Structure of the Electromagnetic Field

BRUNO BERTOTTI
Institute for Advanced Study, Princeton, New Jersey
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The structure of a curved space-time in which a source-free electromagnetic field is present is studied according to the "already unified field theory" of Misner and Wheeler, and the geometrical meaning of the differential conditions imposed on the Ricci tensor is established. Four independent equations equivalent to the algebraic conditions are stated, and the case of a null electromagnetic field is also clarified.

1. INTRODUCTION

THE wealth and the physical possibilities which a Riemannian geometry can allow have been once more stressed in a recent paper by Misner and Wheeler[1,2]: They showed that electromagnetism is contained in, and can be deduced from, the geometry itself of a particular kind of Riemannian manifold. The charge can in fact be fully understood in terms of a topology of space-time different from the usual one; while the electromagnetic field can be extracted—and computed—from the metric. The conditions required for this involve the fundamental tensor only and fall naturally into two groups. If[3]

$$R_{\mu\nu}\equiv R^{\rho}{}_{\mu\nu\rho}\equiv\Gamma^{\rho}{}_{\mu\nu,\rho}-\Gamma^{\rho}{}_{\mu\rho,\nu}+\Gamma^{\rho}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\mu\nu}-\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} \quad (1)$$

is the Ricci tensor, the second-order conditions read

$$R_{\mu}{}^{\mu}=0, \quad (2)$$

$$4R_{\mu}{}^{\rho}R_{\rho\nu}=g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma}\neq 0. \quad (3)$$

To these we must also add a qualitative demand, that the quadratic form

$$R_{\mu\nu}l^{\mu}l^{\nu}$$

be positive definite for any time-like l^{μ}. The fourth-order conditions demand just that the vector defined by

$$\alpha_{\mu}\equiv(-g)^{\frac{1}{2}}\epsilon_{\mu\nu\rho\sigma}R^{\nu\tau/\rho}R_{\tau}{}^{\sigma}/R_{\varphi\psi}R^{\varphi\psi} \quad (4)$$

should be the gradient of a scalar α, namely

$$\alpha_{\mu,\nu}-\alpha_{\nu,\mu}=0. \quad (5)$$

For any such manifold a bivector field $f_{\mu\nu}$ can be determined uniquely (up to a constant of integration) which satisfies the source-free Maxwell's equations. The case

$$R_{\mu\nu}R^{\mu\nu}=0 \quad (6)$$

corresponds to a null electromagnetic field, for which the two invariants,

$$I_1(f)=\mathbf{E}^2-\mathbf{H}^2, \quad (7)$$

$$I_2(f)=\mathbf{E}\cdot\mathbf{H}, \quad (8)$$

both vanish; but then Eq. (4) loses its meaning. The null case is not completely understood so far.

It is of great importance to assess the structure of this type of Riemannian manifold and to understand it in terms of simpler and fewer concepts which in some way reduce the number of choices necessary to arrive at it. We propose in this paper (a) to add a small contribution to our knowledge of the algebraic equations by expressing them in a form in which they are independent from one another; (b) to find a simple geometrical interpretation of the curl condition (5) in terms of a Pfaff's problem; (c) to clarify the case of a null field.

2. ALGEBRAIC CONDITIONS

It is known[4] that the algebraic conditions (2) and (3) are equivalent to saying that the matrix R (by which we mean $R_{\mu}{}^{\nu}$) has (in the non-null case) a spectral decomposition of the type

$$R=\rho(p-q),\quad \rho>0 \quad (9)$$

where p and q are the projection operators on the two orthogonal "blades" into which the electromagnetic field can be decomposed (and ρ is a scalar). They satisfy

$$p^2=p;\quad q^2=q;\quad pq=0;\quad p+q=1. \quad (10)$$

In the null case we have, instead,

$$R=p_{\xi}, \quad (11)$$

p_{ξ} being the projection operator on a null vector ξ. We want to find the independent conditions required for a matrix R to fall into one of these two types. First of all, in order that its eigenvalues be $(\rho,\rho,-\rho,-\rho)$ (with $\rho=0$ in the null case), it is necessary and sufficient that its secular equation be of the form

$$(\lambda^2-\rho^2)^2=0.$$

By means of standard formulas of matrix theory[5] it can be easily shown that this is the case when the fol-

[1] C. W. Misner and J. A. Wheeler, Ann Phys. 2, 525 (1957).

