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Euclidean Quantum Electrodynamics

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Quantum electrodynamics is transcribed into a Euclidean metric. A review is presented of the quantum action-principle approach to quantization, with its automatic emphasis on the dynamical variables associated with the physical degrees of freedom. Green's functions of the radiation gauge are defined, and then characterized by differential equations and boundary conditions. These Green's functions are of direct physical significance but involve a distinguished time-like direction. A gauge transformation is then performed to eliminate this dependence, introducing thereby the Green's functions of the Lorentz gauge, which lack immediate physical interpretation. The latter functions are now primarily defined by differential equations and boundary conditions, and form the basis for the analytic extension which is the change from space-time to Euclidean metric. Some properties of anticommuting matrices are discussed in relation to this metric transformation. Real Euclidean Green's functions are defined by correspondence with the Lorentz gauge functions and the appropriate differential equations obtained. Invariance properties of the Euclidean functions are discussed. The individual Euclidean Green's functions are given an operator construction and then combined into a generating Green's functional which is interpreted as the wave function, in a canonical field representation, of a state characterized by the Euclidean action operator. Differential operator realizations and some other benefits of a canonical variable description are exhibited.

INTRODUCTION

In a recent note[1] the author has remarked on the possibility of establishing a correspondence between the quantum theory of fields in space-time, and a mathematical structure that employs a four-dimensional Euclidean coordinate manifold. In that note the simplifying fiction was adopted, for the purposes of exposition, that all field components are kinematically independent in the standard form of the action principle that produces first order differential field equations. While this is true of the only known kind of F.D. (Fermi-Dirac) field, with spin $\frac{1}{2}$, there is no B.E. (Bose-Einstein) field of this type. Accordingly, we must supply some assurance that the discussion applies to real systems, and the Maxwell field is naturally indicated as an example of more than routine interest. Thus the content of this paper is the detailed transcription of quantum electrodynamics into the Euclidean formulation.

THE ACTION PRINCIPLE

We shall need to review some aspects of the development of quantum electrodynamics from the action principle. (Perhaps one should record at this point the author's opinion that the currently popular indefinite-metric quantization of the electromagnetic field is unphysical and unnecessary.) The Lagrange function for the system of Maxwell and Dirac fields is

$$\mathcal{L}=-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}\psi\beta\gamma^\mu\partial_\mu\psi +\tfrac{1}{2}im\psi\beta\psi-\tfrac{1}{2}ie\psi\beta\gamma^\mu q\psi A_\mu,$$

where symmetrized or antisymmetrized multiplication is to be understood for B.E. and F.D. terms, respectively. The electric current operator formed from the Hermitian field ψ,

$$j^\mu=-\tfrac{1}{2}ie\psi\beta\gamma^\mu q\psi,$$

is a B.E. quantity in the latter context. The Dirac matrices γ^μ have the algebraic property

$$\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu},$$
$$g^{00}=-1,\quad g^{kl}=\delta^{kl},$$

while

$$\beta=i\gamma^0$$

is a real, antisymmetrical matrix and the matrices $\beta\gamma^\mu$ are symmetrical and imaginary. The imaginary, antisymmetrical charge matrix q possesses integer eigenvalues. This way of defining the charge characteristics of the ψ field is designed to emphasize that the coupling constant e is primarily a property of the electromagnetic field. Indeed, by a suitable scale change of the fields, e is removed from the coupling term to reappear only in the field strength term:

$$\tfrac{1}{4}e^2F^{\mu\nu}F_{\mu\nu}.$$

With the latter field definition, the gauge invariance of the theory refers to the purely kinematical transformation

$$\psi\rightarrow e^{iq\lambda}\psi,$$
$$A_\mu\rightarrow A_\mu+\partial_\mu\lambda.$$

The application of the stationary action principle extracts from the Lagrange function the field equations

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad \partial_\nu F^{\mu\nu}=j^\mu,$$
$$[\gamma^\mu(-i\partial_\mu-eqA_\mu)+m]\psi=0,$$

in which the product $A\psi$ is to be symmetrized, and the infinitesimal generator

$$G=\int_\sigma d\sigma_\mu[-F^{\mu\nu}\delta A_\nu+\tfrac{1}{2}\psi\beta\gamma^\mu\delta\psi]$$
$$=\int d\sigma[-F^{0k}\delta A_k+\tfrac{1}{2}i\psi\delta\psi],$$

* This paper was largely written during the summer of 1958 at the University of Wisconsin, Madison. The hospitality of the Department of Physics is gratefully acknowledged.

[1] J. Schwinger, Proc. Nat. Acad. Sci. U. S. 44, 956 (1958); and *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958).

where the second form refers to the special coordinate system that identifies n_μ, the unit time-like vector normal to the surface σ, with the time axis. To facilitate the interpretation of this generator it should be noted that, while all components of ψ obey explicit equations of motion, the only Maxwell field equations of that type are

$$\partial_0 A_k = \partial_k A_0 + F_{0k}, \quad \partial_0 F^{k0} = j^k - \partial_l F^{kl},$$

the remaining equations,

$$F_{kl} = \partial_k A_l - \partial_l A_k, \quad \partial_k F^{0k} = j^0,$$

containing no time derivatives. Thus the magnetic field (in this coordinate system) is not an independent dynamical quantity, and the longitudinal part of the electric field is an explicit function of the Dirac field through the charge density, as given by

$$^L F^{0k}(x) = -\partial^k \int (dx') \mathfrak{D}(x-x') j^0(x'),$$

where

$$-[\partial^2 + (n\partial)^2]\mathfrak{D}(x-x') = -\partial^k \partial_k \mathfrak{D}(x-x') = \delta(x-x').$$

From the longitudinal component of the equation of motion for A_k, we learn that

$$^L A_k(x) = \partial_k \Lambda(x),$$

$$A_0(x) = \int (dx') \mathfrak{D}(x-x') j_0(x') + \partial_0 \Lambda(x),$$

in which $\Lambda(x)$ must remain arbitrary to express the freedom of gauge transformations. Hence it is only the gauge-invariant transverse vector potential $^T A_k$, together with the complementary variables $-{}^T F^{0k}$, that qualify as the independent dynamical variables of the electromagnetic field.

Since a variation of the longitudinal part of the vector potential is a gauge transformation

$$\delta_\Lambda {}^L A_k = \partial_k \delta\Lambda,$$

we remove from $\delta\psi$ the corresponding infinitesimal gauge transformation

$$\delta_\Lambda \psi = ieq\delta\Lambda\psi,$$

and observe that these contributions to the infinitesimal generator are

$$\int d\sigma[-F^{0k}\partial_k \delta\Lambda - \tfrac{1}{2}e\psi q\psi\delta\Lambda] = \int d\sigma[\partial_k F^{0k} - j^0]\delta\Lambda = 0.$$

Thus the generator of independent variations of the dynamical variables at a given time is

$$G = \int d\sigma[-{}^T F^{0k}\delta^T A_k + \tfrac{1}{2}i\psi\delta\psi]$$

where the $\delta^T A_k$ and $\delta\psi$ possess the properties of commutativity or anticommutativity characteristic of the field statistics. Only one set of complementary variables for the Maxwell field is changed in this form and the interpretation of the generator[2] appropriate to that circumstance yields for the nonvanishing commutators

$$\delta(x^0 - x^{0\prime})\frac{1}{i}[{}^T A_k(x), -{}^T F^{0l}(x')]$$
$$= {}^T(\delta_k{}^l \delta(x-x')) = \delta_k{}^l \delta(x-x') - \partial_k \partial^{l\prime} \mathfrak{D}(x-x'),$$

while the treatment of all components of ψ on the same footing implies that

$$\delta(x^0 - x^{0\prime})\{\psi(x), \psi(x')\} = \delta(x-x').$$

Various additional commutation properties which can be derived from these fundamental ones will be stated as they are needed.

