Adiabatic Invariants of Periodic Classical Systems*

C. S. GARDNER
Poulter Laboratories, Stanford Research Institute, Menlo Park, California
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Recently there has been renewed interest in adiabatic invariants of simply-periodic classical systems subject to perturbation by slow variation of parameters. In several interesting cases it has been shown that if the system varies slowly from one steady state to a different steady state, the appropriate adiabatic invariant is constant to an arbitrarily high order in a slowness parameter.

It is shown here how these and similar results may be derived by systematic use of a technique of perturbation theory of classical Hamiltonian systems. The method is essentially iteration of the transformation to action and angle variables.

I. INTRODUCTION

RECENTLY there has been renewed interest in the subject of adiabatic invariants of periodic classical systems perturbed by slow variation of parameters. In two interesting cases, Kruskal[1] and Lenard[2] have shown that if the system varies slowly from one steady state to a different steady state, the appropriate adiabatic invariant is constant, not only to first order, but to *all* orders in a parameter measuring the slowness of variation. Kruskal considered a charged particle in a slowly-varying magnetic field. The adiabatic invariant in this case is the magnetic moment of the current loop described by the particle in one cyclotron gyration. Lenard considered a slowly-varying periodic system with one degree of freedom. The adiabatic invariant here is the familiar action integral. The special case of Lenard's result, in which the system is the harmonic oscillator, was considered previously by Kulsrud[3] and by Hertweck and Schlüter.[4] In what follows we wish to point out how these and similar results can be derived by a perturbation method suggested by Chandrasekhar's treatment of the harmonic oscillator.[5]

II. PERIODIC SYSTEMS WITH ONE DEGREE OF FREEDOM

We consider a system having one degree of freedom, varying slowly with time, so that the Hamiltonian may be written

$$H=H(q,p,\epsilon t),$$

where ϵ is small. We suppose that the lines $H=\text{constant}$ (t being fixed) in the q-p plane are closed curves nested one inside the other, so that they can be distorted into a family of concentric circles by a continuous distortion of the q-p plane. We suppose that H and its derivatives of all orders with respect to q, p, ϵt are continuous.

To obtain an asymptotic solution of the equations of motion by perturbation theory we proceed as follows: We determine an *area-preserving* and (sense-preserving) mapping of the q-p plane into a q'-p' plane such that the curves $H=\text{constant}$ in the q-p plane are mapped into concentric circles in the q'-p' plane with centers at the origin. This mapping may be constructed as follows: Let $J(q,p,\epsilon t)$ be the area of the $H=\text{constant}$ curve through q, p. Let r, θ be the polar coordinates, in the q', p' plane, of the image of the point q, p. Then

$$r=[J/\pi]^{\frac{1}{2}},$$

$$\theta=2\pi\frac{\displaystyle\int^{q,p} dS/|\nabla H|}{\displaystyle\oint dS/|\nabla H|}.$$

The integrals here are line integrals on the $H=\text{constant}$ curve; dS is the element of arc length, and ∇H is the gradient of H. To remove an element of arbitrariness in θ, we specify that the ray from the center of the H-curves in the q, p plane toward positive q maps into the positive q'-axis. Of course J and $\theta/2\pi$ are the usual action and angle variables. Now, using the fact that the Jacobian of q', p' with respect to q, p is unity, it is easily shown that

$$q'dp'+pdq$$

is a complete differential (t being fixed), and so we can find a function $F(q,p',\epsilon t)$ such that

$$q'=\partial F/\partial p',\quad p=\partial F/\partial q.$$

We set the integration constant by specifying $F=0$ for $q'=0$, $p'=0$. We note that H depends on q', p' only via $(q')^2+(p')^2$, i.e., H is of the form

$$H=H_0[(q')^2+(p')^2,\ \epsilon t].$$

Now we define a canonical transformation as follows:

$$q'=\partial F/\partial p',\quad p=\partial F/\partial q,$$

$$H'=H+\partial F/\partial t=H+\epsilon\partial F/\partial(\epsilon t),$$

and we see that H' has the form

$$H'=H_0[(q')^2+(p')^2,\ \epsilon t]+\epsilon H_1(q',p',\epsilon t).$$

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[2] A. Lenard, Ann. Phys. (N. Y.) **6**, 261–276 (1959).

