Moment of Inertia of Interacting Many-Body Fermion Systems*

R. D. AMADO AND K. A. BRUECKNER
University of Pennsylvania, Philadelphia, Pennsylvania
(Received March 9, 1959)

It is shown that the moment of inertia of a noninteracting many-body fermion system moving under periodic boundary conditions has the classical or rigid value when calculated on the "cranking" model of Inglis. By investigating the analogous "pushing" case for the inertial mass we show that the rigid moment can be associated with rigid rotation in spite of apparent surface currents. The effect of particle-particle forces is investigated in the lowest order of perturbation theory. The terms corresponding to a level shift or effective mass are just compensated by other terms and there is no change in the moment of inertia. The possible general validity of these results and their consequence is discussed.

I. INTRODUCTION

THE experimental determination of the moment of inertia of large nuclei[1] has in the past few years excited a variety of theoretical attempts to account for the magnitude of the moment.[2] It was first suggested by Teller and Wheeler[3] that the identity of the nuclear particles would tend to prevent rotation in states with small rotational quantum number. This would lead to large rotational excitation energies and hence to small effective moments of inertia. An alternative version of this view was given by Bohr and Mottelson[4] who argued that shell structure and particle indistinguishability would lead to irrotational flow. On this view one expects that

$$\mathcal{I} \approx \beta^2 \mathcal{I}_{\text{rigid}}, \tag{1}$$

where $\mathcal{I}_{\text{rigid}}$ is the moment of inertia for rigid rotation and β is a distortion parameter measuring the departure from sphericity. Predictions based on Eq. (1) give much too small moments of inertia, showing that the picture of irrotational flow is incorrect.

Inglis[5] put forward an alternative approach based on the very reasonable "cranking" model, in which the nucleus is constrained to rotate in an external field with fixed angular momentum and its rotational energy then determined. Inglis found there markable result that for particles moving independently in a harmonic oscillator potential one obtains the irrotational flow value for a nonspherical nucleus with closed-shell occupation numbers, but that if only one or more particles are added in the unfilled shell, one obtains the rigid moment.[6] It has been suggested that any large independent-particle system should yield the rigid moment of inertia on the "cranking" model, independent of the potential form.[6] In Sec. II we show that this is the case for free fermions in a periodic box, but we know of no general proof.

Many attempts have been made to determine the effects of interactions on the rigid moment. In particular Blin-Stoyle[7] has suggested that the change in nuclear level density due to interactions may replace the nucleon mass, M, by an effective mass, M^*, thereby decreasing the moment of inertia to roughly $0.6\mathcal{I}_{\text{rigid}}$, in better agreement with experiment. The procedure leads, however, to a similar reduction in the translational or inertial mass of the nucleus—hardly in agreement with experiment. Recently Belyaev[8] and Migdal[9] have considered interaction effects associated with the abnormal level structure in a Fermi gas with weak attractive interactions, the anomaly being similar to the change in electron level density in a superconductor.[10]

We wish in this paper to consider interaction effects and in particular to determine the effects of changes in level density. In the interest of clarity we shall discuss in parallel the "cranking" and "pushing" models to insure that our methods give the correct results for the translation case. In Sec. II we obtain the rigid moment for a free Fermi gas. We also discuss the physical interpretation of the motion and show that in both rotation and translation the motion is indeed rigid. In Sec. III the effects on the moment of inertia of a large system of an interaction in lowest perturbation order are investigated. It is shown that those terms corresponding to a level shift—that is, to an effective mass—are just compensated by other terms so that there is *no change* in the moment of inertia.

II. ROTATION AND TRANSLATION IN THE NONINTERACTING SYSTEM

We first consider the problem without interactions. The calculation is elementary and gives useful insight

* Supported in part by the National Science Foundation.

[1] See Alder, Bohr, Huus, Mottelson, and Winter, Revs. Modern Phys. **28**, 432 (1956).

[2] A full list of references is given in the recent review articles: F. Villars, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, 1957), Vol. 7, p. 185; S. A. Moszkowski, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 39.

[3] E. Teller and J. A. Wheeler, Phys. Rev. **53**, 778 (1938).

[4] A. Bohr, *Rotational States of Atomic Nuclei*, Doctoral thesis (Ejnar Munksgaard Forlag. Copenhagen, 1954); A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1954).

[5] D. R. Inglis, Phys. Rev. **96**, 1059 (1954); **103**, 1786 (1956).

[6] This result was also obtained by Bohr and Mottelson [A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955).

[7] R. J. Blin-Stoyle, Nuclear Phys. **2**, 169 (1956–57).

[8] V. I. Belyaev, Nuclear Phys. (to be published).