GREEN'S FUNCTIONS

We now proceed to the Green's functions of the Maxwell-Dirac system. Although the definitions and relevant properties of these functions are most naturally and compactly obtained from the device of external sources used in conjunction with the action principle, that procedure may not provide, for some, the conviction that accompanies the following explicit considerations. The Green's functions are symmetrical functions of B.E. coordinates, $\xi_1 \cdots \xi_\nu$, and antisymmetrical functions of F.D. coordinates, $x_1 \cdots x_{2n}$, defined as vacuum expectation values of time ordered products:

$$G_\pm(x_1 \cdots x_{2n}, \xi_1 \cdots \xi_\nu)$$
$$= \langle(\psi(x_1) \cdots \psi(x_{2n}) \bar{A}(\xi_1) \cdots \bar{A}(\xi_\nu))_\pm\rangle \epsilon_\pm(x).$$

The vacuum referred to here is the lowest-energy state of the fully interacting system[3]—no use is made of the device of adiabatic decoupling. Positive or negative time ordering is the assignment of multiplication order in conformity with the sequence of projections on a time-like vector n_μ, the positive sense of multiplication being from right to left. The quantities

$$\epsilon_+(x_1 \cdots x_{2n}) = \epsilon_-(x_{2n} \cdots x_1),$$

are antisymmetrical functions of the F.D. coordinates that assume the value $+1$ when the time-ordered arrangement coincides with the written order. The symbols

$$\bar{A}_\mu(\xi) = A_\mu(\xi) \pm in_\mu n_\nu \int (d\xi') \mathfrak{D}(\xi - \xi')\delta/\delta A_\nu(\xi')$$

are used to give a compact expression to a sum of terms.

[2] J. Schwinger, Phil. Mag. **44**, 1171 (1953).
[3] Thus there are no apparent difficulties of physical interpretation, such as P.A.M. Dirac [Quantum Mechanics (Clarendon Press, Oxford, 1958), fourth edition,] believes exist, which arise from an unwarranted physical identification of states that do not include the effects of interaction.

The simplest example is

$$G_\pm(\xi_1\xi_2)_{\mu_1\mu_2}=\langle(A_{\mu1}(\xi_1)A_{\mu2}(\xi_2))_\pm\rangle\pm in_\mu n_\nu\mathfrak{D}(\xi_1-\xi_2).$$

The operators $A_\mu(\xi)$ are defined, in the special coordinate system when the unit time-like vector n_μ coincides with the time axis, by assigning the value zero to the arbitrary gauge function $\Lambda(\xi)$. Thus A^0 is the instantaneous Coulomb potential of the charges, and A_k is entirely transverse, which properties characterize the *radiation gauge.* The property of transversality is described more generally by

$$\boldsymbol{\partial}_\mu A^\mu=\boldsymbol{\partial}A=0,$$
$$\boldsymbol{\partial}_\mu=\partial_\mu+n_\mu n^\nu\partial_\nu,$$

which also applies to the symbolic quantities $\bar{A}$, since

$$n\boldsymbol{\partial}=0.$$

Hence

$$(\boldsymbol{\partial})_{\xi_\alpha}G_\pm(x,\xi)=0,\quad \alpha=1\cdots\nu$$

distinguishes the Green's functions of the radiation gauge, $G^{(R)}$.

The two classes of Green's functions are complex conjugate,

$$G_-=G_+^*,$$

and, in the simplest case of two points, the real functions formed by addition of G_+ and G_- become definite functions on removing the alternating sign factor,

$$G_+^{(R)}(\xi_1\xi_2)+G_-^{(R)}(\xi_1\xi_2)=\langle\{A(\xi_1),A(\xi_2)\}\rangle\geq 0,$$
$$\epsilon_+(x_1x_2)[G_+^{(R)}(x_1x_2)+G_-^{(R)}(x_1x_2)]=\langle\{\psi(x_1),\psi(x_2)\}\rangle\geq 0.$$

The differential equations obeyed by the Green's functions combine the field equations and the commutation relations, the latter appearing to characterize the discontinuities encountered at equal times. In applying the Maxwell field equations the following commutators are required:

$$\begin{aligned}\delta(x^0-x^{0\prime})i[F^{\mu0}(x),A_\nu(x')]&=(\delta_\nu{}^\mu+n^\mu n_\nu)\delta(x-x')+\boldsymbol{\partial}^\mu\boldsymbol{\partial}_\nu\mathfrak{D}(x-x')\\&=[\delta_\nu{}^\mu+n^\mu n_\nu-\boldsymbol{\partial}^\mu\boldsymbol{\partial}_\nu(\boldsymbol{\partial}^2)^{-1}]\delta(x-x'),\end{aligned}$$

and

$$\delta(x^0-x^{0\prime})[F^{\mu0}(x),\psi(x')]=-\boldsymbol{\partial}^\mu\mathfrak{D}(x-x')eq\psi(x'),$$

which express the operator properties of the transverse and longitudinal electric field, respectively. The latter result has been derived from, and in turn implies, the charge density commutator

$$\delta(x^0-x^{0\prime})[j^0(x),\psi(x')]=-\delta(x-x')eq\psi(x').$$

The Maxwell differential equations for the Green's functions $G_+^{(R)}$ emerge as

$$\begin{aligned}(\partial\partial-\partial^2)_{\xi_1}G_+^{(R)}(x,\xi)&=(1-\boldsymbol{\partial}\partial(\boldsymbol{\partial}^2)^{-1})_{\xi_1}(1/i)\delta(\xi_1-\xi_2)G_+^{(R)}(x,\xi_3\cdots)+\cdots\\&\quad+(\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1})_{\xi_1}\sum_{a=1}^{2n}\delta(\xi_1-x_a)eq_aG_+^{(R)}(x,\xi_2\cdots)\\&\quad+\mathrm{tr}\tfrac{1}{2}i\beta\gamma eqG_+^{(R)}(x_1\cdots x_{2n}\xi_1\xi_1,\xi_2\cdots),\end{aligned}$$

where the dots following the first right-hand term signify the $\nu-2$ similar ones containing $\delta(\xi_1-\xi_\alpha)$, $\alpha=3\cdots\nu$. These terms are produced partly by the transverse field commutation properties, and partly by the differential operator contained in $\bar{A}$, with the aid of the relation

$$(\partial\partial-\partial^2)nn(\boldsymbol{\partial}^2)^{-1}=-nn+\boldsymbol{\partial}(\boldsymbol{\partial}-\partial)(\boldsymbol{\partial}^2)^{-1}.$$

The last term of this equation is the Green's function expression for

$$\langle(\psi(x_1)\cdots\psi(x_{2n})j(\xi_1)\bar{A}(\xi_2)\cdots)_+\rangle\epsilon_+(x),$$

which is obtained on writing

$$\begin{aligned}j(x)&=-\tfrac{1}{2}ie\psi(x)\beta\gamma q\psi(x)=\mathrm{tr}\tfrac{1}{2}i\beta\gamma eq\psi(x)\psi(x)\\&=\lim_{x'\to x}\mathrm{tr}\tfrac{1}{2}i\beta\gamma eq(\psi(x)\psi(x'))_+\epsilon_+(xx').\end{aligned}$$

The limiting approach of x' to x can be performed symmetrically from the past and the future.