[3] R. M. Kulsrud, Phys. Rev. **106**, 205 (1957).

[4] F. Hertweck and A. Schlüter, Z. Naturforsch. **12A**, 844 (1957).

[5] S. Chandrasekhar, in *The Plasma in a Magnetic Field*, edited by R. Landshoff (Stanford, 1958).

The level lines of H' in the q'-p' plane are now nearly circles. We can make them circles by an area-preserving mapping that is nearly the identity mapping—given by

$$q''=\frac{\partial}{\partial p''}[q'p''+\epsilon F'(q',p'';\epsilon t)],$$

$$p'=\frac{\partial}{\partial q'}[q'p''+\epsilon F'(q',p'';\epsilon t)],$$

where we make $F'=0$ at $q''=0$, $p''=0$. Using $q'p''+\epsilon F'$ as generating function, we perform a canonical transformation, and obtain

$$H''=H_0[(q'')^2+(p'')^2,\epsilon t]+\epsilon H_1[(q'')^2+(p'')^2,\epsilon t]+\epsilon^2 H_2.$$

This procedure can be repeated *ad libitum*. We obtain, for any given N, variables q^N, p^N and a Hamiltonian H^N which is a function of ϵt and $(q^N)^2+(p^N)^2$ plus terms of order ϵ^N. Using Hamilton's equations of motion,

$$dp^N/dt=-\partial H^N/\partial q^N,$$
$$dq^N/dt=\partial H^N/\partial p^N,$$

we see that

$$\frac{d}{dt}[(p^N)^2+(q^N)^2]=O(\epsilon^N).$$

Hence J^N, given by

$$J^N=\pi[(p^N)^2+(q^N)^2],$$

is invariant to any desired order in ϵ.

From the construction, it is clear that if at any time the derivatives, of all orders, of H with respect to ϵt are zero, then q^N, p^N are identical with q', p', and J^N is identical with J. Thus we have the following result (which was proved in another manner by Lenard): Suppose that at $\epsilon t=0$ and at $\epsilon t=1$ the derivatives, of all orders, of H with respect to ϵt are zero. Consider a motion defined by initial conditions

$$q=q_0,\quad p=p_0\quad \text{for}\quad \epsilon t=0,$$

where q_0, p_0 are independent of ϵ. Let q_1, p_1 be the values of q, p at $\epsilon t=1$. (Then q_1, p_1 are functions of ϵ.) Then we have

$$J(q_1,p_1,1)=J(q_0,p_0,0)+O(\epsilon^{N-1})$$

for any integer N, however large.

For the sake of mathematical rigor, sufficient assumptions should be made about the function H so that we can be sure the mapping of q, p into q', p' and also the inverse mapping are continuously differentiable any number of times.

III. THE CHARGED PARTICLE IN A SLOWLY VARYING ELECTROMAGNETIC FIELD

To apply the perturbation-theory approach to the motion of a charged particle in a slowly-varying electromagnetic field, some preliminary preparations are required. We assume $\mathbf{E}$, $\mathbf{B}$ are slowly-varying functions of the space coordinates x_i and the time t, that $\mathbf{E}\cdot\mathbf{B}$ is small, and that B is bounded away from zero. We express these conditions by assuming that the vector and scalar potentials have the form

$$\mathbf{A}=\frac{1}{\epsilon}\mathbf{A}'(\epsilon x_i,\epsilon t),$$

$$\phi=\frac{1}{\epsilon}\phi'(\epsilon x_i,\epsilon t),$$

and

$$\mathbf{E}\cdot\mathbf{B}=O(\epsilon).$$

The Hamiltonian $\bar{H}$ is (for simplicity we assume $e=1$, $m=1$)

$$\bar{H}=\frac{1}{\epsilon}\phi'+\tfrac{1}{2}\sum_{i=1}^{3}\left(\bar{p}_i-\frac{1}{\epsilon}A_i'\right)^2$$