[9] A. B. Migdal (private communication).

[10] Bardeen, Cooper and Schrieffer, Phys. Rev. **108**, 1175 (1957).

into the wave function of the rotating or translating system.

In the case of rotation we choose as unperturbed wave functions plane waves, periodic in a cubic box of side L. The "cranking" model instructs us to find the energy required to "crank" the boundary on which the wave function is periodic. For rotation about the z axis with angular frequency ω, the Hamiltonian in the rotating coordinate system becomes[5]

$$H=H_0-\omega L_z, \tag{2}$$

where L_z is the operator for the angular momentum about the z axis. The moment of inertia of the system in the state "0" is

$$\mathcal{J}=-2\sum_{n\neq 0}\frac{|\langle\psi_0|L_z|\psi_n\rangle|^2}{E_0-E_n}. \tag{3}$$

A single-particle plane wave state in the box, labeled by the positive or negative integers l,m,n is

$$|lmn\rangle=\frac{1}{L^{\frac{3}{2}}}\exp\left[\frac{2\pi i}{L}(lx+my+nz)\right]. \tag{4}$$

The energy of such a state is

$$(1/2M)(2\pi\hbar/L)^2(l^2+m^2+n^2). \tag{5}$$

The matrix elements of L_z then are

$$\langle lmn|L_z|l'm'n'\rangle$$
$$=\delta_{n,n'}\frac{\hbar}{i}\left[\frac{m}{l'-l}(-1)^{l-l'}\delta_{m,m'}(1-\delta_{l,l'}) -\frac{l}{m'-m}(-1)^{m-m'}\delta_{l,l'}(1-\delta_{m,m'})\right]. \tag{6}$$

Substituting into Eq. (3) and using the symmetry of the m,l sums, we have for the moment of inertia

$$\mathcal{J}=\frac{2Ml^2}{\pi^2}\sum_{m,n,l}\sum_{l\neq l'}\frac{m^2}{(l'-l)^2(l'^2-l^2)}. \tag{7}$$

The sums in Eq. (7) are greatly simplified if we note that the major contribution comes from l' very close to l. Due to the exclusion principle this can occur only when $|l|$ and $|l'|$ are close to the maximum value of $|l|$, which is

$$\Lambda=(F^2-m^2-n^2)^{\frac{1}{2}}, \tag{8}$$

F being the magnitude of $(l^2+m^2+n^2)^{\frac{1}{2}}$ at the Fermi surface. For a large system Λ will be large, so we may write

$$|l^2-l'^2|=2\Lambda|l-l'|, \tag{9}$$

and also introduce the new summation variable

$$q=l'-l. \tag{10}$$

Making use of the symmetry of contributions from negative and positive l, we find for Eq. (7)

$$\mathcal{J}=\frac{2ML^2}{\pi^2}\sum_{m,n}\frac{m^2}{\Lambda}\sum_{q=1}^{\infty}\sum_{l=\Lambda-q}^{\Lambda}\frac{1}{q^3}. \tag{11}$$

The sum over l and q can now be reduced to a zeta function,

$$\sum_{q=1}^{\infty}\sum_{l=\Lambda-q}^{\Lambda}\frac{1}{q^3}=\sum_{q=1}^{\infty}\frac{1}{q^2}=\frac{\pi^2}{6}. \tag{12}$$

The remaining sums in Eq. (11) may be done as integrals. The result is

$$\sum_{m=-F}^{F}m^2\sum_{n=n_{\min}}^{n_{\max}}\frac{1}{\Lambda}=\tfrac{2}{3}\pi F^3=\tfrac{1}{2}N, \tag{13}$$

where N is the total number of particles. Thus combining Eqs. (11), (12), and (13), we find

$$\mathcal{J}=MN(L^2/6), \tag{14}$$

which is the rigid moment or classical value. It is interesting to note that the same result can be obtained if the exclusion principle is ignored in the intermediate states, since the symmetry of the sums in l and l' is such as to cancel all terms in which the excited state lies within the Fermi sea. A similar circumstance occurs in the work of Inglis for the harmonic oscillator.[5]

The fact that the moment of inertia turns out to have the rigid value is a little surprising, as Inglis has already emphasized, since in his calculation and again in ours, the rotational energy appears to be carried entirely by particles moving in a few states very near the Fermi surface. It is to be emphasized, however, that because the particles are identical every particle participates in these few states and a rigid rotation of the system can be completely equivalent to motion in which only a small fraction of the particles appear to be changing their state.