[2] Compare also G. Y. Rainich, Trans. Am. Math. Soc. 27, 106 (1925).

[3] Greek indices μ, ν, ρ, σ, etc. range from 0 to 3. $\epsilon_{\mu\nu\rho\sigma}$ is the alternating tensor whose components are 0 and ± 1. A comma and a stroke signify, respectively, ordinary and covariant differentiation. $\eta^{\mu\nu}$ is the flat-space metric, of the type $(1,-1,-1,-1)$.

[4] As far as the structure of the electromagnetic field *at one point* is concerned, compare J. L. Synge, *Relativity: The Special Theory* (North-Holland Publishing Company, Amsterdam, 1956), p. 326.

[5] The quickest way is to follow the method proposed by H. E. Fettis, Quart. Appl. Math. 8, 206 (1950).

lowing conditions are fulfilled:

$$\mathrm{Tr}\,R=\mathrm{Tr}\,R^3=0,$$
$$\mathrm{Tr}\,R^4=\tfrac{1}{4}(\mathrm{Tr}\,R^2)^2 \quad (=4\rho^2). \tag{12}$$

But this is not enough; we must consider the possibility of a degenerate matrix in which the eigenvectors do not span the entire space. Let us envisage first the non-null case and suppose that, starting from a general matrix Q (with $Q_{\mu\nu}=Q_{\nu\mu}$) with distinct eigenvalues, we go over into R with a continuous transition. Since the eigenvectors of Q are orthogonal (in the Minkowskian sense), the only thing that may happen—and we want to avoid—is that a space-like and a time-like eigenvector collapse into a null eigenvector, always remaining orthogonal to one another and to the other two. In a particular frame of reference we must therefore exclude the case in which there are *only* three eigenvectors, say (1,1,0,0), (0,0,1,0), and (0,0,0,1), with respective eigenvalues ρ, $-\rho$ and $-\rho$. It can be seen by a direct computation that this amounts to having

$$R=\begin{vmatrix} \sigma & \rho-\sigma & 0 & 0 \\ \sigma-\rho & 2\rho-\sigma & 0 & 0 \\ 0 & 0 & -\rho & 0 \\ 0 & 0 & 0 & -\rho \end{vmatrix}, \tag{13}$$

with $\sigma\neq\rho$. Hence we must demand instead

$$\rho=\sigma. \tag{14}$$

In the null case we do allow coalescing eigenvectors, but we require at the same time $\rho=0$. Thus, in general the condition

$$\rho(\rho-\sigma)=0, \tag{15}$$

together with (12) suffice to insure (9). Although (15) has been deduced with a particular choice of the basis vectors, its meaning is patently invariant.

3. CURL CONDITION

The fourth-order conditions (5) can be written in a much simpler way[6] if the metric tensor is expressed through four orthogonal unit vectors $\lambda^{(\alpha)}$ [7] lying in the two blades determined by the Ricci tensor $R_{\mu\nu}$ (see Fig. 1):

$$g_{\mu\nu}=\eta_{\alpha\beta}\lambda^{(\alpha)}{}_\mu\lambda^{(\beta)}{}_\nu, \tag{16}$$

with

$$R\lambda^{(\alpha)}=\pm\rho\lambda^{(\alpha)}. \tag{17}$$

The essential quantities one has to deal with are the structure coefficients

$$c_{\alpha\beta\gamma}=[\lambda^{(\alpha)}{}_{\mu,\nu}-\lambda^{(\alpha)}{}_{\nu,\mu}]\lambda^{(\beta)\mu}\lambda^{(\gamma)\nu}=-c_{\alpha\gamma\beta}, \tag{18}$$