(where we denote the momentum conjugate to x_i by the symbol $\bar{p}_i$). We can find geometric parameters α, β, functions of ϵx_i, ϵt, so that

$$\mathbf{B}=(1/\epsilon^2)\nabla\alpha\times\nabla\beta.$$

Therefore with the proper gauge,[6] we can write

$$A_i'=(1/\epsilon)\alpha(\partial\beta/\partial x_i).$$

We define a parameter S, a function of ϵx_i, ϵt, by saying that S/ϵ is the *arc length* along a line of force. We now define coordinates and momenta q_i, p_i and their associated Hamiltonian H by

$$q_i=\partial\bar{F}/\partial p_i,$$
$$\bar{p}_i=\partial\bar{F}/\partial x_i,$$
$$H=\bar{H}+\partial\bar{F}/\partial t,$$

where the generating function is

$$\bar{F}(x_i;p_i)=\frac{S}{\epsilon}p_2+\frac{\beta}{\epsilon}p_1+\frac{\alpha}{\epsilon}p_3-p_1p_3.$$

We obtain

$$\bar{p}_i-\frac{1}{\epsilon}A_i'=\frac{1}{\epsilon}\frac{\partial S}{\partial x_i}p_2-\frac{1}{\epsilon}\frac{\partial\beta}{\partial x_i}q_3+\frac{1}{\epsilon}\frac{\partial\alpha}{\partial x_i}p_3,$$

and

$$\alpha=\epsilon p_1+\epsilon q_3,$$
$$\beta=\epsilon q_1+\epsilon p_3,\qquad(1)$$
$$S=\epsilon q_2.$$

We note that p_2, q_3, p_3 are essentially components of velocity, and are thus $O(1)$, and ϵq_2, ϵp_1, ϵq_1 are essentially geometric coordinates, and are thus $O(1)$.

[6] The use of the parameters α and β was suggested by Professor H. Grad of New York University.

The Hamiltonian H can now be written down. We have

$$H=\frac{1}{\epsilon}\left[\phi'+\frac{1}{\epsilon}\frac{\partial\beta}{\partial t}(\epsilon p_1)\right]+\frac{1}{2}\left\{\left|\frac{1}{\epsilon}(\nabla S)p_2-\frac{1}{\epsilon}(\nabla\beta)q_3+\frac{1}{\epsilon}(\Delta\alpha)p_3\right|^2+\frac{1}{\epsilon}\left(\frac{\partial S}{\partial t}\right)p_2+\frac{1}{\epsilon}\frac{\partial\alpha}{\partial t}p_3\right\}.$$

The coefficients here are understood to be evaluated in terms of q's, p's by use of the relations (1). It is appropriate to expand in powers of q_3, p_3, since the terms ϵq_3, ϵp_3 in (1) are of higher order than the terms ϵp_1, ϵq_1. If we carry out this expansion, we see that H has the form

$$H=\frac{1}{\epsilon}H_{-1}+H_0+\epsilon H_1+\cdots, \tag{2}$$

where H_{-1} is a function of ϵq_1, ϵp_1, ϵq_2, ϵt but

$$\frac{\partial}{\partial(\epsilon q_2)}H_{-1}=O(\epsilon) \tag{3}$$

[this follows from the assumption $\mathbf{E}\cdot\mathbf{B}=O(\epsilon)$ and the relation $\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\phi$]; and where H_0, H_1, etc., are functions of ϵq_1, ϵp_1, ϵq_2, ϵt and also of p_2, q_3, p_3. Note that since ϕ', $\partial\beta/\partial t$ depend on α, β, which in turn contain terms ϵq_3, ϵp_3, the first term in H above contributes terms to H_0.