To clarify better the significance of the apparent surface flow in the rotation, we next carry out an evaluation of the translational, inertial mass using a method exactly paralleling the rotational calculation.[11] For this we use the "pushing" model, which gives for the inertial mass

$$\mathfrak{M}=-2\sum_{n\neq 0}\frac{|\langle\psi_0|P_z|\psi_n\rangle|^2}{E_0-E_n}, \tag{15}$$

where P_z is the linear momentum operator. Equation (15) can be evaluated formally using the relation between P_z and the commutator of z with H,[12] but we

[11] This analog has also been considered by Inglis [D. R. Inglis, Nuclear Phys. 8, 125 (1958)].

[12] A similar formal approach is lacking in the rotational case since there is no operator whose commutator with H is L_z. It might be possible to find an operator whose commutator with H is approximately L_z but care must be taken that this operator be proper. For example, the angular coordinate conjugate to L_z is not a proper operator since it is not periodic in itself.

carry out a direct evaluation since the comparison with the "cranking" model is illuminating.

As basic wave functions for the "pushing" case we cannot use plane waves since these are eigenfunctions of P_z. An appropriate choice is standing waves in a box of side L. The matrix element of P_z between single-particle states in the box is

$$\langle mnl|P_z|m'n'l'\rangle=\delta_{m,m'}\delta_{n,n'}\frac{2ll'\hbar i}{(l'^2-l^2)L}[1-(-1)^{l-l'}]. \quad (16)$$

The energy of a single particle state is

$$E_{m,n,l}=\frac{\hbar^2\pi^2}{2ML^2}(m^2+n^2+l^2). \quad (17)$$

Substituting into Eq. (15) we find

$$\mathfrak{M}=\frac{16M}{\pi^2}\sum_{l,l'}\frac{(ll')^2[1-(-1)^{l-l'}]^2}{(l'^2-l^2)^3}. \quad (18)$$

The principal contribution to the sums comes from l and l' close to the maximum value of l which is

$$\Lambda=(F^2-m^2-n^2)^{\frac{1}{2}}. \quad (19)$$

Thus we again write

$$l'^2-l^2=2\Lambda(l'-l)=2\Lambda q. \quad (20)$$

Equation (18) now becomes

$$\mathfrak{M}=\frac{8M}{\pi^2}\sum_{m,n}\Lambda\sum_{q(\mathrm{odd})=1}^{\infty}\sum_{l=\Lambda-q}^{\Lambda}\frac{1}{q^3}. \quad (21)$$

The sum over q and l again gives a zeta function,

$$\sum_{q(\mathrm{odd})=1}^{\infty}\sum_{l=\Lambda-q}^{\Lambda}\frac{1}{q^3}=\sum_{q(\mathrm{odd})=1}^{\infty}\frac{1}{q^2}=\tfrac{1}{8}\pi^2, \quad (22)$$

and the sums over m and n give the number of particles, N. Thus we obtain the expected result

$$\mathfrak{M}=NM. \quad (23)$$

We see again that the translation energy comes from small values of q [see Eq. (21)] or from states very near the Fermi surface. In this case, however, it is clear that this description is simply an alternative way of describing the full, rigid mass translation. The translation of the Fermion system is represented in momentum space by a simple displacement of the entire Fermi sphere; clearly this may be achieved by moving all the states, or by taking a few from one side of the sphere and putting them over on the other.

The perturbation approach corresponds to the second of these equivalent descriptions. The wave function for translation of a system with velocity v is the internal state multiplied by plane wave for center-of-mass motion $\exp(-iMv\sum_i z_i/\hbar)$. Using the formal commutator method mentioned above, it can be shown that the perturbation series for the wave function in the "pushing" model gives just the series expansion in powers of the velocity of this center-of-mass plane wave, and hence corresponds to rigid translation in spite of the apparent surface flow.

III. ROTATION IN THE INTERACTING SYSTEM

In this section we shall prove that the rigid moment of inertia obtained in the previous section is not altered by an interparticle interaction to first order in the interaction strength. The proof is straightforward if tedious but unfortunately we have found no way of making it more elegant or of generalizing it to higher orders, if indeed this is possible. First order is sufficient, however, to show that the result holds even if the level density of the system is altered by the interaction, i.e., if an effective mass M^* must be used to characterize the energy levels.[13]