In consequence of the conservation, and of the operator properties, of the current vector we have

$$\begin{aligned}(\partial_\mu)_{\xi_1}\,&\mathrm{tr}\tfrac{1}{2}i\beta\gamma^\mu eqG_+^{(R)}(\cdots x_{2n}\xi_1\xi_1,\xi_2\cdots)\\&=-\sum_{a=1}^{2n}\delta(\xi_1-x_a)eq_aG_+^{(R)}(x,\xi_2\cdots),\end{aligned}$$

which enables the Maxwell differential equations for the Green's functions to be presented as

$$\begin{aligned}(\partial\partial-\partial^2)_{\xi_1}&G_+^{(R)}(x,\xi)\\&=(1-\boldsymbol{\partial}\partial(\boldsymbol{\partial}^2)^{-1})_{\xi_1}[(1/i)\delta(\xi_1-\xi_2)G_+^{(R)}(x,\xi_3\cdots)\\&\qquad+\cdots+\mathrm{tr}\tfrac{1}{2}i\beta\gamma eqG_+^{(R)}(\cdots\xi_1\xi_1,\xi_2\cdots)].\end{aligned}$$

These equations still need to be supplemented by the condition of transversality characteristic of the radiation gauge. This requirement is explicitly satisfied in the following form of the Maxwell equations,

$$\begin{aligned}(-\partial^2)_{\xi_1}&G_+^{(R)}(x,\xi)\\&=[(1-\partial\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1})(1-\boldsymbol{\partial}\partial(\boldsymbol{\partial}^2)^{-1})]_{\xi_1}\\&\qquad\times[(1/i)\delta(\xi_1-\xi_2)G_+^{(R)}(x,\xi_3\cdots)\\&\qquad+\cdots+\mathrm{tr}\tfrac{1}{2}i\beta\gamma eqG_+^{(R)}(\cdots\xi_1\xi_1,\xi_2\cdots)].\end{aligned}$$

Complex conjugation produces the analogous differential equation for $G_-^{(R)}$,

$$\begin{aligned}(-\partial^2)_{\xi_1}&G_-^{(R)}(x,\xi)\\&=[(1-\partial\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1})(1-\boldsymbol{\partial}\partial(\boldsymbol{\partial}^2)^{-1})]_{\xi_1}\\&\qquad\times[i\delta(\xi_1-\xi_2)G_-^{(R)}(x,\xi_3\cdots)\\&\qquad+\cdots-\mathrm{tr}\tfrac{1}{2}i\beta\gamma eqG_-^{(R)}(\cdots\xi_1\xi_1,\xi_2\cdots)].\end{aligned}$$

The Dirac equations for the Green's functions are

$$\begin{aligned}(\beta\gamma(1/i)\partial+m\beta)_{x_1}G_\pm^{(R)}(x,\xi)-(\beta\gamma eq)_1G_\pm^{(R)}(x,x_1\xi_1\cdots)\\=\delta(x_1-x_2)G_\pm^{(R)}(x_3\cdots,\xi)-\cdots,\end{aligned}$$

where the dots indicate the $2n-2$ similar terms containing $(-1)^a\delta(x_1-x_a)$, $a=3\cdots 2n$. Involved here are the Dirac field commutation properties and the relation

$$\delta(x^0-x^{0\prime})[\psi(x),A_\mu(x')]=n_\mu n_\nu\mathfrak{D}(x-x')ieq\beta\gamma^\nu\psi(x).$$

The latter is the source of the additional terms necessary to convert

$$\langle(\tfrac{1}{2}\{A(x_1),\psi(x_1)\}\psi(x_2)\cdots\bar{A}(\xi_1)\cdots)_+\rangle\epsilon_+(x)$$

into the Green's function $G_+^{(R)}(x_1\cdots x_{2n},x_1\xi_1\cdots\xi_\nu)$.

The Green's functions are also characterized by their spectral properties. When all time coordinates are distinct, the Green's functions assume the general form[1]

$$\langle(\chi(z_1)\chi(z_2)\cdots)_\pm\rangle\epsilon_\pm,$$

where $\chi(z)$ refers either to the B.E. field $A(\xi)$ or to the F.D. field $\psi(x)$. Hence $G_+^{(R)}$ contains no negative frequencies in its dependence upon the differences of the consecutive time coordinates, while $G_-^{(R)}$ contains no positive frequencies. The connection between the spectral characteristics of the two types of Green's functions is expressed by complex conjugation, and also by the analytic continuation[1]

$$G_-^{(R)}(z)=(-1)^nG_+^{(R)}(-e^{-\pi i}z).$$

The verification of the latter involves more than the comparison of the two time-ordered forms at distinct times since the Green's functions also contain delta functions of the time differences. The differential equations take account of these terms and we observe that the operation $-e^{-\pi i}$, which reverses the sign of all delta functions, together with the factor $(-1)^n$, produces the G_- differential equations from those for G_+.

THE LORENTZ GAUGE

We now begin the task of subjecting the radiation-gauge Green's functions to a gauge transformation that is designed to remove the explicit dependence upon the unit time-like vector n_μ and thereby introduce the Green's functions of the Lorentz gauge. The preliminary transformation,

$$G^{(R)}(x,\xi)=\prod_{\alpha=1}^{\nu}(1-\partial\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1})_{\xi_\alpha}G(x,\xi),$$

exhibits radiation-gauge functions in terms of new functions which, in their dependence upon each variable ξ_α, remain arbitrary to the extent of added gradients. If the latter functions are restricted by the differential equations,

$$\begin{aligned}(-\partial^2)_{\xi_1}&G_+(x,\xi)\\ &=-i\delta(\xi_1-\xi_2)G_+(x,\xi_3\cdots)+\cdots\\ &\qquad+(1-\boldsymbol{\partial}\partial(\boldsymbol{\partial}^2)^{-1})_{\xi_1}\operatorname{tr}\tfrac{1}{2}i\beta\gamma eqG_+(\cdots\xi_1\xi_1,\xi_2\cdots),\end{aligned}$$

the radiation-gauge differential equations will be reproduced. To present the analogous Dirac field equations most conveniently, we introduce the symbol $\mathcal{A}(\xi)$, which is defined by

$$\mathcal{A}(\xi)G(x,\xi_1\cdots\xi_\nu)=G(x,\xi\xi_1\cdots\xi_\nu).$$

According to the resulting product

$$\mathcal{A}(\xi)\mathcal{A}(\xi')G(x,\xi_1\cdots\xi_\nu)=G(x,\xi\xi'\xi_1\cdots\xi_\nu),$$

and the symmetry of the Green's functions in B.E. coordinates, these symbols are commutative (we are now imitating the external source procedure). The Dirac differential equations for the functions then appear as

$$\begin{aligned}(\beta\gamma(1/i)\partial+\beta m)_{x_1}G_+(x,\xi)&-(\beta\gamma eq)_1G_+(x,x_1\xi_1\cdots)\\ &+(\beta\gamma eq\partial\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1}\mathcal{A})_1G_+(x,\xi)\\ &=\delta(x_1-x_2)G_+(x_3\cdots,\xi)-\cdots,\end{aligned}$$

which indicates the Dirac field gauge transformation implied by that of the Maxwell field,

$$G_+(x,\xi)=\prod_{a=1}^{2n}\exp[-ieq_a\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1}\mathcal{A}(x_a)]G_+^{(L)}(x,\xi).$$

The utility of the $\mathcal{A}$ symbols is clearly shown in this result which constructs the G functions by means of an infinite series of $G^{(L)}$ functions, with increasing numbers of B.E. coordinates. The new Green's functions obey the Dirac equation

$$\begin{aligned}(\beta\gamma(1/i)\partial+m\beta)_{x_1}G_+^{(L)}(x,\xi)&-(\beta\gamma eq)_1G_+^{(L)}(x,x_1\xi_1\cdots)\\ &=\delta(x_1-x_2)G_+^{(L)}(x_3\cdots,\xi)-\cdots,\end{aligned}$$

and the same form applies to the complex conjugate functions $G_-^{(L)}(x,\xi)$.