$$H_{-1}=\phi'+\frac{\alpha}{\epsilon}\frac{\partial\beta}{\partial t}, \tag{4}$$

$$H_0=\frac{1}{2}\left\{v_\perp^2+p_2^2+\frac{2}{\epsilon}\left[\frac{\partial S}{\partial t}+\frac{(\mathbf{E}\times\mathbf{B})}{B^2}\cdot\nabla S\right]p_2-\frac{E^2}{B^2}\right\}, \tag{5}$$

where the vector $\mathbf{v}_\perp$ is given by

$$\mathbf{v}_\perp=\frac{\nabla S}{\epsilon}p_2-\frac{\nabla\beta}{\epsilon}q_3+\frac{\nabla\alpha}{\epsilon}p_3-p_2\frac{\mathbf{B}}{B}-\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{6}$$

or

$$\mathbf{v}_\perp=-\frac{\nabla\beta}{\epsilon}\left[q_3+\frac{1}{\epsilon^2}\left(p_2\frac{\mathbf{B}}{B}+\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\cdot\frac{(\nabla S\times\nabla\alpha)}{B}\right]+\frac{\nabla\alpha}{\epsilon}\left[p_3+\frac{1}{\epsilon^2}\left(p_2\frac{\mathbf{B}}{B}+\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\cdot\frac{(\nabla S\times\nabla\beta)}{B}\right]. \tag{7}$$

Here it is understood that the coefficients are to be evaluated in terms of the q's and p's by setting

$$\alpha=\epsilon p_1,$$
$$\beta=\epsilon q_1,$$
$$S=\epsilon q_2.$$

The Hamiltonian function is constant on closed curves (ellipses) in the q_3p_3 plane if all other variables are fixed; and is a slowly-varying function of time and also is a slowly-varying function of the additional variables q_2, q_1, p_1. These circumstances permit a perturbation-theory treatment modeled on the method of the first part of the paper. The first step, as before, is to find an area-preserving mapping of the q_3, p_3 plane into a q_3', p_3' plane which takes the curves $H_0=$constant into circles. Let us exhibit the dependence of H_0 on the various variables as follows:

$$H_0=H_0(\epsilon q_1,\epsilon p_1;\ \epsilon q_2,p_2;\ q_3,p_3;\ \epsilon t).$$

Then we can find a function F, where

$$F=q_1p_1'+q_2p_2'+\psi(\epsilon q_1,\epsilon p_1';\ \epsilon q_2,p_2';\ q_3,p_3';\ \epsilon t),$$

such that if we determine q_3', p_3' by solving

$$q_3'=\partial F/\partial p_3',$$
$$p_3=\partial F/\partial q_3,$$

we find that the curves in the q_3'-p_3' plane given by

$$H_0(\epsilon q_1,\epsilon p_1';\ \epsilon q_2,p_2';\ q_3,p_3;\ \epsilon t)=\text{constant}$$

are concentric circles with center $q_3'=0$, $p_3'=0$.

Now we perform a canonical transformation as follows:

$$q_i'=\partial F/\partial p_i',$$
$$p_i=\partial F/\partial q_i,$$
$$H'=H+\partial F/\partial t=H+\epsilon\partial F/\partial(\epsilon t).$$

We then obtain

$$\epsilon q_1=\epsilon q_1'+O(\epsilon^2),$$
$$\epsilon p_1=\epsilon p_1'+O(\epsilon^2),$$
$$\epsilon q_2=\epsilon q_2'+O(\epsilon),$$
$$p_2=p_2'+O(\epsilon),$$

and it follows [making use of (3)] that H' has the form

$$H'=(1/\epsilon)H_{-1}+H_0'+\epsilon H_1'+\ldots,$$

and now H_0' depends on q_3', p_3' only via the combination $(q_3')^2+(p_3')^2$.

The procedure can now be repeated; by an appropriate infinitesimal canonical transformation we can bring it about that H'' depends on q_3'', p_3'' only via $(q_3'')^2+(p_3'')^2$ except for terms of order ϵ^2, and so forth.