We now include in the Hamiltonian, Eq. (2), a perturbation

$$v=\tfrac{1}{2}\sum_{i,j}v(|\mathbf{r}_i-\mathbf{r}_j|). \quad (24)$$

The matrix elements of v for a large system do not strictly conserve total momentum; we can, however, to first order in the size of the system, neglect the matrix elements nondiagonal in the total momentum. To first order in the interaction, the shift in energy of the system due to rotation may be written

$$\Delta E=-\tfrac{1}{2}\mathscr{I}\omega^2=\omega^2\Bigg[\sum_n\frac{|\langle\psi_0|L_z|\psi_n\rangle|^2}{E_0-E_n}+\sum_{n,m}\frac{\langle\psi_0|L_z|\psi_n\rangle(\langle\psi_n|v|\psi_m\rangle-\langle\psi_0|v|\psi_0\rangle\delta_{m,n})\langle\psi_m|L_z|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}$$
$$+2\sum_{n,m}\frac{\langle\psi_0|L_z|\psi_n\rangle\langle\psi_n|L_z|\psi_m\rangle\langle\psi_m|v|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}\Bigg]. \quad (25)$$

The first term gives just the rigid moment, as we have seen, and thus we may write[14]

$$\mathscr{I}-\mathscr{I}_{\mathrm{rigid}}=-2\sum_{n,m}\frac{\langle\psi_0|L_z|\psi_n\rangle(\langle\psi_n|v|\psi_m\rangle-\langle\psi_0|v|\psi_0\rangle\delta_{m,n})\langle\psi_m|L_z|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}-4\sum_{n,m}\frac{\langle\psi_0|L_z|\psi_n\rangle\langle\psi_n|L_z|\psi_m\rangle\langle\psi_m|v|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)}. \quad (26)$$

[13] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[14] Using the formal commutation relations, one can easily show that Eq. (26) gives zero in the "pushing" case.

The matrix elements may be simplified since for a given term in v, v_{12} say, only the angular momentum operators for the pair 1 and 2 give nonvanishing contributions. There are three distinct terms then, and we write for Eq. (26)

$$\mathcal{I}-\mathcal{I}_{\text{rigid}}=\Delta_1+\Delta_2+\Delta_3,$$

where

$$\Delta_1=-2\sum_{1,2}\sum_{n}\frac{\langle\psi_0|L_z(1)|\psi_n\rangle(\langle\psi_n|v_{12}|\psi_n\rangle-\langle\psi_0|v_{12}|\psi_0\rangle)\langle\psi_n|L_z(1)|\psi_0\rangle}{(E_0-E_n)(E_0-E_n)},$$

$$\Delta_2=-2\sum_{1,2}\sum_{n\neq m}\frac{\langle\psi_0|L_z(1)|\psi_n\rangle\langle\psi_n|v_{12}|\psi_m\rangle\langle\psi_m|L_z(2)|\psi_0\rangle}{(E_0-E_n)(E_0-E_m)},$$

$$\Delta_3=-2\sum_{1,2}\sum_{nm}\frac{[\langle\psi_0|L_z(1)|\psi_n\rangle\langle\psi_n|L_z(2)|\psi_m\rangle+\langle\psi_0|L_z(2)|\psi_n\rangle\langle\psi_n|L_z(1)|\psi_m\rangle]}{(E_0-E_n)(E_0-E_m)}\langle\psi_m|v_{12}|\psi_0\rangle. \quad (27)$$

These three terms may also be expressed diagramatically as shown in Fig. 1.

To evaluate Eq. (27) we use the matrix elements of L_z between single-particle states as given in Eq. (6). The matrix elements of v_{12} between particle states are conveniently expressed in terms of the vectors

$$\mathbf{p}_1=\frac{2\pi\hbar}{L}(l_1,m_1,n_1),\quad \mathbf{p}_1'=\frac{2\pi\hbar}{L}(l_1',m_1,n_1), \quad (28)$$

and similarly for $\mathbf{p}_2$ and $\mathbf{p}_2'$. The result is

$$-\Delta_1=\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2n_2}\left(\frac{m_1}{l_1-l_1'}\right)^2\frac{1}{(l_1^2-l_1'^2)^2}[\langle\mathbf{p}_1'\mathbf{p}_2|v|\mathbf{p}_1'\mathbf{p}_2\rangle-\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle],$$

$$-\Delta_2=\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2n_2l_2'}\frac{m_1}{(l_1-l_1')}\frac{m_2}{(l_2-l_2')}\frac{\langle\mathbf{p}_1'\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2'\rangle}{(l_1^2-l_1'^2)(l_2^2-l_2'^2)}, \quad (29)$$

$$-\Delta_3=-\frac{M^2L^4}{\pi^4\hbar^2}\sum_{l_1m_1n_1l_1'}\sum_{l_2m_2n_2l_2'}\frac{m_1}{(l_1-l_1')}\frac{m_2}{(l_2-l_2')}\left(\frac{1}{l_1^2-l_1'^2}+\frac{1}{l_2^2-l_2'^2}\right)\frac{\langle\mathbf{p}_1'\mathbf{p}_2'|v|\mathbf{p}_1\mathbf{p}_2\rangle}{l_1^2+l_2^2-l_1'^2-l_2'^2}.$$

Each matrix element of v represents the correctly antisymmetrized combination, direct term minus exchange term.

