binding on screening at various energies is needed in order to evaluate measurements in hydrogen. Conduction electrons in metallic absorbers may also affect the screening. In considering measurements and calculations with accuracy $\lesssim 1\%$ it may be necessary to calculate the contribution of the second order diagrams in the perturbation theory. Until such calculations are made, no significance can be given to the slight deviations of experiment from theory seen in the graphs.

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I am grateful to Mr. J. Kirchgessner from whom I inherited the pair spectrometer, and to the operating crews of the Cornell synchrotron whose help was an essential part of this experiment. The late Mr. C. F. Van Amber gave me much assistance in obtaining the high purity absorbers. Above all, I want to thank Professor John DeWire and Professor Albert Silverman, who gave me help, advice, and encouragement throughout the course of the experiment.

Note added in proof.—The final results of Wyckoff and Koch, which Dr. Koch has kindly sent me before publication, seem to indicate a real discrepancy between the experimental and theoretical nuclear-field pair cross sections at energies below 100 Mev. If this discrepancy exists, Figs. 4, 5, and 6 show that it is independent of Z but depends on energy.

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Radiative Muon Capture

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The theory of radiative muon capture is developed. The discussion includes both parity conserving and nonconserving effects. The Gell-Mann weak magnetic term and the induced pseudoscalar are included, along with comparable relativistic effects in the nucleons. The theory is applied to light nuclei and especially to the radiative Godfrey reaction $\mu^- + {}_6C^{12} \rightarrow \nu + \gamma + {}_5B^{12}$. An experiment to detect the induced pseudoscalar directly is proposed.

I. INTRODUCTION

THE theory of radiative K capture has a long history. The first computation was made, at the suggestion of Oppenheimer, by Morrison and Schiff in 1940.[1] They found that the photon spectrum to be expected in allowed electron K capture, neglecting all relativistic and screening effects, is of the form $(1-x)^2 x\,dx$, where x is the photon energy in units of the maximum photon energy. This formula obtains for both Fermi and Gamow-Teller transitions and allows no distinction between them. The first experiments exhibiting this spectrum were done by Bradt *et al.*[2] on the nucleus Fe^{55}. The theory for allowed transitions was refined by Jauch[3] and is most completely given by Glauber and Martin.[4] The latter authors consider relativistic, Coulombic, and screening corrections to the Morrison and Schiff computation and experiments by Lindquist and Wu[4] are in excellent agreement with their elaborate theoretical treatment. The net conclusion of this work is that allowed electron radiative K capture is well described by a four-fermion coupling with photon emission superimposed in the natural way.

The shape of the photon spectrum is, of course, independent of parity conservation or nonconservation in the electron capture event. However, it was realized[5] shortly after the discovery of parity nonconservation, that in a parity-nonconserving interaction the γ's coming from the inner bremsstrahlung of the electron undergoing capture could be circularly polarized, partially or completely. In fact on the two-component theory for V and A the circular polarization is 100% right independent of any details of nuclear matrix elements. Recently Mann *et al.*[6] have measured the circular polarization in K capture in Al^{37} and have obtained close agreement with this prediction of the $V-A$ two-component theory. In general, in a given transition, the degree of circular polarization is a measure of the relative strengths of the covariants involved in the four-fermion coupling. We shall return to this point below when we discuss the "induced" pseudoscalar in radiative muon K capture.