which determine the commutation relations between

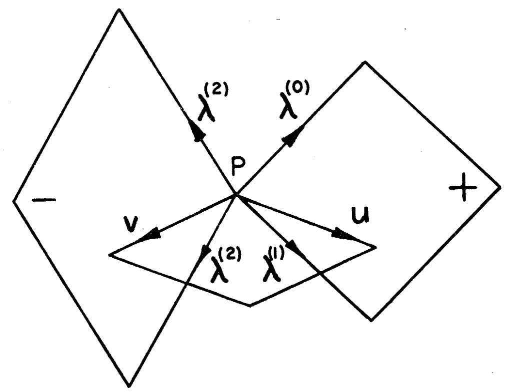

FIG. 1. The two-blade structure of the electromagnetic field.

the differential operators

$$\partial_\alpha=\lambda^{(\alpha)}\cdot\nabla:$$
$$[\partial_\alpha,\partial_\beta]=\eta^{\gamma\delta}c^{\gamma}{}_{\alpha\beta}\partial_\delta. \tag{19}$$

It is interesting to remark that out of the 24 scalars $c_{\alpha\beta\gamma}$ only *four* enter in our equation, namely

$$u_0\equiv c_{023},\quad v_2\equiv c_{210},$$
$$u_1\equiv c_{123},\quad v_3\equiv c_{310}; \tag{20}$$

the intrinsic components of the vector α_μ are just, in fact,

$$\alpha_0=-u_1,\quad \alpha_2=-v_3,$$
$$\alpha_1=-u_0,\quad \alpha_3=v_2. \tag{21}$$

To find the geometrical meaning of these equations, let us forget for the moment the orthogonality properties of the λ's and think of the *affine* properties of our two-bladed structure. The most important quantities one can construct when considering a set of blades [the − blades, say, spanned by all vectors orthogonal to $\lambda^{(0)}$ and $\lambda^{(1)}$] are the scalar densities

$$\mathfrak{u}_0=\lambda^{(0)}{}_{\mu,\nu}\lambda^{(0)}{}_\rho\lambda^{(1)}{}_\sigma\epsilon^{\mu\nu\rho\sigma},$$
$$\mathfrak{u}_1=\lambda^{(1)}{}_{\mu,\nu}\lambda^{(0)}{}_\rho\lambda^{(1)}{}_\sigma\epsilon^{\mu\nu\rho\sigma}. \tag{22}$$

Their vanishing is the necessary and sufficient condition in order that the − blades be *integrable*[8]; i.e., that, when connected, they form a family of two-dimensional surfaces. In general $\mathfrak{u}_0$ and $\mathfrak{u}_1$ describe how, so to speak, the blade rotates around itself on going from one point to another. It can be easily proved by direct substitution that when the basis vectors $\lambda^{(0)}$ and $\lambda^{(1)}$ are replaced by a linear combination thereof,

$$\lambda'^{(0)}=a\lambda^{(0)}+b\lambda^{(1)},$$
$$\lambda'^{(1)}=c\lambda^{(0)}+d\lambda^{(1)},\quad (D=ad-bc\neq0)$$

[6] G. Rosen, Phys. Rev. **114**, 1179 (1959). The geometrical meaning of the condition α=const was partially illustrated by E. T. Whittaker, Proc. Roy. Soc. Edinburgh **42**, 1 (1921).

[7] The first letters of the Greek alphabet (α, β, γ $\cdots$) only number the "legs" of the "vierbein."