We first calculate Δ_1. We see that as in the noninteracting case the important contributions to the l_1 and l_1' sum come from excitations very near the Fermi surface, where

$$l_1'^2-l_1^2=2q\Lambda_1=2q(F^2-m_1^2-n_1^2)^{\frac{1}{2}}. \quad (30)$$

The difference of the potential matrix elements is zero to lowest order in q and thus we write

$$\langle\mathbf{p}_1'\mathbf{p}_2|v|\mathbf{p}_1'\mathbf{p}_2\rangle-\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle|_{l_1\cong l_1=\Lambda_1}$$
$$\cong q(d/dl_1)\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle|_{l_1=\Lambda_1}. \quad (31)$$

Proceeding now as in Sec. II, and combining the contributions from positive and negative values of l_1 and l_1', we find

$$\Delta_1=-\frac{M^2L^4}{2\pi^4\hbar^2}\sum_{m_1n_1}\frac{m_1^2}{\Lambda_1^2}\left\{\frac{d}{dl_1}\sum_{l_2m_2n_2}\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle\right\}_{l_1=\Lambda_1}$$
$$\times\sum_{l_1=0}^{\Lambda_1}\sum_{q=\Lambda_1-l_1+1}^{\infty}\frac{1}{q^3}. \quad (32)$$

The sum over q and l_1 gives $\pi^2/6$. The sum over l_2, m_2, and n_2 in the Fermi sea gives the single-particle potential, i.e.,

$$V(p_1)=\sum_{l_2m_2n_2}\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle. \quad (33)$$

Since $V(p_1)$ is a function only of the magnitude of $\mathbf{p}_1$,

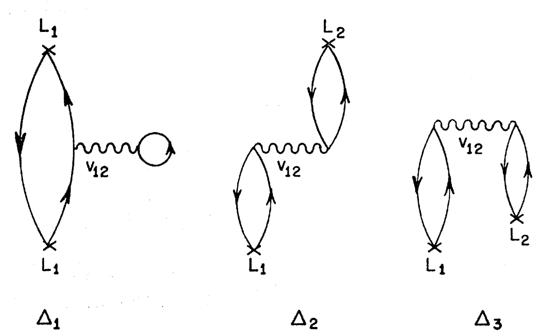

FIG. 1. Diagrams representing corrections to moment of inertia, corresponding to the definitions of Eq. (27). Lines with upward and downward arrows represent particles and holes in the Fermi sea. A diagram similar to Δ_1 but with the interaction with the hole line has not been included. It corresponds to the second term in Δ_1 as given in Eq. (27).

the derivative may be expressed as

$$\left\{\frac{d}{dl_1}V(p_1)\right\}_{l_1=\Lambda_1}=\Lambda_1\left(\frac{2\pi\hbar}{L}\right)^2\frac{1}{p_F}\left(\frac{dV}{dp}\right)_{p=p_F}, \quad (34)$$

where we have used the fact that at $l_1=\Lambda_1$, the momentum vector lies at the Fermi surface, $p_1=p_F$. Since $(dV/dp)_{p=p_F}$ is constant, it may be taken out of the sum and we are left with

$$\Delta_1=-\frac{M^2L^2}{3}\frac{1}{p_F}\frac{dV}{dp_F}\sum_{m_1n_1}\frac{m_1^2}{\Lambda_1}. \quad (35)$$

The sum over m_1 and n_1 gives $\frac{1}{2}N$, the total number of particles. Thus we obtain for Δ_1 the result

$$\Delta_1=-\frac{M}{p_F}\frac{dV}{dp_F}\mathcal{I}_{\text{rigid}}, \quad (36)$$

where $\mathcal{I}_{\text{rigid}}$ is the rigid moment of inertia, Eq. (14).

The contribution Δ_1 has the form of a change in the energy denominator of Eq. (3), and hence it should be the first-order effective-mass term. To see this we combine $\mathcal{I}_{\text{rigid}}$ and Δ_1, giving

$$\mathcal{I}_{\text{rigid}}+\Delta_1=\mathcal{I}_{\text{rigid}}\left(1-\frac{M}{p_F}\frac{dV}{dp_F}\right). \quad (37)$$

Treating the correction as a small perturbation, we can also write this as

$$\mathcal{I}_{\text{rigid}}+\Delta_1\cong\mathcal{I}_{\text{rigid}}\Big/\left(1+\frac{M}{p_F}\frac{dV}{dp_F}\right). \quad (38)$$

