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Haag's Theorem and Clothed Operators

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Haag's theorem is proved. This theorem states—essentially—that a given relativistic field which, at a fixed time, is related by a unitary transformation to the free field, is completely equivalent to the free field throughout all space-time. Previously it had been proved that the vacuum expectation values of the given field equal the free-field ones up to and including the fourfold vacuum expectation value. A corollary to Haag's theorem is derived. The corollary shows that a certain type of relativistic, clothed operator is equivalent to the free field everywhere.

1. INTRODUCTION

WE denote by Haag's theorem[1,2] the statement that any quantum field theory which has the following four properties: I—relativistic transformation properties, II—unique, normalizable, invariant, vacuum state Ψ_0 and no negative-energy states or states of spacelike momenta,[3] III—canonical commutation relations at equal times, and IV—being related to the free-field theory[4] at a given time by a unitary transformation, is completely equivalent to the free-field theory. Haag[1] first stated the essential physical ideas of this theorem. The theorem was later more precisely discussed and generalized, but not completely proved, by Hall and Wightman.[2] In this paper, we will complete the proof of Haag's theorem in the (ungeneralized) form stated above. We will also prove a corollary to this theorem; namely that a theory in which the field satisfies I, II, and, at a given time, has an expansion in terms of annihilation and creation operators belonging to the no-particle representation of the (momentum space) canonical commutation rules, is completely equivalent to the free-field theory. Finally we note that for a clothed theory of the type discussed previously,[5] the requirement that the field transform relativistically places the theory under the assumptions of the corollary, and thus forces the equivalence of such theories to the free-field theory.

In Sec. 2 we give precise statements of properties I through IV, and summarize the relevant results of references 1 and 2. Section 3 contains the proof of Haag's theorem. Section 4 proves a corollary to Haag's theorem, and notes the application to clothed operators. Finally, Sec. 5 contains a remark about clothed operators.

[1] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 12 (1955).
[2] D. W. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 5 (1957).
[3] A. S. Wightman, Phys. Rev. **101**, 860 (1956).
[4] A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955).
[5] O. W. Greenberg and S. S. Schweber, Nuovo cimento **8**, 378 (1958).

2. ASSUMPTIONS OF HAAG'S THEOREM, AND RESULTS OF HAAG, AND HALL AND WIGHTMAN

Property I requires that fields $A_j(x)$ ($j=0$, 1 throughout) transform under the corresponding unitary representation of the inhomogeneous Lorentz group, $U_j(a,\Lambda)$, where a is a space-time translation, and $\Lambda\epsilon L_+^\uparrow$ (the orthochronous group of Lorentz transformations of determinant one), as a scalar field

$$U_j(a,\Lambda)A_j(x)U_j(a,\Lambda)^{-1}=A_j(\Lambda x+a). \tag{1}$$

Property II requires that

$$U_j(a,\Lambda)\Psi_{0j}=\Psi_{0j}, \tag{2a}$$

or

$$P_j^\mu\Psi_{0j}=0, \tag{2b}$$

where Ψ_{0j} is the unique, invariant, normalizable, vacuum state of each theory, and P_j^μ is the energy-momentum operator of each theory. In addition the spectrum of P_j^0 must be bounded from below by zero. Property III requires (a) the existence of canonically conjugate fields $\pi_j(\mathbf{x},\tau)$ at a given time τ (τ is always fixed), which have Euclidean transformation properties:

$$U_j(\mathbf{a},R)\pi_j(\mathbf{x},\tau)U_j(\mathbf{a},R)^{-1}=\pi_j(R\mathbf{x}+\mathbf{a},\tau), \tag{3}$$

where $\mathbf{a}$ is a space translation and R a proper space rotation; and (b) that one of the pairs of conjugate fields satisfy the equal-time, canonical commutation relations

$$[A_0(\mathbf{x},\tau),\pi_0(\mathbf{y},\tau)]=i\delta(\mathbf{x}-\mathbf{y}), \tag{4a}$$

$$[A_0(\mathbf{x},\tau),A_0(\mathbf{y},\tau)]=[\pi_0(\mathbf{x},\tau),\pi_0(\mathbf{y},\tau)]=0. \tag{4b}$$

