*][\Phi_{kk''}-\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''}]}{(\nu_k+\nu_{k'})}\Big)\Big]\Big\}\\
&\qquad+\{\mathbf{k}'\leftrightarrow\mathbf{k}''\}^*\\
&=\Big\{\tfrac{1}{2}\delta_{k'k''}+\frac{1}{\nu_{k'}-\nu_{k''}-i\epsilon}[\Phi_{k''k'}^*+\tfrac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}^*\\
&\quad-\sum_k\Phi_{kk''}\Big[\Phi_{kk'}^*\Big(\frac{1}{\nu_k-\nu_{k'}+i\epsilon}-\frac{1}{\nu_k+\nu_{k'}}\Big)\\
&\quad+\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk'}^*\Big(\frac{1}{\nu_k-\nu_{k'}+i\epsilon}+\frac{1}{\nu_k+\nu_{k'}}\Big)\Big]\\
&\quad-\sum_k\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''}\Big[\Phi_{kk'}^*\Big(\frac{1}{\nu_k-\nu_{k'}+i\epsilon}+\frac{1}{\nu_k+\nu_{k'}}\Big)\\
&\quad+\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})L_{k,k'}^*\Big(\frac{1}{\nu_k-\nu_{k'}+i\epsilon}-\frac{1}{\nu_k+\nu_{k'}}\Big)\Big]\Big\}\\
&\qquad+\{\mathbf{k}'\leftrightarrow\mathbf{k}''\}^*\\
&=\{\tfrac{1}{2}\delta_{k'k''}+\frac{1}{\nu_{k'}-\nu_{k''}-i\epsilon}[\Phi_{k''k'}^*+\tfrac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}^*\\
&\quad-\textstyle\sum_k\Phi_{kk''}[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*+\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*-\delta_{kk'}]\\
&\quad-\textstyle\sum_k\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''}[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*\\
&\qquad-\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*-\delta_{kk'}]\}+\{\mathbf{k}'\leftrightarrow\mathbf{k}''\}^*.
\end{aligned}$$

Now

$$\begin{aligned}
&\textstyle\sum_k\frac{1}{2}l(\mathbf{k},\mathbf{Q})L_{kk''}[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*-\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*]\\
&\quad=\sum_{k,k'''}\tfrac{1}{2}l(\mathbf{k},\mathbf{Q})V(\mathbf{k},\mathbf{k}''')l(\mathbf{k}''',\mathbf{Q})\\
&\qquad\times[\alpha_1(\mathbf{k}''',\mathbf{k}'',\mathbf{Q})-\beta_1(\mathbf{k}''',\mathbf{k}'',\mathbf{Q})]\\
&\qquad\times[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*-\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*].
\end{aligned}$$

This remains unchanged when $\mathbf{k}'$ and $\mathbf{k}''$ are interchanged and the complex conjugate is taken $[V(k,k')=V(k',k)^*]$. Hence this sum disappears from the final result. Similarly, the term

$$\textstyle\sum_k\Phi_{kk''}[\alpha_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*+\beta_1(\mathbf{k},\mathbf{k}',\mathbf{Q})^*]$$

does not contribute. Hence

$$\begin{aligned}
&[X_{k''}(\mathbf{Q}),X_{k'}(\mathbf{Q})^*]\\
&=\Big\{\tfrac{1}{2}\delta_{k'k''}+\frac{1}{\nu_{k'}-\nu_{k''}-i\epsilon}[\Phi_{k''k'}^*+\tfrac{1}{2}l(\mathbf{k}'',\mathbf{Q})L_{k''k'}^*\\
&\qquad+\Phi_{k'k''}+\tfrac{1}{2}l(\mathbf{k}',\mathbf{Q})L_{k'k''}]\Big\}+\{\mathbf{k}'\leftrightarrow\mathbf{k}''\}^*=\delta_{k'k''},
\end{aligned}$$

as was to be proved.