[8] Compare, e.g., E. Cartan, *Les systèmes différentielles extérieurs et leurs applications géométriques* (Paris, 1945), Chap. III.

the $\mathfrak{u}$'s undergo the transformation

$$\mathfrak{u}_0 \to \mathfrak{u}_0' = D(a\mathfrak{u}_0 + b\mathfrak{u}_1),$$
$$\mathfrak{u}_1 \to \mathfrak{u}_1' = D(c\mathfrak{u}_0 + d\mathfrak{u}_1).$$

For this reason $\mathfrak{u}_0$ and $\mathfrak{u}_1$ determine a direction on the + blade; if in fact for a definite choice of $\boldsymbol{\lambda}^{(0)}$ and $\boldsymbol{\lambda}^{(1)}$ we define the vector $\mathbf{u}$ by taking its intrinsic components $u_\alpha = \mathbf{u} \cdot \boldsymbol{\lambda}^{(\alpha)}$ to be proportional to $(\mathfrak{u}_0, \mathfrak{u}_1, 0, 0)$, the same proportionality will hold for any other choice. In the same way the two scalar densities,

$$\mathfrak{v}_2 = \lambda^{(2)}{}_{\mu,\nu}\lambda^{(2)}{}_\rho\lambda^{(3)}{}_\sigma\epsilon^{\mu\nu\rho\sigma},$$
$$\mathfrak{v}_3 = \lambda^{(3)}{}_{\mu,\nu}\lambda^{(2)}{}_\rho\lambda^{(3)}{}_\sigma\epsilon^{\mu\nu\rho\sigma}, \tag{23}$$

define (up to a factor) on the − blade a controvariant vector $\mathbf{v}$ through its intrinsic components

$$v_\alpha = \mathbf{v} \cdot \boldsymbol{\lambda}^{(\alpha)} \propto (0, 0, \mathfrak{v}_2, \mathfrak{v}_3).$$

We are then led naturally to consider another blade, the one defined by $\mathbf{u}$ and $\mathbf{v}$ (0-blade) and to look for its integrability conditions. It turns out that, as a consequence of Eqs. (20), the 0-blades fulfill a Pfaff's condition *less* restrictive than the full integrability: if we go from one event to another following the 0-blade, we always stay on a *three*-dimensional hypersurface, on which α is constant.

Let us first notice that

$$\lambda^{(2)\mu}\lambda^{(3)\nu} - \lambda^{(2)\nu}\lambda^{(3)\mu} = (-g)^{-\frac{1}{2}}\epsilon^{\mu\nu\rho\sigma}\lambda^{(0)}{}_\rho\lambda^{(1)}{}_\sigma, \tag{24}$$

and

$$\lambda^{(0)\mu}\lambda^{(1)\nu} - \lambda^{(0)\nu}\lambda^{(1)\mu} = -(-g)^{-\frac{1}{2}}\epsilon^{\mu\nu\rho\sigma}\lambda^{(2)}{}_\rho\lambda^{(3)}{}_\sigma, \tag{25}$$

as one can check easily in the frame of reference in which the matrix $\|\lambda^{(\alpha)}{}_\mu\|$ is diagonal at a given point. Hence we can write, completing the definition of the scalars u_α and v_α consistently with (20) and (18),

$$\mathfrak{u}_0 = (-g)^{\frac{1}{2}}u_0, \quad \mathfrak{v}_0 = (-g)^{\frac{1}{2}}v_0,$$
$$\mathfrak{u}_1 = (-g)^{\frac{1}{2}}u_1, \quad \mathfrak{v}_1 = (-g)^{\frac{1}{2}}v_1. \tag{26}$$

Our Eqs. (20) can then be written in a more concise form as follows:

$$\alpha_{+\beta} = \epsilon_{+\beta\gamma}\eta_+{}^{\gamma\delta}u_{+\delta},$$
$$\alpha_{-\beta} = \epsilon_{-\beta\gamma}\eta_-{}^{\gamma\delta}v_{-\delta}. \tag{27}$$

The suffixes + and − indicate that only the components in the corresponding blade are taken into account; $\epsilon_{+\beta\gamma}$ and $\epsilon_{-\beta\gamma}$ are the Levi-Civita symbols (values ± 1 and 0) in two dimensions.