Before obtaining the Maxwell field differential equations obeyed by the $G^{(L)}$, which are the desired Lorentz-gauge functions, we must notice another aspect of the symbols $\mathcal{A}(\xi)$. If we compare the differential equation characterizing $G_+(x,\xi_1\cdots\xi_\nu)$ with the one for $G_+(x,\xi_1\cdots\xi_\nu\xi)=\mathcal{A}(\xi)G_+(x,\xi_1\cdots\xi_\nu)$, we recognize that

$$[(-\partial^2)_{\xi_1},\ \mathcal{A}(\xi)]\mathcal{A}(\xi_1)=-i\delta(\xi-\xi_1).$$

The required form of this property is

$$\begin{aligned}\Big\{\exp\Big[-\int(d\xi)\lambda(\xi)\mathcal{A}(\xi)\Big](-\partial^2)_{\xi_1}&\\ \times\exp\Big[\int(d\xi)\lambda(\xi)\mathcal{A}(\xi)\Big]-(-\partial^2)_{\xi}\Big\}\mathcal{A}(\xi_1)&=-i\lambda(\xi_1),\end{aligned}$$

where we choose

$$-i\lambda(\xi)=\textstyle\sum_a eq_a\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1}\delta(\xi-x_a).$$

Accordingly, the insertion of

$$G_+=\exp\Big[\int(d\xi)\lambda(\xi)\mathcal{A}(\xi)\Big]G_+^{(L)}$$

into the Maxwell differential equations yields a similar set for the $G_+^{(L)}$ that contains in the left-hand member the additional term $-i\lambda(\xi_1)\times G_+^{(L)}(x,\xi_2\cdots)$. But in view of the relation that follows directly from the Dirac Green's function equation (in contrast with the deduction of the identical radiation gauge result from operator properties),

$$(\partial_\mu)_{\xi_1}\,\mathrm{tr}\tfrac{1}{2}i\beta\gamma^\mu eqG_+^{(L)}(\cdots\xi_1\xi_1,\xi_2\cdots) = -\sum_{a=1}^{2n}\delta(\xi_1-x_a)eq_aG_+^{(L)}(x,\xi_2\cdots),$$

this additional term finds an exact counterpart already present, and

$$(-\partial^2)_{\xi_1}G_\pm^{(L)}(x,\xi)=\mp i\delta(\xi_1-\xi_2)G_\pm^{(L)}(x,\xi_3\cdots)+\cdots \pm\mathrm{tr}\tfrac{1}{2}i\beta\gamma eqG_\pm^{(L)}(x\xi_1\xi_1,\xi_2\cdots)$$

is the desired set of Maxwell differential equations for the Green's functions of the Lorentz gauge. To these equations must be added boundary conditions that will reproduce the spectral characteristics of the radiation-gauge functions. We specify the Lorentz gauge completely by requiring that the $G_+^{(L)}$ contain no negative frequencies and the $G_-^{(L)}$ no positive frequencies in their dependence upon the differences of consecutive time coordinates. The two classes of Lorentz-gauge functions are then connected by complex conjugation and by the analytic continuation

$$G_-^{(L)}(z)=(-1)^nG_+^{(L)}(-e^{-\pi i}z).$$

We have now shown[4] that the radiation-gauge Green's functions, which have a direct operator definition and corresponding physical interpretation, but involve a distinguished time-like direction, can be constructed from Lorentz-gauge Green's functions,

$$G_\pm^{(R)}(x,\xi)=\prod_{\alpha=1}^{\nu}(1-\partial\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1})_{\xi_\alpha} \times\prod_{a=1}^{2n}\exp[-ieq\boldsymbol{\partial}(\boldsymbol{\partial}^2)^{-1}\mathfrak{A}]_{x_a}G_\pm^{(L)}(x,\xi),$$

where the latter functions do not depend upon the time-like vector n_μ but have no immediate physical significance. The simplest examples of this construction are

$$G_\pm^{(R)}(\xi_1\xi_2) =\int(d\xi_1')(d\xi_2')[\delta(\xi_1-\xi_1')+\partial\boldsymbol{\partial}\mathfrak{D}(\xi_1-\xi_1')] \times[\delta(\xi_2-\xi_2')+\partial\boldsymbol{\partial}\mathfrak{D}(\xi_2-\xi_2')]G_\pm^{(L)}(\xi_1'\xi_2'),$$

and

$$G_\pm^{(R)}(x_1x_2) =\exp\left\{ie\int(d\xi)[q_1\boldsymbol{\partial}\mathfrak{D}(x_1-\xi) +q_2\boldsymbol{\partial}\mathfrak{D}(x_2-\xi)]\mathfrak{A}(\xi)\right\}G_\pm^{(L)}(x_1x_2)$$

$$=G_\pm^{(L)}(x_1x_2)+ie\int(d\xi)[q_1\boldsymbol{\partial}\mathfrak{D}(x_1-\xi) +q_2\boldsymbol{\partial}\mathfrak{D}(x_2-\xi)]G_\pm^{(L)}(x_1x_2,\xi)$$
$$-\tfrac{1}{2}e^2\int(d\xi)(d\xi')[q_1\boldsymbol{\partial}\mathfrak{D}(x_1-\xi)+q_2\boldsymbol{\partial}\mathfrak{D}(x_2-\xi)] \times[q_1\boldsymbol{\partial}\mathfrak{D}(x_1-\xi')+q_2\boldsymbol{\partial}\mathfrak{D}(x_2-\xi')]G_\pm^{(L)}(x_1x_2,\xi\xi')+\cdots.$$

The same connections between radiation-gauge and Lorentz-gauge functions apply to the linear combinations $G_+(\xi_1\xi_2)+G_-(\xi_1\xi_2)$, and $\epsilon_+(x_1x_2)[G_+(x_1x_2)+G_-(x_1x_2)]$, the latter relations also containing the infinite sequence of functions

$$\epsilon_+(x_1x_2)[G_+^{(L)}(x_1x_2,\xi_1\cdots\xi_\nu)+G_-^{(L)}(x_1x_2,\xi_1\cdots\xi_\nu)],\quad \nu=1,2\cdots.$$

The combinations of radiation-gauge functions are non-negative, but there is no assurance that such properties extend to the Lorentz-gauge functions. Indeed, the general loss of the positiveness conditions that accompany physical realizability is made evident by the attempt to supply a time-ordered operator construction for the Green's functions of the Lorentz gauge.

The differential equations characterizing the $G^{(L)}$ are satisfied by the following structure:

$$G_\pm^{(L)}(x,\xi)=\langle(\psi(x_1)\cdots\psi(x_{2n})A(\xi_1)\cdots A(\xi_2))_\pm\rangle\epsilon_\pm(x)$$

where the operators $\psi(x)$ and $A_\mu(\xi)$ obey

$$[\gamma((1/i)\partial-eqA)+m]\psi=0,$$
$$-\partial^2A=-\tfrac{1}{2}ie\psi\beta\gamma q\psi,$$

and (among others)

$$\delta(\xi^0-\xi^{0\prime})i[\partial_0A_\mu(\xi),A_\nu(\xi')]=g_{\mu\nu}\delta(\xi-\xi'),$$
$$\delta(x^0-x^{0\prime})\{\psi(x),\psi(x')\}=\delta(x-x'),$$
$$\delta(\xi^0-x^0)[A(\xi),\psi(x)]=0.$$

The symbol $\langle\ \rangle$ signifies a linear mapping of operators onto numbers, including the correspondence $\langle 1\rangle=1$, which must possess the property

$$\langle\chi_1\cdots\chi_k\rangle^*=\langle\chi_k\cdots\chi_1\rangle$$

in order to reproduce the complex conjugate relationship of $G_+^{(L)}$ and $G_-^{(L)}$. The spectral requirements on the $G^{(L)}$ indicate that the space-time variation of the fields is represented by

$$\chi(x)=e^{-iPx}\chi e^{iPx},$$
$$(1/i)\partial_\mu\chi(x)=[\chi(x),P_\mu],$$

[4] This result was first obtained some years ago by K. Johnson and the author (unpublished), using the method of external currents.

where the operator P^0 has a non-negative eigenvalue spectrum, and the more specific interpretation is attached to $\langle$ and $\rangle$ of the left and right eigenvector of P^0 associated with the eigenvalue zero. The field operators are self-adjoint with respect to the operation that interchanges the left and right eigenvectors. We now remark on the following consequence of the commutation relations,

$$\delta(\xi^0-\xi^{0\prime})\langle A_\mu(\xi)P^0A_\nu(\xi')+A_\nu(\xi')P^0A_\mu(\xi)\rangle=g_{\mu\nu}\delta(\xi-\xi'),$$

which is to be compared with the implications of the hypothesis that the adjoint operation is Hermitian conjugation. The evident contradiction between the non-negative nature of the left-hand side and the value $g_{00}=-1$ demands a more general interpretation of the adjoint, corresponding to the introduction of an indefinite metric in the vector space. We shall not continue this approach, with its inevitable requirement that the consistency of the various assumptions concerning operator properties be established for the physical situation of interacting fields. It is our view that the physical operator basis used in the definition of the radiation gauge Green's functions is entirely adequate, the introduction of the Lorentz-gauge functions being an application of the freedom of gauge transformations, and not an occasion for a somewhat dubious reconstruction of the mathematical foundation of the theory.