The other conjugate pair satisfies (4a) and (4b) by virtue of property IV. The operators $A_j(\mathbf{x},\tau),\pi_j(\mathbf{x},\tau)$ are assumed to form an irreducible set for each theory, which we take to mean that $A_j(\mathbf{x},\tau)$, $\pi_j(\mathbf{x},\tau)$ give a complete description of each theory. Property IV requires that there be a unitary operator V such that

$$A_1(\mathbf{x},\tau)=VA_0(\mathbf{x},\tau)V^{-1}, \tag{5a}$$

$$\pi_1(\mathbf{x},\tau)=V\pi_0(\mathbf{x},\tau)V^{-1}. \tag{5b}$$

For simplicity, it is assumed that the fields are neutral:[6]

$$A_j(x)=A_j(x)^*, \tag{6a}$$

$$\pi_j(\mathbf{x},\tau)=\pi_j(\mathbf{x},\tau)^*. \tag{6b}$$

For a theory with the properties I through IV, three results were proved[1,2]: first, that the vacuum states of each theory are related[7] by V,

$$\Psi_{01}=V\Psi_{00}; \tag{7}$$

secondly, that the equal-time vacuum expectation values of the canonical operators are equal for all n,

$$(\Psi_{01},A_1(\mathbf{x}_1,\tau)\cdots A_1(\mathbf{x}_n,\tau)\Psi_{01}) =(\Psi_{00},A_0(\mathbf{x}_1,\tau)\cdots A_0(\mathbf{x}_n,\tau)\Psi_{00}), \tag{8}$$

and similar equations when any number of the A_j are replaced on both sides of (9) by the corresponding π_j; and, finally, that the vacuum expectation values $F_j^{(n)}$ of the two theories are identical throughout all space-time for $n\leqslant 4$, where

$$F_j^{(n)}(\xi_1,\xi_2,\cdots\xi_{n-1}) =(\Psi_{0j},A_j(x_1)A_j(x_2)\cdots A_j(x_n)\Psi_{0j}), \tag{9}$$

and $\xi_k=x_k-x_{k+1}$ throughout this paper.

Two additional facts which we will need are the Lorentz invariance[3] of the $F_j^{(n)}$,

$$F_j^{(n)}(\xi_1,\xi_2,\cdots\xi_{n-1}) =F_j^{(n)}(\Lambda\xi_1,\Lambda\xi_2,\cdots\Lambda\xi_{n-1}),\Lambda\epsilon L_+^{\uparrow}, \tag{10}$$

and the fact that the $F_j^{(n)}$ are determined uniquely throughout all space-time by their values in the neighborhood of one set of vectors, $(\rho_1,\rho_2,\cdots\rho_{n-1})$, which we call a Jost point, which have the property that the convex set of vectors generated from them are all space-like,[3,8]

$$\left(\sum_{i=1}^{n-1}\lambda_i\rho_i\right)^2<0,\quad \lambda_i\geqslant 0,\quad \sum_{i=1}^{n-1}\lambda_i=1. \tag{11}$$

The references cited discuss the analytic continuation arguments upon which this last fact is based.

The results of Hall and Wightman are valid regardless of whether either of the fields A_j is the free field. As mentioned in Sec. 1, we will discuss only the (ungeneralized) case where one of the fields is a free field[4] of some given mass m. Until further notice we take A_0 to be this free field, call its vacuum state Φ_0, and we drop the subscript from the other field theory. Now we are ready to prove Haag's theorem as stated in Sec. 1, with properties I through IV as defined in Sec. 2.

3. PROOF OF HAAG'S THEOREM

Theorem.—Any relativistic theory with properties I through IV of Sec. 2 is equivalent to the free-field theory.

Proof.—We give an inductive proof in which two more vacuum expectation values of the theory of the A field are proved equal to the free-field vacuum expectation values in each cycle of the argument. We first give the argument[9] for the case where we know that $F^{(n)}=F_0^{(n)}$, $n\leqslant 2$; and then proceed to the mth step of the induction, where it has been proved that $F^{(n)}=F_0^{(n)}$, $n\leqslant 2m$.