Now using the definition of the effective mass,[13]

$$\frac{1}{M^*}=\frac{1}{M}+\frac{1}{p_F}\frac{dV}{dp_F}, \quad (39)$$

we find

$$\mathcal{I}_{\text{rigid}}+\Delta_1\cong(M^*/M)\mathcal{I}_{\text{rigid}}. \quad (40)$$

We next evaluate Δ_2 and Δ_3. Δ_3 may be simplified using the identity

$$\left(\frac{1}{l_1^2-l_1'^2}+\frac{1}{l_2^2+l_2'^2}\right)\frac{1}{l_1^2+l_2^2-l_1'^2-l_2'^2}=\frac{1}{(l_1^2-l_1'^2)(l_2^2-l_2'^2)}. \quad (41)$$

We also impose total momentum conservation in the matrix elements of v, which leads to the conditions

$$l_1'-l_1=l_2'-l_2 \quad \text{in } \Delta_2,$$
$$l_1'-l_1=l_2-l_2' \quad \text{in } \Delta_3. \quad (42)$$

Again the momenta l_1, l_1' and l_2, l_2' must lie close to their values at the Fermi surface. Using this simplifying feature, we find

$$\Delta_2+\Delta_3=-\frac{M^2L^4}{2\pi^4\hbar^2}\sum_{m_1n_1,m_2n_2}\frac{m_1m_2}{\Lambda_1\Lambda_2}\times[\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle+\langle\mathbf{p}_1-\mathbf{p}_2|v|\mathbf{p}_1-\mathbf{p}_2\rangle]_{p_1=p_2=p_F}\times\sum_{l=0}^{\Lambda_1}\sum_{q=1}^{\Lambda_1-l_1+1}\sum_{l_2=\Lambda_2-q+1}^{\Lambda_2}\frac{1}{q^4}, \quad (43)$$

where once again the symmetry of the contributions from positive and negative l has been taken into account. The vector $-\mathbf{p}_2$ is the vector $\mathbf{p}_2$ with only its l component reversed. This reversal arises from the conditions of Eq. (42) which relate l_1', l_1 and l_2', l_2 with opposite sign for Δ_2 and Δ_3. The sum over l_1, l_2, and q again gives $\pi^2/6$. The remaining sums must be done so that the vector $\mathbf{p}$ lies on the positive-l hemisphere of the Fermi surface. The sums are thus most conveniently done as surface integrals, making the coordinate transformation

$$m_1=F\cos\theta_1, \qquad m_2=F\cos\theta_2,$$
$$n_1=F\sin\theta_1\cos\phi_1, \quad n_2=F\sin\theta_2\cos\phi_2,$$
$$dm_1dn_1=F^2\sin\theta_1\sin\phi_1 d\Omega_1.$$

The two terms from Δ_2 and Δ_3 may now be combined and the $d\Omega_2$ integral extended over the entire Fermi sphere. The $d\Omega_1$ integral may similarly be extended if we divide by two, since its contribution is symmetric. The result is

$$\Delta_2+\Delta_3=-\frac{M^2L^4F^4}{24\pi^2\hbar^2}\int_{4\pi}d\Omega_1\int_{4\pi}d\Omega_2\cos\theta_1\cos\theta_2\times\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle_{p_1=p_2=p_F}. \quad (44)$$

To evaluate this integral, we use the fact that at the Fermi surface, $\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle$ can only depend on the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and we make the spherical harmonic decomposition

$$\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle=\textstyle\sum_l v_l(p_1,p_2)P_l(\cos\theta_{12}). \quad (45)$$

Using the relation

$$\cos\theta_{12}=\cos\theta_1\cos\theta_2+\cos\phi_{12}\sin\theta_1\sin\theta_2, \quad (46)$$

the integral for $\Delta_2+\Delta_3$ gives

$$\Delta_2+\Delta_3=-\frac{2M^2L^4}{27\hbar^2}F^4v_1(p_F,p_F). \quad (47)$$

The result may be cast into a convenient form if we use the fact that $p_F=(2\pi\hbar F/L)$ and the definitions of v_1 and $\mathcal{I}_{\text{rigid}}$. The result is

$$\Delta_2+\Delta_3=-\mathcal{I}_{\text{rigid}}Mp_F\left(\frac{L}{2\pi\hbar}\right)^3 2\pi\int_{-1}^{1}d(\cos\theta_{12})\times\langle\mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle|_{p_1=p_2=p_F}\cos\theta_{12}. \quad (48)$$

In the Appendix it is shown that

$$M p_F \left(\frac{L}{2\pi\hbar}\right)^3 2\pi \int_{-1}^{1} d(\cos\theta_{12}) \times \langle \mathbf{p}_1\mathbf{p}_2 | v | \mathbf{p}_1\mathbf{p}_2 \rangle_{p_1=p_2=p_F} \cos\theta_{12} = -\frac{M}{p_F}\left(\frac{dV}{dp_1}\right)_{p_1=p_F}, \quad (49)$$

so that $\Delta_2+\Delta_3$ exactly cancels Δ_1.