With the advent of intense muon beams[7] one may contemplate the study of radiative muon K capture experimentally. We shall see later that the anticipated

[1] P. Morrison and L. I. Schiff, Phys. Rev. **58**, 24 (1940).
[2] H. Bradt *et al.*, Helv. Phys. Acta **19**, 222 (1946).
[3] J. M. Jauch, Oak Ridge National Laboratory Report ORNL-1102, 1957 (unpublished).
[4] R. J. Glauber and P. C. Martin, J. phys. radium **16**, 573 (1955); Glauber, Martin, Lindquist, and Wu, Phys. Rev. **101**, 905 (1956); P. C. Martin and R. J. Glauber, Phys. Rev. **109**, 1307 (1958).
[5] R. E. Cutkosky, Phys. Rev. **107**, 330 (1957).
[6] L. G. Mann *et al.*, Phys. Rev. Letters **1**, 34 (1958).
[7] An intensity of 10^5 muons/cm^2 sec is now obtainable.

ratio of radiative to nonradiative muon capture rates is about 10^{-4} and hence, as an example, in the Godfrey reaction,[8] $\mu^- + {}_6C^{12} \rightarrow {}_5B^{12} + \nu$, where the ordinary muon capture rate to the ground state is observed to be about 10^4 sec^{-1},[9] we expect an absolute radiative rate of about one a second. An encouraging feature of the process is the relative abundance of high-energy gammas in the allowed spectrum, Fig. 1. This spectrum has its maximum at $x=\frac{1}{3}$ or about 30 Mev in the case of the Godfrey experiment where the maximum photon energy is 91.4 Mev—taking the muon mass as 103.8 Mev. However, it is clear from Fig. 1 that a considerable fraction of the γ's lie above 50 Mev which is approximately the maximum photon energy emitted in external or internal bremsstrahlung during the process $\mu^- \rightarrow e^- + \nu + \bar{\nu}$; i.e., ordinary muon decay. The high-energy γ's also lie well above all of the background from nuclear transitions. Hence, if one observes such high-energy photons, $E_\gamma > 50$ Mev, they must come from the muon capture process. Experiments are now under way to detect them.[10]

The *theory* of radiative muon K capture is much more complex, and hence more interesting, than that of radiative electron capture. This has, of course, to do with the large ratio of muon to electron mass, ~ 208. Because of it the muon Bohr orbit is some two hundred times smaller than the electron Bohr orbit, and the neutrino wavelength is comparable to or smaller than the nuclear radius. For carbon, $\nu R \sim 2$, where R is the nuclear radius and for heavier nuclei it is still larger. Thus the usual beta-decay expansion into degrees of forbiddeness is quite slowly convergent, even for carbon, and hence sizable departures from the allowed spectral shape, $(1-x)^2 x\, dx$, are to be expected for all nuclei. This clearly recognized by Cantwell[11] who, in 1956, gave the spectral shapes for radiative muon capture from various light nuclei. The case of carbon, the only example of muon capture to a particular final state which has been studied experimentally, was not considered by him and we shall give the spectral shape below. Cantwell also discussed corrections to radiative muon capture which arise when the nuclear extension and shape are taken into account in the wave function of the muon after it has radiated. In electron K capture, from medium heavy nuclei, the intermediate wave function can be taken to be that of a Dirac electron in a point Coulomb field and this is what Glauber and Martin[4] do. In muon K capture, due to the small Bohr radius, the extension of the Coulomb field must be considered even for medium-heavy nuclei. For light nuclei, like carbon, we can with very little error ignore all Coulomb effects in the intermediate states and this is done in the calculations below. We emphasize that these results are meant to apply to nuclei for which $Z\alpha < 1$. Moreover, we shall replace the K-shell muon wave function by its value at the origin, an excellent approximation for carbon, but entirely inadmissible for heavy nuclei.

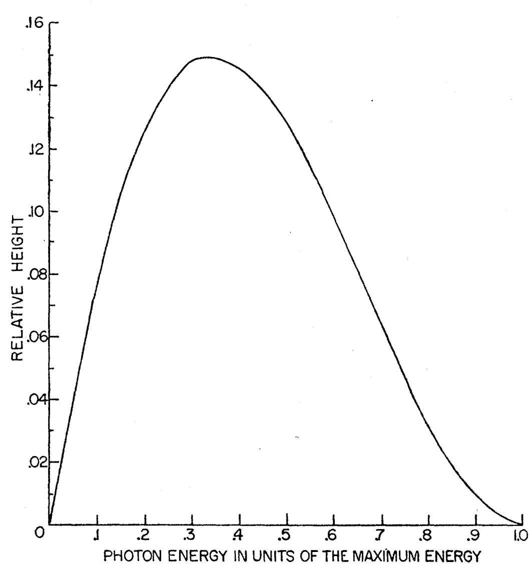

FIG. 1. The "allowed" photon spectrum.