Define the operator[10] $A_\tau(x)$, which satisfies the Klein-Gordon equation, and extrapolates the operator $A(x)$ off the surface $x^0=\tau$:

$$A_\tau(x)=\int_{y^0=\tau} d^3\mathbf{y}\left(A(y)\overleftrightarrow{\frac{\partial}{\partial y^0}}\Delta(x-y)\right), \tag{12}$$

where $i\Delta(x-y)=[A_0(x),A_0(y)]$ is the mass-m free-field commutator, and

$$A\overleftrightarrow{\frac{\partial}{\partial x}}B=A\frac{\partial B}{\partial x}-\frac{\partial A}{\partial x}B.$$

We will show that

$$A_\tau(x)\Psi_0=A(x)\Psi_0,\quad \text{all } x. \tag{13}$$

Consider the norm

$$\left\|\int d^4x\, f(x)(A_\tau(x)-A(x))\Psi_0\right\|^2 =\int d^4xd^4y\,\bar{f}(x)f(y)\{(\Psi_0,A_\tau(x)A_\tau(y)\Psi_0) -(\Psi_0,A_\tau(x)A(y)\Psi_0)-(\Psi_0,A(x)A_\tau(y)\Psi_0) +(\Psi_0,A(x)A(y)\Psi_0)\}, \tag{14}$$

where $f(x)$ is any testing function in C^∞, the set of functions with an infinite number of continuous derivatives, for which

$$i\int\bar{f}(y)\Delta^{(+)}(y-x)f(x)d^4xd^4y<\infty.$$

All four of the vacuum expectation values in (14) are equal to $F_0^{(2)}(x-y)$, because replacing an operator

[6] The assumptions that a single pair of canonical operators is an irreducible set, that the fields $A_j(x)$ transform as scalar (rather than as spin-$\frac{1}{2}$, etc.) fields and that the fields are neutral can all be removed straightforwardly; all statements in this paper remain true for theories with a finite number of charged or neutral fields transforming as scalar, spinor, vector, etc., fields.

[7] The constant of modulus one which can appear on the right-hand side of (7) has been absorbed in V with no loss of generality.

[8] R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[9] This case is chosen to shorten the formulas; we could have started with the case $n\leqslant 4$.

[10] For a general field $A(x)$ the operator $A_\tau(x)$ might not exist. However, provided it is first integrated with suitable testing functions as indicated below Eqs. (14) and (18), A_τ will exist here on the domain of A.

$A(x_i)$ by $A_\tau(x_i)$ in any vacuum expectation value $F^{(n)}$ equal to the free-field one $F_0^{(n)}$ just reproduces the $F_0^{(n)}$. This statement is equivalent to the formula

$$A_0(x)=A_{0\tau}(x)=\int_{y^0=\tau} d^3\mathbf{y}\left(A_0(y)\frac{\overleftrightarrow{\partial}}{\partial y^0}\Delta(x-y)\right), \quad (15)$$

which we will use explicitly below in proving (23).

Next we show that

$$\pi(\mathbf{x},\tau)\Psi_0=\dot{A}(\mathbf{x},\tau)\Psi_0. \quad (16)$$

We will prove (16) by a method which will carry over to the general case in the induction, rather than giving a shorter argument which will not apply to the general case. The next paragraph develops the method to be used in proving (16).

Consider the set of vectors $\{\chi_n\}$,

$$\chi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau)=A(\mathbf{x}_1,\tau)\cdots A(\mathbf{x}_n,\tau)\Psi_0,$$

and construct the set $\{\Psi_n\}$, where Ψ_n is a linear combination of χ_k for $k\leqslant n$,

$$\Psi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau)=(n!)^{-\frac{1}{2}}\Big(A(\mathbf{x}_1,\tau)\cdots A(\mathbf{x}_n,\tau) - \sum_{\text{all factor pairings}} A(\mathbf{x}_1,\tau)\cdots A(\mathbf{x}_n,\tau)\Big)\Psi_0,$$

where the factor pairing[11] of $A(\mathbf{x}_i,\tau)$ and $A(\mathbf{x}_j,\tau)$ is defined to be

$$(\Psi_0,A(\mathbf{x}_i,\tau)A(\mathbf{x}_j,\tau)\Psi_0)=i\Delta^{(+)}(x_i-x_j)\,|_{x_i^0=x_j^0=\tau}.$$