In this way we can construct an asymptotic integral of the equations of motion, which is constant to any desired order in ϵ. Using (5) and (7) which show explicitly how H_0 depends on q_3 and p_3, and referring to the definition of the invariant as an area, we see that, to lowest order, the invariant μ we construct is

$$\mu\sim\frac{\epsilon^2v_\perp^2}{[(\nabla\alpha)^2(\Delta\beta)^2-(\nabla\alpha\cdot\nabla\beta)^2]^{\frac{1}{2}}}=\frac{\epsilon^2v_\perp^2}{(\nabla\alpha\times\nabla\beta)}=\frac{v_\perp^2}{B}. \tag{8}$$

If **E** and **B** are constant in space and time, we can choose α, β, and S to be rectangular coordinates; and it is seen that the perturbation process terminates with H'; and so in this case

$$\mu = v_\perp^2/B$$

exactly. This observation furnishes a proof of the theorem of Kruskal, which can be stated as follows:

If before the time $\epsilon t=0$ and after the time $\epsilon t=1$, it happens that **E**, **B** are constant vectors (independent of space coordinates and of time), then the value of $v_\perp^2/B$ at $\epsilon t=1$ differs from its value at $\epsilon t=0$ by a quantity which is $O(\epsilon^N)$, however large N may be.

It can be seen from the above developments that in $H^{(N)}$ we may neglect terms of order ϵ^N and replace $(p_3^{(N)})^2+(q_3^{(N)})^2$ in what remains by its initial value and thereby commit errors which are small of any desired order in ϵ, if N is large enough. We then have a Hamiltonian of a system with *two* degrees of freedom. This can be interpreted as the Hamiltonian describing the motion of the guiding center[7] of the particle. If the guiding-center motion is approximately periodic in the q_2-p_2 plane, it can happen that a second adiabatic invariant can be defined. By an application of perturbation theory to this system it can be proved, under appropriate sufficient conditions, that the second adiabatic invariant is also constant to all orders. Such a second adiabatic invariant has been considered for the case of the particle in a double-mirror field, adiabatically trapped on a line of force between two maxima of B.[8] Where a second adiabatic invariant exists, by perturbation theory we can reduce the system to one having *one* degree of freedom.

To indicate how this second adiabatic invariant may be treated according to the methods of the present paper, let us assume that (1) time variations are extremely slow—so that we may suppose that **E**, **B** depend on $\epsilon^2 t$, and (2) the component of **E** perpendicular to **B** (as well as the component parallel to **B**) is $O(\epsilon)$. Then the guiding-center Hamiltonian is

$$H=\frac{1}{2}\left\{p_2^2+\mu B+2\left[\frac{\phi'}{\epsilon}+\alpha\frac{\partial\beta}{\partial(\epsilon^2 t)}\right]\right\}+O(\epsilon).$$

Here use has been made of (8). The coefficients are evaluated at

$$\alpha=\epsilon p_1,\quad \beta=\epsilon q_1,\quad S=\epsilon q_2.$$

The above Hamiltonian depends *more slowly* on t and on q_1 and p_1 than it does on q_2 and p_2. If α, β, and t are fixed, we see that the Hamiltonian would describe the one-dimensional motion of a particle of unit mass in a force field with potential

$$\tfrac{1}{2}\mu B-\int E_{\parallel}\frac{dS}{\epsilon}.$$

If this potential rises enough as S increases or decreases, that is, as one goes either way along a line of force (fixing α, β determines a line of force) at a fixed time, the particle is trapped. The level lines of H in the q_2p_2 plane will then be nested closed curves. The perturbation treatment we have described may then be used to construct an integral I given by a series

$$I=I_0+\epsilon I_1+\epsilon^2 I_2+\ldots,$$

which is constant to all orders in ϵ; and to lowest order we have

$$I_0=\oint p_2 dq_2,$$

where the path of integration is a curve $H=$constant in the q_2p_2 plane, where α, β, t are given fixed values. This is the second adiabatic invariant as usually defined.

[7] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950), Chap. II.

[8] Chew, Goldberger, and Low, Proc. Roy. Soc. (London) **A236**, 112 (1956).