EUCLIDEAN GREEN'S FUNCTIONS

The Lorentz-gauge Green's functions referring to $2n+\nu=p$ space-time points involve $p-1$ linearly independent coordinate differences, and these appear in $p!$ distinct functional forms corresponding to the various time orderings. Each continuous function associated with a particular time order, $t^{(1)}>\cdots>t^{(p)}$, is formed from harmonic functions of the time differences, $t^{(\alpha)}-t^{(\alpha+1)}$, which contain only non-negative frequencies (G_+), or only nonpositive frequencies (G_-). These functions are also defined outside the special time domain where they reproduce the Green's function, and can be identified as boundary values, on the real axis, of complex variable functions which are regular in various half-planes. But the Green's function is more than the union of its several parts. In particular, it possesses possibilities of analytic extension that are not available to the functions[5] associated with a particular time order. Thus, for distinct space-time points, but with no restriction on the p time variables, the Green's function $G_+(t_1\cdots t_p)$ emerges as the boundary value on the positive real axis of a function $G_+(\zeta t_1\cdots\zeta t_p)$ which is regular in the lower-half ζ plane (it is sufficient to let $\zeta\rightarrow+1$). If ζ approaches the limit -1 from the lower half-plane we obtain the function $(-1)^nG_-(-t_1\cdots-t_p)$. Similarly there exists a function of ζ, regular in the upper half-plane, that yields $G_-(t_1\cdots t_p)$ as $\zeta\rightarrow1$ and $(-1)^nG_+(-t_1\cdots-t_p)$ as $\zeta\rightarrow-1$. These analyticity properties imply, in particular, that the values obtained when ζ occupies the appropriate imaginary axis ($\zeta=\pm i$ suffices) completely determine the Green's function, and conversely.

A very simple illustration of these remarks may be helpful. The functions

$$g_\pm(t_1t_2)=\frac{1}{2\omega}e^{\mp i\omega|t_1-t_2|},\quad \omega>0,$$

obey

$$(\partial^2/\partial t_1^2+\omega^2)g_\pm(t_1t_2)=\mp i\delta(t_1-t_2),$$

and have the frequency characteristics appropriate to their designation. As a function of the single time difference $t=t_1-t_2$, $2\omega g_+$ for $t>0$, and $2\omega g_-$ for $t<0$, coincide with the function $e^{-i\omega t}$, which is defined for all real t and is the value on the real axis of a function that is regular and bounded in the lower-half t plane. The quantities $2\omega g_+(t<0)$ and $2\omega g_-(t>0)$ are similarly related to the function $e^{i\omega t}$ which possesses a bounded analytic extension into the upper half plane. The functions $g_\pm$ are represented for all t by

$$g_\pm(t_1t_2)=\mp\frac{i}{2\pi}\int_{-\infty}^{\infty}d\nu\,\frac{e^{-i\nu(t_1-t_2)}}{\omega^2-\nu^2},$$

where the integration contour for g_+ passes below $-\omega$ and above $+\omega$; it is to be reflected in the real axis for g_-. An analytic extension of g_+ is now obtained by making the substitutions $t\rightarrow\zeta t$, $\zeta=\rho e^{i\vartheta}$, $-\pi<\vartheta<0$, together with $\nu\rightarrow\zeta^{-1}\nu$, which gives the contour a positive rotation. The resulting integral extended along the real axis,

$$g_+(\zeta t_1\zeta t_2)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\nu\,\frac{\zeta}{\omega^2\zeta^2-\nu^2}\,e^{-i\nu(t_1-t_2)},$$

defines for all t an analytic function of $\zeta[=(2\omega)^{-1}e^{-i\zeta\omega|t_1-t_2|}]$ which is regular and bounded in the lower half plane and reproduces g_+ as $\zeta\rightarrow+1$, or g_- as $\zeta\rightarrow-1$. Similarly, the substitutions $t\rightarrow\zeta t$, $\nu\rightarrow\zeta^{-1}\nu$, $\zeta=\rho e^{i\theta}$, $\pi>\theta>0$, performed in the integral representation of g_- yields

$$g_-(\zeta t_1\zeta t_2)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\nu\,\frac{\zeta}{\omega^2\zeta^2-\nu^2}\,e^{-i\nu(t_1-t_2)},$$

and this is a regular, bounded function of ζ, in the upper half-plane, for all $t[=(2\omega)^{-1}e^{i\zeta\omega|t_1-t_2|}]$. As ζ approaches $+1$ or -1, we obtain g_- or g_+. The evident relation between these analytic extensions of g_+ and g_- is such that

$$g_+(-it_1-it_2)=g_-(it_1it_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\nu\,\frac{e^{-i\nu(t_1-t_2)}}{\nu^2+\omega^2}$$

$$=\frac{1}{2\omega}e^{-\omega|t_1-t_2|},$$

[5] When the Green's function is directly connected with a vacuum expectation value of time-ordered field operator products, these functions are the unordered product expectation values that have been discussed particularly by A. Wightman.

which is a bounded quantity for all t. In contrast, the real functions that are analogously produced from $e^{\pm i\omega t}$ are not bounded for all values of t. It should also be noted that

$$(-\partial^2/\partial t_1^2+\omega^2)g_\pm(\mp it_1\mp it_2)=\delta(t_1-t_2),$$

from which we infer the formal transformation

$$\mp i\delta(\mp it)=\delta(t).$$

To facilitate the conversion of the Green's function differential equations to the Euclidean metric, some remarks about the structure of Dirac matrices are needed. The matrices γ^μ have simple algebraic properties, as determined by the metric tensor $g^{\mu\nu}$, while it is the matrices $\beta\gamma^\mu$ that possess the property of symmetry. The matrix β appears here as representative of the indefinite Lorentz (or Minkowski) metric. Accordingly, the replacement of the latter by the Euclidean metric will introduce matrices α_μ that are symmetrical and have the simple algebraic property

$$\{\alpha_\mu,\alpha_\nu\}=2\delta_{\mu\nu}.$$

We shall inquire generally about the possibility of constructing such matrices. It is well known that $2n$ anticommuting matrices of unit square generate an algebra with the dimensionality

$$1+2n+2n(2n-1)/2+\cdots+2n+1=2^{2n},$$

corresponding to the enumeration of the operator basis formed by the independent products. The last element of this collection is

$$\alpha_{2n+1}=i^{-n}\alpha_1\cdots\alpha_{2n},$$

which extends by one the set of anticommuting matrices. We now want to recognize that when these $2n+1$ matrices are irreducible and possess a particular symmetry, $n+1$ of them are symmetrical, and the remaining n are antisymmetrical.

An elementary proof employs the construction of the algebra as the product of n independent algebras of dimensionality 2^2, represented by the 2×2 Pauli matrices, 1, σ_1, σ_2, σ_3. The ascent from $2n+1$ anticommuting matrices α_κ to $2n+3$ such matrices is produced, for example, by

$$\sigma_1,\ \sigma_2,\ \sigma_3\alpha_\kappa,\quad \kappa=1\cdots2n+1.$$

If σ_3 is symmetrical the set of $2n+1$ matrices has the same symmetry distribution as the α_κ, $\kappa=1\cdots2n+1$. The two additional matrices constitute one symmetrical and one antisymmetrical matrix. Hence the number of each type grows by one when n is increased by unity. The stated result now follows from the remark that we can begin with unity, for $n=0$, and it is symmetrical. The particular construction method employed does not influence the dimensionality of the two symmetry categories.