The unitary relations (5a) and (7) imply that

$$\Psi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau)=V\Phi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau),$$

where Φ_n is constructed from A_0 and Φ_0 in analogy to Ψ_n. Because A_0 is the free field,

$$\Phi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau)=(n!)^{-\frac{1}{2}}A_0^{(-)}(\mathbf{x}_1,\tau)\cdots A_0^{(-)}(\mathbf{x}_n,\tau)\Phi_0,$$

where $A_0^{(-)}$ is the negative-frequency part of A_0,

$$A_0^{(-)}(x)=\int_{y^0=x^0} d^3\mathbf{y}\left(A_0(y)\frac{\overleftrightarrow{\partial}}{\partial y^0}\Delta^{(-)}(x-y)\right),$$

$$-i\Delta^{(-)}(x-y)=(\Phi_0,A_0(y)A_0(x)\Phi_0).$$

The vectors Φ_n (a) are orthogonal for different values of n, (b) span the Hilbert space of the free field theory at time τ, and (c) obey the completeness identity

$$\sum_{n=0}^{\infty}\int d^3\mathbf{x}_1\cdots d^3\mathbf{x}_n d^3\mathbf{y}_1\cdots d^3\mathbf{y}_n|\Phi_n(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\tau)\rangle \times\Delta_n^{(+)}(\mathbf{x}_1,\cdots\mathbf{x}_n;\mathbf{y}_1,\cdots\mathbf{y}_n)\langle\Phi_n(\mathbf{y}_1,\cdots\mathbf{y}_n;\,\tau)|=1, \quad (17)$$

where

$$\Delta_n^{(+)}(\mathbf{x}_1,\cdots\mathbf{x}_n;\,\mathbf{y}_1,\cdots\mathbf{y}_n) =\prod_{j=1}^{n}\left[4i\frac{\partial^2}{\partial x_j^0\partial y_j^0}\Delta^{(+)}(x_j-y_j)\Big|_{x_j^0=y_j^0=\tau}\right].$$

Because of their unitary relation to the Φ_n, the set $\{\Psi_n\}$ have the analogous three properties for the theory of the field $A(\mathbf{x},\tau)$ at time τ. Now we can prove Eq. (16).

Consider the norm

$$\left\|\int d^3\mathbf{x} f(\mathbf{x})(\pi(\mathbf{x},\tau)-\dot{A}(\mathbf{x},\tau))\Psi_0\right\|^2$$

$$=\int d^3\mathbf{x}d^3\mathbf{y}\bar{f}(\mathbf{x})f(\mathbf{y})\{(\Psi_0,\pi(\mathbf{x},\tau)\pi(\mathbf{y},\tau)\Psi_0) -(\Psi_0,\pi(\mathbf{x},\tau)\dot{A}(\mathbf{y},\tau)\Psi_0)-(\Psi_0,\dot{A}(\mathbf{x},\tau)\pi(\mathbf{y},\tau)\Psi_0) +(\Psi_0,\dot{A}(\mathbf{x},\tau)\dot{A}(\mathbf{y},\tau)\Psi_0)\}, \quad (18)$$

where $f(\mathbf{x})$ is any function in C^∞, for which

$$\int\bar{f}(\mathbf{y})\left[\frac{\partial^2}{\partial x^0\partial y^0}i\Delta^{(+)}(y-x)\Big|_{x^0=y^0=\tau}\right]f(\mathbf{x})d^3\mathbf{x}d^3\mathbf{y}<\infty.$$

The first and last matrix elements in (18) can be evaluated simply. For the first,

$$(\Psi_0,\pi(\mathbf{x},\tau)\pi(\mathbf{y},\tau)\Psi_0)=(\Phi_0,\dot{A}_0(\mathbf{x},\tau)\dot{A}_0(\mathbf{y},\tau)\Phi_0)$$

by (8); for the last,

$$(\Psi_0,\dot{A}(\mathbf{x},\tau)\dot{A}(\mathbf{y},\tau)\Psi_0)=(\Phi_0,\dot{A}_0(\mathbf{x},\tau)\dot{A}_0(\mathbf{y},\tau)\Phi_0),$$

because $F^{(n)}=F_0^{(n)}$ for $n\leqslant 2$. The remaining two matrix elements can be evaluated using the identity (17),

$$(\Psi_0,\pi(\mathbf{x},\tau)\dot{A}(\mathbf{y},\tau)\Psi_0)=\sum_{k=0}^{\infty}\int d^3\mathbf{x}_1'\cdots d^3\mathbf{x}_k'd^3\mathbf{y}_1'\cdots \times d^3\mathbf{y}_k'(\Psi_0,\pi(\mathbf{x},\tau)\Psi_k(\mathbf{x}_1',\cdots\mathbf{x}_k';\,\tau)) \times\Delta_k^{(+)}(\mathbf{x}_1',\cdots\mathbf{x}_k';\,\mathbf{y}_1',\cdots\mathbf{y}_k') \times(\Psi_k(\mathbf{y}_1',\cdots\mathbf{y}_k';\,\tau),A(\mathbf{y},\tau)\Psi_0). \quad (19)$$