This result shows that the interaction effects appearing through Δ_1 [Eq. (36)] as an effective-mass correction are to first order in the particle-particle interaction cancelled by Δ_2 and Δ_3. These terms may be interpreted as giving the alteration of the matrix elements of L_z due to the interactions.

IV. DISCUSSION

We have seen in Sec. II that a noninteracting Fermi gas of many particles in a box with periodic boundaries has the rigid moment on the "cranking" model, and that this may be interpreted as coming from a rigid rotation of the system in spite of apparent surface flow. We feel that this result must hold for any large noninteracting system, independent of the boundary conditions, but unfortunately we know of no general proof. It has been suggested[6] that this result follows simply from van Leeuwen's theorem on the vanishing of the magnetic susceptibility of a dynamical system, or in particular from the absence of diamagnetism for electrons.[15] However, these results hold only for Boltzmann distributions, and are explicitly violated in the quantum case. Further, it is possible that even with classical statistics the presence of collisions, plus some constraint like fixed angular momentum, will invalidate the general theorem. Lastly, although the analogy between the rotating nuclear system and the magnetic one is suggestive, the exact correspondence is somewhat obscure and thus we consider the matter of general proof as still open.

The status of the proof in the presence of collisions, or interactions, is even less satisfactory. Here we have been able to show that in the large Fermi gas with periodic boundaries, the interactions cause no shift in the moment in first order. In this case the problem exists both of the effect of the boundary conditions on the result and of investigating the higher orders in the interaction strength. Whereas the boundaries probably have no effect, it does not seem obvious what one should expect from the higher orders in the interaction. Unfortunately our method does not lend itself simply to extension to even the next order.

Even without generalization, the result of Sec. III is interesting since it shows that no simple change in level structure that has a first-order part, like the effective mass, can change the moment of inertia. The apparent change from the effective mass is just cancelled by terms which may be thought of as changes in the angular momentum operator as a result of the interaction. One might then ask how to account for the experimentally observed departures from the rigid moment in finite nuclei. For small values of the deformation these departures are quite large and are probably to be associated with failure of the adiabatic approximation and hence to failure of the "cranking" model. It is, of course, possible that for known deformations the "cranking" model is never valid, but we shall assume it is for large deformations. If we assume the validity of the "cranking" model, the departures from the rigid moment may be thought of as having two sources. The first source is finite-size effects which may both change the value of the noninteracting moment and change the effect of the interactions. The second source is effects independent of the nuclear size such as we have investigated here. Although there is cancellation of the interaction effects in first order, and it has been conjectured[16] that this cancellation persists to all finite orders of perturbation theory, the moment of inertia may still be affected if perturbation theory for the interaction does not converge, leading to anomalous level changes of the type considered by Migdal[9] and Belyaev.[8]

APPENDIX

We wish to prove Eq. (49). To do this, first consider the single-particle potential defined by Eq. (33). Going to an integral over $\mathbf{p}_2$, and changing variable to

$$\mathbf{s}=(\mathbf{p}_1-\mathbf{p}_2)/2, \quad (A1)$$

we find

$$V(p_1)=\left(\frac{L}{2\pi\hbar}\right)^3 8\int_{|\mathbf{p}_1-2\mathbf{s}|\leq p_F} d\mathbf{s}\, v(s), \quad (A2)$$

where we have written for simplicity

$$v(s)=\langle \mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle. \quad (A3)$$

The angular integral in Eq. (A2) is easily done, giving

$$V(p_1)=\left(\frac{L}{2\pi\hbar}\right)^3 16\pi\left[2\int_0^{\frac{1}{2}(p_F-p_1)} s^2 ds\, v(s) + \int_{\frac{1}{2}(p_F-p_1)}^{\frac{1}{2}(p_F+p_1)} s^2 v(s) ds \left(1+\frac{p_F^2-p_1^2-4s^2}{4p_1 s}\right)\right]. \quad (A4)$$

The derivative of $V(p_1)$ at $p_1=p_F$ is then

$$\frac{dV}{dp_F}=\left(\frac{L}{2\pi\hbar}\right)^3 8\pi\int_0^{p_F} s v(s) ds\left(\frac{2s^2-p_F^2}{p_F^2}\right). \quad (A5)$$

[15] J. H. Van Vleck, *Theory of Electric and Magnetic Susceptibilities* (Oxford University Press, Oxford, 1932), pp. 94–104.