We first consider $n=2$, which produces the familiar set of five anticommuting 4×4 matrices. Since there are only three symmetrical matrices, this set is adapted to the 3+1 Lorentz space but cannot be applied to the four-dimensional Euclidean space. To have four symmetrical anticommuting matrices, we must choose $n\geq3$, corresponding to 8×8 matrices, at least. Thus the requirement of a Euclidean formulation excludes the simplest field in space-time, the four-component Hermitian spin-$\frac{1}{2}$ field (Majorana). In this context, a trivial observation may be worth repeating—a four-component Hermitian field is fully equivalent to a two-component non-Hermitian field.

For $n=3$, the three symmetrical matrices $i\gamma^1$, $i\gamma^2$, $i\gamma^3$ and the antisymmetrical matrix γ^0 are supplemented by $i\gamma_5 l_3$, $i\gamma_5 l_1$, $i\gamma_5 l_2$, of which the first is symmetrical. The matrices $l_1 l_2 l_3$ are 2×2 Pauli matrices, with l_3 antisymmetrical, and we have defined

$$\gamma_5=\gamma^0\gamma^1\gamma^2\gamma^3.$$

There are a variety of unitary transformations that will effectively change the order of these seven anticommuting matrices and supply four symmetrical real matrices followed by three antisymmetrical imaginary matrices. Thus, the unitary transformation by the matrix $\exp[\frac{1}{2}\pi i\gamma^0\gamma_5 l_3]$ produces the sequence: $i\gamma^1$, $i\gamma^2$, $i\gamma^3$, $i\gamma_5 l_3$; $-\gamma^0$, $i\gamma_5 l_1$, $i\gamma_5 l_2$. Subsequent unitary-orthogonal transformations will not alter this symmetry partitioning. The transformation matrix $\exp[\frac{1}{2}\pi i\gamma^0]$, for example, supplies the list: $\gamma^0\gamma^1$, $\gamma^0\gamma^2$, $\gamma^0\gamma^3$, $\gamma^0\gamma_5 l_3$; $-\gamma^0$, $\gamma^0\gamma_5 l_1$, $\gamma^0\gamma_5 l_2$. With $n>3$, we gain the possibility of describing additional internal symmetry properties, which appear rather differently in Lorentz or Euclidean form, but we shall not discuss it here.

The analytic extension of the time variables to the appropriate imaginary axis introduces the Euclidean metric,

$$G_\pm:\quad x^0\leftrightarrow\mp ix_4.$$

The Green's functions also undergo additional matrix transformations. For the vector indices related to the Maxwell field the transformation is just that formally associated with the redefinition of the coordinates, and we shall not indicate it explicitly. With this understanding, a suitable correspondence is

$$\prod_{a=1}^{2n}(\exp(\pm\tfrac{1}{4}\pi i\gamma^0\gamma_5 l_3\mp\tfrac{1}{4}\pi i))_a G_\pm{}^{(L)}(x)\leftrightarrow G^{(E)}(x),$$

where, as the right side suggests, the same real Euclidean Green's function is obtained from the two complex Lorentz Green's functions. The effect of this substitution on the various Dirac matrices is indicated by

$$\begin{aligned}\exp(\mp\tfrac{1}{4}\pi i\gamma^0\gamma_5 l_3)(1/i)\beta(\gamma^k,\pm i\gamma^0,1)\exp(\mp\tfrac{1}{4}\pi i\gamma^0\gamma_5 l_3)\\ =\gamma^0\exp(\pm\tfrac{1}{4}\pi i\gamma^0\gamma_5 l_3)(\gamma^k,\pm i\gamma^0,1)\exp(\mp\tfrac{1}{4}\pi i\gamma^0\gamma_5 l_3)\\ =\gamma^0\gamma^k,\ \gamma^0\gamma_5 l_3,\ \gamma^0,\end{aligned}$$

and the resulting set can be labelled consistently as α_μ,

$\mu=1\cdots5$. The real differential equations thus obtained are

$$(\alpha\partial+mi\alpha_5)_{x1}G^{(E)}(x,\xi)-e(\alpha iq)_1G^{(E)}(x,x_1\xi) =\delta(x_1-x_2)G^{(E)}(x_3\cdots,\xi)-\cdots,$$

and

$$(-\partial^2)_{\xi 1}G^{(E)}(x,\xi)+e\,\mathrm{tr}\tfrac{1}{2}\alpha iqG^{(E)}(x\xi_1\xi_1,\xi_2\cdots) =\delta(\xi_1-\xi_2)G^{(E)}(x,\xi_3\cdots)+\cdots.$$

The accompanying regularity conditions demand the boundedness of each $G^{(E)}(x_1\cdots x_{2n},\xi_1\cdots\xi_\nu)$ when neighborhoods of coincidence of the $2n+\nu$ points are excluded.

The Euclidean Green's functions are invariant under a variety of transformations, in the sense of

$$\prod_{\alpha=1}^{p}(R)_\alpha G(r^{-1}z)=G(z).$$

Four-dimensional rotational invariance is described in the evident manner with the Dirac field spin matrices given by $\frac{1}{2}\sigma_{\mu\nu}$, where each

$$\sigma_{\mu\nu}=(1/2i)[\alpha_\mu,\alpha_\nu],\quad \mu,\nu=1\cdots4$$

is antisymmetrical and imaginary. If the matrix q is invariant, the reflection of any coordinate axis implies the corresponding vector transformation for the Maxwell field, while the Dirac field reflection matrices can be chosen as

$$R_\mu=i\alpha_\mu\alpha_6,\quad \mu=1\cdots4$$

which are real, symmetrical, anticommuting matrices. We also observe that the geometrical connection between reflection and rotation is correctly described,

$$R_\mu R_\nu=e^{\frac{1}{2}\pi i\sigma_{\mu\nu}}.$$

Invariance under the coordinate-independent transformation of the Dirac field that is generated by the imaginary antisymmetrical rotation matrix

$$(1/i)\alpha_6\alpha_7=l(=l_3),$$

implies the conservation of the fermionic charge represented by l. There is a similar transformation associated with the electric charge matrix q. The two charges may be identical, but we need not insist on this. Indeed, the coordinate reflection transformation that we have described is an invariance operation only when l and q are independent. If these charges are the same (or are coupled together) the reflection transformation must be accompanied by an additional sign reversal of the Maxwell field, for each R_μ induces a reflection of the fermionic charge. This is a combined coordinate and charge reflection transformation.

The matrix α_6, or α_7, induces a coordinate-independent invariance transformation that also implies fermionic charge reflection. Hence one could combine R_μ with either α_6 or α_7, yielding $i\alpha_\mu$ or $l\alpha_\mu$, which describe coordinate reflections without charge reflection. It is interesting that the matrices now under discussion are imaginary, and yet the reality of the Green's functions is not disturbed since a transformation involves an even number of such Dirac matrix factors. Let us also note here that the algebraic sign of m is without physical significance since the Green's function transformation described by the imaginary Dirac matrix α_5, or $-il\alpha_5=\alpha_1\alpha_2\alpha_3\alpha_4$, has no other effect than to reverse this sign.