Now the unitary relations (5a, 6) and (7) imply that[12]

$$(\Psi_0,\pi(\mathbf{x},\tau)\Psi_k(\mathbf{x}_1',\cdots\mathbf{x}_k';\,\tau)) =(\Phi_0,\dot{A}_0(\mathbf{x},\tau)\Phi_k(\mathbf{x}_1',\cdots\mathbf{x}_k';\,\tau))=0,\quad k>1. \quad (20)$$

Thus the sum in (19) goes only from 0 to 1 (in the general case it would go from 0 to m). Since Ψ_k, by

[11] F. J. Dyson, Phys. Rev. **82**, 428 (1951).

[12] The vanishing of the matrix elements in (20) follows if we use the expansion of the free field in annihilation and creation operators. Then $\dot{A}_0$ is linear in the annihilation operators, and since the state Φ_k has the form of k creation operators acting on Φ_0, the matrix element vanishes for $k>1$. If the matrix element in (20) had m $\dot{A}_0$ operators, then this product would contain at most m annihilation operators and the matrix element would vanish for $k>m$.

construction, involves at most k $A(\mathbf{x},\tau)$ operators acting on Ψ_0, we can evaluate the matrix elements involving $\dot{A}$ in the sum $\sum_0^1$ in (19) (in general, in the sum $\sum_0^m$). These matrix elements equal the corresponding free-field matrix elements because the $F^{(n)}=F_0^{(n)}$ for $n\leqslant 2$ (in general, there will be m operators $\dot{A}$, and $F^{(n)}=F_0^{(n)}$, for $n\leqslant 2m$),

$$(\Psi_k(\mathbf{y}_1';\tau),\dot{A}(\mathbf{y},\tau)\Psi_0)=(\Phi_k(\mathbf{y}_1',\tau),\dot{A}_0(\mathbf{y},\tau)\Phi_0). \quad (21)$$

Now we replace the matrix elements in (19) by the free-field ones according to (20) and (21), and we can restore the sum to $\sum_0^\infty$ since the terms from 2 to ∞ (in general, from $m+1$ to ∞) vanish. Thus

$$\begin{aligned}(\Psi_0,\pi(\mathbf{x},\tau)\dot{A}(\mathbf{y},\tau)\Psi_0)=\sum_{k=0}^{\infty}\int d^3\mathbf{x}_1'\cdots d^3\mathbf{x}_k'd^3\mathbf{y}_1'\cdots \\ \times d^3\mathbf{y}_k'(\Phi_0,A_0(\mathbf{x},\tau)\Phi_k(\mathbf{x}_1',\cdots\mathbf{x}_k';\tau)) \\ \times\Delta_k^{(+)}(\mathbf{x}_1',\cdots\mathbf{x}_k';\mathbf{y}_1',\cdots\mathbf{y}_k') \\ \times(\Phi_k(\mathbf{y}_1',\cdots\mathbf{y}_k';\tau),\dot{A}_0(\mathbf{y},\tau)\Phi_0) \\ =(\Phi_0,\dot{A}_0(\mathbf{x},\tau)\dot{A}_0(\mathbf{y},\tau)\Phi_0), \quad (22)\end{aligned}$$

where the final line of (22) again makes use of the identity (17) (in the general case, (22) would involve m π and m $\dot{A}$ operators on the left and $2m$ $\dot{A}_0$ operators on the right). A similar discussion proves that the remaining matrix element also equals the corresponding free field one. Thus the four matrix elements in (18) are equal and cancel, the norm (18) vanishes, and we have completed the proof of (16).

Using (13), (12), (16), (5a, b), (7), and (15), we find

$$\begin{aligned}A(x)\Psi_0=A_\tau(x)\Psi_0=\int_{y^0=\tau}d^3\mathbf{y}\left(A(y)\frac{\overleftrightarrow{\partial}}{\partial y^0}\Delta(x-y)\right)\Psi_0 \\ =\int_{y^0=\tau}d^3\mathbf{y}\left(A(y)\frac{\partial}{\partial y^0}\Delta(x-y)-\pi(y)\Delta(x-y)\right)\Psi_0 \\ =V\int_{y^0=\tau}d^3\mathbf{y}\left(A_0(y)\frac{\overleftrightarrow{\partial}}{\partial y^0}\Delta(x-y)\right)\Phi_0=VA_0(x)\Phi_0, \\ \text{for all } x. \quad (23)\end{aligned}$$