[16] A. Bohr and A. B. Migdal (private communication).

Next consider the expression

$$\int_{-1}^{1} d(\cos\theta_{12})\langle \mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_1\mathbf{p}_2\rangle|_{p_1=p_2=p_F}\cos\theta_{12}. \qquad (A6)$$

This we evaluate by introducing the variable

$$s=|\mathbf{p}_1-\mathbf{p}_2|/2, \qquad (A7)$$

so that at $p_1=p_2=p_F$,

$$4sds=-p_F^2d(\cos\theta_{12}), \qquad (A8)$$

and

$$\cos\theta_{12}=(p_F^2-2s^2)/p_F^2. \qquad (A9)$$

Equation (A6) now becomes

$$\frac{4}{p_F^2}\int_0^{p_F} sds\, v(s)\left(\frac{p_F^2-2s^2}{p_F^2}\right). \qquad (A10)$$

Combining Eqs. (A10) and (A5), we obtain the desired result.

PHYSICAL REVIEW VOLUME 115, NUMBER 4 AUGUST 15, 1959

Proposed Direct Test of the Uncertainty Principle

P. R. RYASON
California Research Corporation, Richmond, California

(Received May 15, 1957: revised manuscript received April 28, 1959)

The behavior of single particles is a central issue in the interpretation of the quantum theory. Yet, the observation of single particles under well-controlled conditions has been difficult. Field-ion and field-emission microscopy permit the ready observation of single particles. It is proposed to test the relation $\Delta E\Delta t\approx\hbar$ by the pulsed-field desorption of single particles from the tip of a field-emission microscope. The conditions for such an experiment are briefly discussed.

ONE of the most frequently cited illustrations of the uncertainty principle is the simultaneous measurement of position and momentum of an electron by scattering a photon from it. There are no methods presently available to measure the position and momentum of a *single* electron in separate experiments which are sufficiently precise to evaluate the errors in these quantities measured simultaneously. Similarly, determinations of the wavelength and position of a single photon are measurements not presently feasible. The wavelength measurement in particular can probably never be sufficiently accurate. The most accurate measurements of wavelength involve interference phenomena, not observable with a single photon. Suffice to say, the observation of single particles under well-controlled conditions is very difficult. Yet, the behavior of *single* particles under assumed well-controlled conditions is precisely the central issue in the interpretation of the quantum theory.[1] The "Gedanken" experiments proposed do not constitute tests in that they cannot be performed experimentally and more often than not multiparticle results are used to derive the uncertainties in various quantities. For example, the analysis of the photon-electron experiment uses an expression for the uncertainty in position which assumes an interference pattern. Such cannot occur for a single photon. This note proposes an experiment which permits a close approximation to the ideal one-particle experiment.

Müller has described the field desorption[2] of adsorbed atoms from fine tungsten tips in a field-ion microscope. It is possible to pulse the desorption field and remove adatoms. The proposed experiment would test the relation $\Delta E\Delta t\approx\hbar$ by the pulsed-field desorption of adatoms on a metal tip at very low temperatures. The adatom must be adsorbed as an ion, e.g., barium on tungsten. In brief, the experiment would involve evaporating a few atoms onto the tip of the field-ion microscope, demonstrating their presence by either field-ion microscopy[3–5] or pulsed-field emission microscopy,[6] and then field-desorbing these adatoms in a short pulse, and examining the tip again. The desorption field would be a single pulse, initially of long duration to determine the desorption energy Q_0. For the test of the uncertainty principle, the pulse width would be reduced to 10^{-12} second. The applied potential would have to be known to 1 part in 10^5. One of the possible modes of operation is indicated schematically in Fig. 1. In this experiment, the applied potential has been reduced slightly, resulting in a small potential barrier ΔE_{exp}. If the maximum uncertainty in the time of desorption is 10^{-12} second, then the particle should fluctuate in energy at least by an amount $\Delta E=6.3\times10^{-4}$ ev. The applied potential is adjusted until ΔE_{exp} is equal to this value. If the uncertainty principle holds, the particle should

[1] D. Bohm, *Casuality and Chance in Modern Physics* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1957).

[2] E. W. Müller, Phys. Rev. **102**, 618 (1956).
[3] E. W. Müller, J. Appl. Phys. **27**, 474 (1956).
[4] E. W. Müller and K. Bahadur, Phys. Rev. **102**, 624 (1956).
[5] E. W. Müller, J. Appl. Phys. **28**, 1 (1957).
[6] W. P. Dyke and J. P. Barbour, J. Appl. Phys. **27**, 356 (1956).