The full equivalence of all directions in the Euclidean space makes it unnatural to relate the Green's functions to operators ordered by coordinate projection on some line, and one might seek to introduce an invariant ordering parameter (proper time). We shall follow a different course, however, which also has its counterpart in space-time where it is the formulation to which the source techniques lead. To avoid emulation of the latter procedure, we set down directly the following operator construction of the Euclidean Green's functions

$$G(x,\xi)=\langle 0|\psi(x_1)\cdots\psi(x_{2n})A(\xi_1)\cdots A(\xi_\nu)|W\rangle/\langle 0|W\rangle,$$

where

$$[A_\mu(\xi),A_\nu(\xi')]=[A_\mu(\xi),\psi(x)]=\{\psi(x),\psi(x')\}=0,$$

throughout the four-dimensional space. The symmetry properties of the Green's functions are thereby reproduced. In addition to these commutative or anticommutative fields, there is a complementary set of fields, $B_\mu(\xi)$, $\phi(x)$, which are also everywhere commutative or anticommutative, and obey

$$i[B_\mu(\xi),A_\nu(\xi')]=\delta_{\mu\nu}\delta(\xi-\xi'),$$
$$\{\phi(x),\psi(x')\}=\delta(x-x').$$

All other commutators vanish. The state $\langle 0|$ is characterized by zero eigenvalues of the second operator set,

$$\langle 0|B(\xi)=0,\quad \langle 0|\phi(\xi)=0,$$

while $|W\rangle$ is described by

$$[(\alpha\partial+mi\alpha_5)\psi(x)-eiq\alpha A(x)\psi(x)+\phi(x)]|W\rangle=0,$$

and

$$[-\partial^2A(\xi)-\tfrac{1}{2}e\psi(\xi)iq\alpha\psi(\xi)+iB(\xi)]|W\rangle=0.$$

One verifies immediately that the differential equations are reproduced by these definitions. In virtue of the complementary field commutation properties, the operator definitions of the vector $|W\rangle$ are satisfied by the construction

$$|W\rangle=e^{-W}|0\rangle,$$

where

$$W=\tfrac{1}{2}\int(dx)[(\partial_\mu A_\nu)^2+\psi(\alpha_\mu\partial_\mu+mi\alpha_5)\psi-eA_\mu\psi iq\alpha_\mu\psi],$$

and $|0\rangle$ is the right eigenvector of B and ϕ associated with zero eigenvalues. If these states are to exist it is necessary that the B and A operators be Hermitian. That property, together with the choice of ψ and ϕ as mutually Hermitian conjugate operators,

$$\phi(x)=\psi(x)^{\dagger},$$

assures the reality of the Green's functions. We verify the latter statement by remarking that

$$G(x,\xi)=\frac{\langle\phi'=B'=0|\psi(x_1)\cdots\psi(x_{2n})A(\xi_1)\cdots A(\xi_\nu)e^{-W[\psi,A]}|0\rangle}{\langle 0|e^{-W[\psi,A]}|0\rangle},$$

implies

$$G(x,\xi)^*=\frac{\langle\psi'=B'=0|i\phi(x_1)\cdots i\phi(x_{2n})A(\xi_1)\cdots A(\xi_\nu)e^{-W[i\phi,A]}|0\rangle}{\langle 0|e^{-W[i\phi,A]}|0\rangle}$$

where the factor of i associated with each F.D. field refers to the sign reversal induced in every pair of such field products by the adjoint operation. Since the conversion of $i\phi$ to ψ and the interchange of the null-eigenvalue states is a canonical transformation, the equivalence of the two forms, and the reality of the Green's functions, follows.

The latter discussion shows that while the Euclidean action operator W is not Hermitian, it is connected with its Hermitian adjoint by a unitary transformation. In contrast with this expression of reality, the Green's function invariance properties to which we have referred are associated with unitary transformations that leave the action operator invariant. An example that has not been mentioned is provided by the unitary operator

$$U=\exp(i\epsilon_\mu P_\mu),$$

where P_μ, the Euclidean total linear momentum vector, is the Hermitian operator

$$P_\mu=\int(dx)\left[-B_\nu\partial_\mu A_\nu+\phi\frac{1}{i}\partial_\mu\psi\right]$$
$$=\int(dx)\left[A_\nu\partial_\mu B_\nu+\psi\frac{1}{i}\partial_\mu\phi\right].$$

Evidently,

$$[A,P_\mu]=(1/i)\partial_\mu A,\quad [\psi,P_\mu]=(1/i)\partial_\mu\psi,$$

and generally

$$U^{-1}\chi(z)U=\chi(z+\epsilon).$$

The action operator is invariant under this transformation, and since

$$\langle 0|P_\mu=0,\quad P_\mu|0\rangle=0,$$

we verify that the Green's functions are translationally invariant. We give this property another form on replacing the coordinate variables with the complementary momentum variables. This is expressed by the transformation

$$G(x_1\cdots x_{2n},\xi_1\cdots\xi_\nu)=\int\frac{(dp_1)}{(2\pi)^2}\cdots\frac{(dk_\nu)}{(2\pi)^2}\exp[i(p_1x_1+\cdots+k_\nu\xi_\nu)]\times G(p_1\cdots p_{2n},k_1\cdots k_\nu).$$

Under the related field transformation, the action operator acquires the form

$$W[\psi,A]=\tfrac{1}{2}\int(dk)A(-k)k^2A(k)+\tfrac{1}{2}i\int(dp)\psi(-p)(\alpha p+m\alpha_5)\psi(p)-\tfrac{1}{2}ie\int(dp)(dp')(dk)\times\frac{1}{(2\pi)^2}\delta(p+p'+k)A(k)\psi(p)q\alpha\psi(p').$$

The translational invariance of $G(x,\xi)$ requires that $G(p,k)$ contain the factor $\delta(p_1+\cdots p_{2n}+k_1+\cdots+k_\nu)$, and this follows directly from the invariance of the operator expression for $G(p,k)$ under the transformation

$$U^{-1}A(k)U=e^{i\epsilon k}A(k),$$
$$U^{-1}\psi(p)U=e^{i\epsilon p}\psi(p).$$

A compact expression of the totality of Green's functions is obtained by defining the generating function(al)

$$G[\phi'B']=1+\sum_{n\nu}\int\frac{(dx_1)\cdots(dx_{2n})}{(2n)!}\frac{(d\xi_1)\cdots(d\xi_\nu)}{\nu!}\times\frac{1}{i}B(\xi_1)'\cdots\frac{1}{i}B(\xi_\nu)'\phi(x_{2n})'\cdots\phi(x_1)'\times G(x_1\cdots x_{2n},\xi_1\cdots\xi_\nu).$$

Here the $B_\mu(\xi)'$ are a continuous set of arbitrary real numbers, while the $\phi(x)'$ are completely anticommuting symbols formed from an algebra external to that of the F.D. fields and thus are commutative with the latter. We shall also use the notation $\phi(x)'$ for reference to symbols that are anticommutative with the operator fields $\psi(x)$, $\phi(x)$. The connection between the two sets is produced by the operator that generates the canonical transformation of sign reversal of all F.D. fields, under which the action operator is invariant, namely

$$\rho=\exp\left[\pi i\int(dx)\phi(x)\psi(x)\right]=\exp\left[-\pi i\int(dx)\psi(x)\phi(x)\right].$$

The equivalence of these two forms depends explicitly on the even number of components possessed by the

F.D. field. (The integrals are given a meaning by an equivalent summation over arbitrarily small four-dimensional cells.) When acting upon the zero-eigenvalue states of the operators ϕ, or ψ, the field reflection operator ρ exhibits the eigenvalue $+1$,

$$\langle 0|\rho=\langle 0|, \quad \rho|0\rangle=|0\rangle.$$

Accordingly, we write

$$\phi(x)'\langle 0|=\langle 0|\phi(x)',$$

and

$$|0\rangle\phi(x)'=\phi(x)'|0\rangle,$$

where $\phi(x)'$ in the position of an operator contains ρ, and thus is anticommutative with all ψ and ϕ operators. The summation defining the generating function, or Green's functional, can now be performed and we get

$$G[\phi'B']=\left\langle 0\left|\exp\left[-i\int(d\xi)B_\mu(\xi)'A_\mu(\xi)\right]\right.\right.$$

$$\left.\left.\times\exp\left[\int(dx)\phi(x)'\psi(x)\right]\right|W\right\rangle\bigg/\langle 0|W\rangle,$$

or

$$G[\phi'B']=\langle\phi'B'|W\rangle/\langle 0|W\rangle,$$

in which we have recognized that the exponentials are operators of the special canonical group,[6] which translate eigenvalues of canonical variables. Indeed,

$$\begin{aligned}\langle\phi'B'|B_\mu(\xi)&=\langle\phi'0|e^{-iB'A}B_\mu(\xi)\\&=\langle\phi'B'|B_\mu(\xi)',\end{aligned}$$

and

$$\begin{aligned}\langle\phi'B'|\phi(x)&=\langle 0B'|e^{\phi'\psi}\phi(x)\\&=\langle\phi'B'|\phi(x)'.\end{aligned}$$

Thus the generating function is exhibited as the wave function representing the state $|W\rangle$ in the $\phi'B'$ description, apart from a constant that normalizes $G[00]$ to unity.