Now we consider,

$$\begin{aligned}F^{(3)}(\xi_1,\xi_2)=(A(x_1)\Psi_0,A(\mathbf{x}_2,\tau)A(x_3)\Psi_0) \\ =(VA_0(x_1)\Phi_0,VA_0(\mathbf{x}_2,\tau)V^{-1}VA_0(x_3)\Phi_0)=F_0^{(3)}(\xi_1,\xi_2), \\ \text{for all } \xi_1,\ \xi_2. \quad (24)\end{aligned}$$

Thus (23) immediately implies $F^{(3)}=F_0^{(3)}$. Equation (23) also implies

$$F^{(4)}(\xi_1,\xi_2,\xi_3)=F_0^{(4)}(\xi_1,\xi_2,\xi_3),\quad \xi_2^0=0, \quad \text{otherwise } \xi_i \text{ arbitrary}. \quad (25)$$

But (10) allows the equality in (25) to be extended to all ξ_2 for which $\xi_2^2<0$. In particular, this region includes all the Jost points, and thus suffices to prove the equality (25) everywhere.[3,8] (The neighborhood of any Jost point is actually sufficient for this purpose.) Thus we have completed a chain of arguments which increases by two the number of vacuum expectation values which are equal everywhere. The argument now proceeds by induction from the case where $F^{(n)}=F_0^{(n)}$, $n\leqslant 2m$, following these steps:

1—Prove that

$$A_\tau(x_1)\cdots A_\tau(x_m)\Psi_0=A(x_1)\cdots A(x_m)\Psi_0$$

in analogy to (13) and the remarks following it.

2—Prove that

$$\pi(\mathbf{x}_1,\tau)\cdots\pi(\mathbf{x}_m,\tau)\Psi_0=\dot{A}(\mathbf{x}_1,\tau)\cdots\dot{A}(\mathbf{x}_m,\tau)\Psi_0,$$

and similar equalities where one or more of the $\pi(\mathbf{x}_i,\tau)$ and the corresponding $\dot{A}(\mathbf{x}_i,\tau)$ are replaced by $A(\mathbf{x}_i,\tau)$ on both sides, in analogy to (16) and the paragraphs following it.

3—Prove that

$$A(x_1)\cdots A(x_m)\Psi_0=VA_0(x_1)\cdots A_0(x_m)\Psi_0,$$

in analogy to (23).

4—Note that step 3 implies immediately that

$$F^{(2m+1)}(\xi_1,\cdots\xi_{2m})=F_0^{(2m+1)}(\xi_1,\cdots\xi_{2m}),\quad \text{for all } \xi_i.$$

5—Note that step 3 implies immediately that

$$F^{(2m+2)}(\xi_1,\cdots\xi_{2m+1})=F_0^{(2m+2)}(\xi_1,\cdots\xi_{2m+1}), \quad \text{for } \xi_m^0=0 \text{ and otherwise } \xi_i \text{ arbitrary},$$

that Lorentz invariance (10) extends this region to $\xi_m^2<0$, and that analytic continuation allows the extension to all ξ_m, and thus the complete equality $F^{(2m+2)}=F_0^{(2m+2)}$. Therefore, by induction, $F^{(n)}=F_0^{(n)}$, for all n, and by a theorem of Wightman[3] the two theories are completely equivalent.

4. COROLLARY TO HAAG'S THEOREM AND APPLICATION TO CLOTHED OPERATORS

Corollary.—A field theory which has the properties I, II, and a further property V (given below) is equivalent to the free-field theory throughout all space-time. Property V requires that, at a fixed time, the field $A(\mathbf{x},\tau)$, where

$$\begin{aligned}A(\mathbf{x},\tau)=(2\pi)^{-\frac{3}{2}}\int d^3\mathbf{k}(2k^0)^{-\frac{1}{2}} \\ \times[a(\mathbf{k},\tau)e^{-ikx}+a^*(\mathbf{k},\tau)e^{ikx}], \quad (26) \\ k^0=(\mathbf{k}^2+m^2)^{\frac{1}{2}},\quad x^0=\tau,\end{aligned}$$

has annihilation and creation operators, $a(\mathbf{k},\tau)$ and $a^*(\mathbf{k},\tau)$, which belong to the no-particle representation of the commutation relations,