The meaning of (27) can now be read directly: To get the vector gradα, one must rotate in each blade the two vectors $\mathbf{u}$ and $\mathbf{v}$ by a right angle (a hyperbolic right angle for the time-like blade); gradα, aside from a constant factor, is the sum of the vectors obtained in this way. It is therefore clear that gradα is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and hence to any linear combination thereof, which is just the same as saying that the 0-blade is contained in the hypersurface α=const. Equations (27) imply therefore a property which has nothing to do with the fact that the $\boldsymbol{\lambda}$'s in each blade are orthogonal: it is an *affine* property of the two-blade structure. But this alone is not sufficient to determine the vector gradα; it says only that it must be orthogonal to the 0-blade, leaving ∞^2 free choices: Eqs. (27) supply also the missing identification.

4. NULL FIELD

A null field is a particular case of the simple[9] field ("extremal," or "essentially electric" according to Misner's terminology[1]), in which only the invariant $\mathbf{E}\cdot\mathbf{H}$ vanishes. A simple field defines at any point two orthogonal integrable blades, as one can see by noticing that this corresponds to a constant α (0 or $\pi/2$). This statement holds even for the null case, albeit then α is not defined. The first quadruplet of Maxwell's equations,

$$f_{\mu\nu,\rho} + f_{\rho\mu,\nu} + f_{\nu\rho,\mu} = 0, \tag{28}$$

for the field

$$f_{\mu\nu} = \xi_\mu\eta_\nu - \xi_\nu\eta_\mu \tag{29}$$

(where ξ_μ and η_μ are any two vector fields), implies in fact that

$$\xi_\sigma f_{\mu\nu,\rho}\epsilon^{\mu\nu\rho\sigma} = \xi_{\mu,\rho}\xi_\sigma\eta_\nu\epsilon^{\mu\nu\rho\sigma} = 0,$$
$$\eta_\sigma f_{\mu\nu,\rho}\epsilon^{\mu\nu\rho\sigma} = \eta_{\mu,\rho}\xi_\nu\eta_\sigma\epsilon^{\mu\nu\rho\sigma} = 0, \tag{30}$$

which are the integrability conditions for the $\boldsymbol{\xi}$-$\boldsymbol{\eta}$ blade [see (22)]. The other quadruplet yields the same condition for the other blade.

The null case occurs when the two orthogonal blades have a null vector—say ξ_μ—in common and lie therefore in a hyperplane tangent to the light-cone. The conservation law for the energy-momentum tensor, now of the form

$$R_{\mu\nu} = \xi_\mu\xi_\nu, \quad (\xi^\mu\xi_\mu = 0) \tag{31}$$

means simply that ξ_μ is a field of null geodesic lines,[10] for which the vector $\xi^\mu{}_{|\nu}\xi^\nu$ is parallel to ξ^μ. Moreover the fourth-order condition (5) becomes empty in this case. The definition (4) of α_μ is in fact equivalent to Eq. (74) of reference 1 which, not containing any reference to α, can be established even in the null case; but from (31) it is easily confirmed to amount to 0=0. The energy-momentum tensor bears no trace left of the position of the blades, except for the null vector ξ_μ about which they "swivel;" but Maxwell's equations contain more than that. In the null case, therefore, the field cannot be recovered from the Ricci tensor, except when the field is not everywhere null in a neighborhood of the point we are considering, since then continuity arguments complete uniquely the solution.

[9] For the definition and properties of simple bivectors, compare J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), p. 35.

[10] See also L. Mariot, Compt. rend. **238**, 2055 (1954).