The operators A and ψ can be given differential operator realizations in the $\phi'B'$ description. Infinitesimal eigenvalue changes induce the variation

$$\delta\langle\phi'B'|=\langle\phi'B'|\left[-i\int(d\xi)\delta B_\mu(\xi)'A_\mu(\xi)\right.$$

$$\left.+\int(dx)\delta\phi(x)'\psi(x)\right],$$

which we express as

$$\langle\phi'B'|A_\mu(\xi)=i\frac{\delta}{\delta B_\mu(\xi)'}\langle\phi'B'|$$

and

$$\langle\phi'B'|\psi(x)=\frac{\delta_l}{\delta\phi(x)'}\langle\phi'B'|.$$

The latter form also involves the conversion of the eigenvalue variations into elements of the external algebra,

$$\langle\phi'B'|\delta\phi(x)'=\delta\phi(x)'\langle\phi'B'|.$$

The abstract operator equations defining the state $|W\rangle$ acquire thereby a differential operator realization that characterizes the Green's functional,

$$\left[(\alpha\partial+mi\alpha_5)\frac{\delta_l}{\delta\phi(x)'}+eq\alpha\frac{\delta}{\delta B(x)'}\frac{\delta_l}{\delta\phi(x)'}+\phi(x)'\right]$$

$$\times G[\phi'B']=0,$$

$$\left[-\partial^2\frac{\delta}{\delta B(\xi)'}-\tfrac{1}{2}e\frac{\delta_l}{\delta\phi(\xi)'}q\alpha\frac{\delta_l}{\delta\phi(\xi)'}+B(\xi)'\right]G[\phi'B']=0,$$

and from which the original Green's function equations are recovered with the aid of the correspondence

$$G(x_1\cdots x_{2n},\xi_1\cdots\xi_\nu)$$

$$=\frac{\delta_l}{\delta\phi(x_1)'}\cdots\frac{\delta_l}{\delta\phi(x_{2n})'}$$

$$\times i\frac{\delta}{\delta B(\xi_1)'}\cdots i\frac{\delta}{\delta B(\xi_\nu)'}G[\phi'B']\bigg|_{\phi'=B'=0}.$$

A formal solution of the functional differential equations appears on applying the differential operator realization to the explicit operator construction of the state $|W\rangle$,

$$\langle 0|W\rangle G[\phi'B']=e^{-W[\delta_l/\delta\phi',\, i\delta/\delta B']}\delta[\phi']\delta[B'].$$

We have used the notation

$$\begin{aligned}\delta[\phi']\delta[B']&=\langle\phi'B'|0(\phi'B')\rangle\\&=\langle 0|e^{-iB'A+\phi'\psi}|0\rangle,\end{aligned}$$

which is in conformity with the fundamental property

$$B_\mu(\xi)'\delta[\phi']\delta[B']=\langle\phi'B'|B_\mu(\xi)|0\rangle=0,$$
$$\phi(x)'\delta[\phi']\delta[B']=\langle\phi'B'|\phi(x)|0\rangle=0.$$

An alternative expression of the relation between G and W is obtained by writing

$$\langle 0|W\rangle\langle\phi'B'|G[\phi,B]|0(\psi'A')\rangle=\langle\phi'B'|e^{-W[\psi,A]}|0(\phi'B')\rangle,$$

which utilizes the unit value ascribed to the transformation function

$$\langle\phi'B'|\psi'A'\rangle=e^{\phi'\psi'-iB'A'},$$

when either eigenvalue set is placed equal to zero. As a vector equation,

$$\langle 0|W\rangle G[\phi,B]|0(\psi'A')\rangle=e^{-W[\psi,A]}|0(\phi'B')\rangle$$

can also be displayed in the $\psi'A'$ representation, where

[6] A fuller discussion of this group appears in an article being prepared for publication in the *Encyclopedia of Physics* [Springer-Verlag (to be published)] Vol. 5, Part 2.

it takes the form

$$\langle 0|W\rangle G[\delta_l/\delta\psi',(1/i)\delta/\delta A']\delta[\psi']\delta[A']=e^{-W[\psi'A']},$$

with the aid of the following consequences of the canonical transformation $B\rightarrow A$, $A\rightarrow -B$; $\phi\leftrightarrow\psi$:

$$\langle\psi'A'|\phi'B'\rangle=e^{-\phi'\psi'+iB'A'}$$
$$\langle\psi'A'|\psi''A''\rangle=\delta[\psi'-\psi'']\delta[A'-A''].$$

It should also be mentioned that an integration concept can be devised for both types of field variable, such that

$$\delta[\phi']\delta[B']=\langle 0|e^{\phi'\psi-iB'A}|0\rangle$$
$$=\int d[\psi']d[A']e^{\phi'\psi'-iB'A'},$$

and

$$\int d[\phi']d[B']\delta[\phi']\delta[B']=1.$$

The generating function, for example, thereby acquires the integral representation

$$G[\phi'B']=\frac{\int d[\psi']d[A']e^{-W[\psi'A']+\phi'\psi'-iB'A'}}{\int d[\psi']d[A']e^{-W[\psi'A']}}.$$

The discussion of Euclidean Green's functions will be continued in another publication.

Note added in proof.—It has not been sufficiently emphasized in this paper that the term "Lorentz gauge," as descriptive of a gauge in which there is no distinguished time-like vector, refers to a class rather than just the special gauge used in the paper. One can also introduce, for example, the transverse Lorentz gauge (here "transverse" has a four-dimensional space-time significance), characterized by

$$(\partial)_{\xi\alpha}G_{\pm}^{(L)}(x,\xi)=0,\quad \alpha=1\ .\ .\ .\ \nu$$

together with appropriately modified Maxwell differential equations. The radiation gauge functions constructed from Lorentz gauge Green's functions are clearly independent of the specific Lorentz gauge employed. This subject will be discussed further in a later paper.

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Structure of the Vertex Function

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An integral representation as a function of invariants is found for the Fourier transform of the matrix element between the vacuum and a one-particle state of the retarded commutator of two currents. A special case is a spectral representation for the vertex as a function of momentum transfer. The threshold in this representation is lower than that found in the usual perturbation theory.

INTRODUCTION

WE shall study the structure of the matrix element of the commutator, the retarded commutator, and the time-ordered product of two field operators taken between the vacuum and a single-particle state. Our technique will be to manipulate these functions in the physical region in such a way as to obtain an integral representation for the function in terms of the invariant momentum parameters that characterize its Fourier transform.

We shall discover that, for a special case of this representation, one can continue analytically certain of the invariant parameters out of the physical region and automatically obtain a spectral relation for the vertex function as a function of the momentum transfer.

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The basic representation we shall find for the Fourier transform of the retarded commutator of two currents is

$$\int e^{ikx}\theta(x)dx\langle[j_1(x),j_2(0)]|p\rangle$$
$$=\int d\mu d\beta\frac{H(\mu,\beta,p^2)}{k^2+2\beta pk-\mu+i\epsilon(pk+\beta p^2)},\quad (1)$$

in which the variables are limited by

$$0\geqslant\beta\geqslant-1,\quad \mu\geqslant\max\{f(\beta),\ -\beta^2p^2\},$$

where $f(\beta)$ is a function, which we will specify later, determined by the mass spectra of the intermediate states, and $H(\mu,\beta,p^2)$ is uniquely determined by the Fourier transform in the physical region.

THE REPRESENTATION

We shall begin by deriving a representation for the matrix element of the commutator of two currents. We