$$[a(\mathbf{k},\tau),a^*(\mathbf{k}',\tau)]=\delta(\mathbf{k}-\mathbf{k}'), \tag{27a}$$

$$[a^*(\mathbf{k},\tau),a^*(\mathbf{k}',\tau)]=[a(\mathbf{k},\tau),a(\mathbf{k}',\tau)]=0, \tag{27b}$$

$$a(\mathbf{k},\tau)\Psi_0=0, \tag{28}$$

and have Euclidean transformation properties at a fixed time,

$$U(\mathbf{a},R)a(\mathbf{k},\tau)U(\mathbf{a},R)^{-1}=e^{iR\mathbf{k}\cdot\mathbf{a}}a(R\mathbf{k},\tau), \tag{29a}$$

$$U(\mathbf{a},R)a^*(\mathbf{k},\tau)U(\mathbf{a},R)^{-1}=e^{-iR\mathbf{k}\cdot\mathbf{a}}a^*(R\mathbf{k},\tau). \tag{29b}$$

Proof.—Define

$$\pi(\mathbf{x},\tau)=(2\pi)^{-\frac{3}{2}}i\int d^3\mathbf{k}(k^0/2)^{\frac{1}{2}} \times[-a(\mathbf{k},\tau)e^{-ikx}+a^*(\mathbf{k},\tau)e^{ikx}]. \tag{30}$$

Then $A(\mathbf{x},\tau)$ and $\pi(\mathbf{k},\tau)$ satisfy the commutation relations (4a) and (4b), and π has the Euclidean transformation properties (3). Further, the free field $A_0(\mathbf{k},\tau)$ and its canonical conjugate $\dot{A}_0(\mathbf{k},\tau)$ have an expansion in terms of annihilation and creation operators $a_0(\mathbf{k})$ and $a_0^*(\mathbf{k})$ analogous to (26) and (30) (for the free-field a_0 and a_0^* are time independent); and a_0 and a_0^* belong to the no-particle representation of the commutation relations (27a, b) and (28), where in (28) we understand that the free-field vacuum Φ_0 replaces Ψ_0. Now the work of Gårding and Wightman[13] on the irreducible representations of the canonical commutation relations includes the theorem that the no-particle representation is unique up to a unitary transformation. Thus a and a^* must be related to a_0 and a_0^* by a unitary transformation, and A and π must be related by the same transformation to A_0 and $\dot{A}_0$. But now we have shown that A and π satisfy all the conditions of Haag's theorem and thus $A(x)$ must be equivalent to the free field.

The corollary applies directly to clothed operators of the type discussed previously,[5,14] provided the requirements I, of relativistic invariance and, II, of no negative energy states are added to the clothing properties. These clothing properties are equivalent to V, that the theory belongs to the no-particle representation of the canonical commutation rules, together with the requirement that the creation operators create the physical one-particle state from the vacuum,

$$Ha^*(\mathbf{k},t)\Psi_0=E(\mathbf{k})a^*(\mathbf{k},t)\Psi_0, \tag{31}$$

where H is the total Hamiltonian and $E(\mathbf{k})$ the energy of a single particle of momentum $\mathbf{k}$. Because of the corollary, this type of relativistic, clothed theory must be identical to the free-field theory even without requiring (31).

5. REMARKS ABOUT CLOTHED OPERATORS

It is our opinion that the trivial nature of the relativistic (no-particle representation), canonical, clothed operators discussed in Sec. 4 was to be expected, and is not discouraging for the use of some notion of clothing in the relativistic theory of interacting fields. We are presently investigating, in collaboration with S. S. Schweber, relativistic theories in which properties III and IV are dropped, and replaced by the requirement of local commutativity,

$$[A(x),A(y)]=0, \quad (x-y)^2<0,$$

and (31) is dropped and replaced by[15]

$$A(x)\Psi_0=A^{\mathrm{in,out}}(x)\Psi_0.$$

6. ACKNOWLEDGMENT

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[13] L. Gårding and A. S. Wightman, Proc. Natl. Acad. Sci. U. S. **40**, 622 (1954).

[14] Compare the simple particle interpretations of reference 4.

[15] See Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955), or O. W. Greenberg and A. S. Wightman (to be published) for discussions of the in and out operators.

[16] I have been informed by Professor A. S. Wightman that Professor R. Jost has independently obtained a proof of Haag's theorem.