It is therefore of importance to study the permanence properties of the null field, that is, to find under which conditions Maxwell's equations will keep null a field initially null on a space-like hypersurface. Mariot[11] has in this connection proved the interesting theorem that this is so if the ξ-field is integrable, i.e., if it is orthogonal to a family of hypersurfaces. The geometrical meaning of the conditions imposed on the actual geometry by the existence of a null field is, however, not yet clear.[12]

[11] L. Mariot, thesis, Paris, 1957 (unpublished).

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[12] *Note added in proof.*—Professor V. Hlavatý has kindly informed me of a forthcoming paper of his on this problem.

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Some Consequences of Symmetries of the Strong Interactions*

G. Feinberg and R. E. Behrends
Brookhaven National Laboratory, Upton, New York
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The consequences for the electromagnetic interactions of strange particles of some proposed symmetries of the strong interactions are discussed. A minimal electromagnetic coupling is assumed. It is shown that if the interactions with both π and K mesons of the nucleon and cascade particle are identical, if one neglects the $n-\Xi$ mass difference, and if charge independence is satisfied, then the electromagnetic form factors of the Λ^0 and Σ^0, and in particular their magnetic moments, vanish. This is independent of any assumed relation between the Λ and Σ interactions, such as global symmetry. It is also shown that the same conditions, together with charge conjugation invariance, imply a generalized form of the Pais-Jost-Pugh theorem about processes involving only meson or photon external lines. If there is still more symmetry for the strong interactions, as in the case where the π interactions are globally symmetric and the K interactions also have this doublet structure, it is shown that the K^0 acts as a completely neutral particle insofar as electromagnetic interactions are concerned.

I. INTRODUCTION

There have been a number of proposals that the strong interactions of baryons with π mesons and K mesons possess more internal symmetry than is implied by invariance under conventional isotopic spin rotations.[1,2] In this paper we will examine some of the consequences of these proposed new symmetries for the interaction of these particles with electromagnetic fields. Some of these results have been stated before under more restrictive assumptions than we will make here, and part of our purpose is to indicate how strong the symmetry required to derive a given result may be.

We shall assume the "principle of minimal electromagnetic coupling,"[3] i.e., that the fundamental interaction of the electromagnetic field is only with the current 4-vector of the charged particles. All other interactions of the electromagnetic field are assumed to arise through the combined effects of the strong interactions and these minimal interactions. There is some evidence for this assumption in π-nucleon physics, where the program of calculating the nucleon magnetic moments and form factors without introducing nonminimal terms has been moderately successful.[4] In strange-particle physics, there is very little evidence about this, as the electromagnetic properties of the strange particles, other than their charges, have not been measured. There is, however, the original argument of Gell-Mann and Pais[3] that the minimal principle is sufficient to eliminate strangeness-violating electromagnetic decays like $\Lambda^0 \rightarrow n+\gamma$. In this paper we use the minimal expression for the electromagnetic interaction to determine the transformation properties of the electromagnetic interaction under certain permutations of the baryons and mesons. Any terms which transform the same way could be added to the minimal interaction without changing the conclusions.

In Sec. II of the paper, we discuss the electromagnetic vertex operators, and in particular the magnetic moments, of the baryons. In Sec. III, a theorem proven by Pugh[5] is generalized and proven under different hypotheses than he used. In Sec. IV the electromagnetic vertex of the neutral K mesons is discussed.

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[1] M. Gell-Mann, Phys. Rev. **106**, 1296 (1957); A. Pais, Phys. Rev. **110**, 574 (1958); J. Schwinger, Ann. Phys. **2**, 407 (1957).

[2] G. Feinberg and F. Gürsey, Phys. Rev. **114**, 1153 (1959). This paper will be referred to as I. J. Sakurai, Phys. Rev. **113**, 1679 (1959).

[3] M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1956); A. Pais, Phys. Rev. **86**, 663 (1952).

[4] Federbush, Goldberger, and Treiman, Phys. Rev. **112**, 642 (1958).

[5] R. Pugh, Phys. Rev. **109**, 989 (1958).