An especially interesting feature of radiative muon capture, again springing from the large muon mass, is the presence of induced couplings in the S matrix for the process. If we begin with an underlying Hamiltonian which is $V-A$ in the weak couplings, then, as is now well understood, there can appear terms in the S matrix which have other covariant forms. These terms are induced by the strong couplings of the capturing nucleons to pions. The effective pseudoscalar[12] and the Gell-Mann weak magnetic term[13] are of this character. In the next section we describe the general formalism for taking such terms into account, and our formulas for the spectra include contributions from them and from other relativistic effects in the nucleon velocities of the same order of magnitude.

Parity-nonconserving quantities in radiative muon capture were first discussed by Huang, Yang, and Lee.[14] These authors consider the circular polarization of the γ's and the angular distribution of the γ's relative to the muon spin of a muon supposed 100% polarized. The

[8] T. N. K. Godfrey, Ph.D. thesis, Princeton University, 1954 (unpublished); and Phys. Rev. **92**, 512 (1953).

[9] Harrison, Argo, Kruse, and McGuire, *Gatlinburg Conference on Weak Interactions, October, 1958* (unpublished); Burgman *et al.*, Phys. Rev. Letters **1**, 469 (1958); J. G. Fetkovich *et al.*, *Gatlinburg Conference on Weak Interactions, October, 1958* [Bull. Am. Phys. Soc. Ser. II, 4, 81 (1959)]; W. Love *et al.*, *Gatlinburg Conference on Weak Interactions, October, 1958* [Bull. Am. Phys. Soc. Ser. II, 4, 81 (1959)]. The first reference is particularly impressive since this group observed the γ's coming from captures to excited states of ${}_5B^{12}$. They show that 90% of the captures are to the ground state.

[10] C. York (private communication); C. Rubbia (private communication).

[11] R. M. Cantwell, Ph.D. thesis, Washington University, 1956 (unpublished).

[12] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 355 (1958); L. Wolfenstein, Nuovo cimento **8**, 882 (1958).

[13] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); M. Gell-Mann, Phys. Rev. **111**, 362 (1958); J. Bernstein and R. R. Lewis, Phys. Rev. **112**, 232 (1958).

[14] Huang, Yang, and Lee, Phys. Rev. **108**, 1348 (1957).

two quantities are, in fact, not independent, as we shall show later. It is apparent from the formulas of Huang *et al.* that the $V-A$ theory, without induced couplings, predicts 100% right circularly polarized quanta for radiative muon as well as electron capture, irrespective of nuclear structure. One of the principal results of our work is the observation that the induced pseudoscalar term, *by itself*, yields quanta 100% *left* circularly polarized and hence that the over-all circular polarization of photons in muon capture will be less than 100% and in this way the induced pseudoscalar may be detected directly. The actual circular polarization depends on the nuclear matrix elements, and below we give a prediction for the radiative Godfrey experiment based on an analysis by Fujii and Primakoff[15] for the nonradiative Godfrey reaction. The computation of the induced pseudoscalar and Gell-Mann terms in radiative capture raises some points concerning gauge invariance which are discussed in Secs. III and IV.

In all of this work we shall not consider the effect of the hyperfine interaction between muon and nuclear spins on capture rates, etc.[16] It was felt that at the present stage of experiments this was an irrelevant complication, especially since our principal application was to ${}_6C^{12}$, a spin zero nucleus. It is an entirely straightforward matter to amend the formulas to include the hyperfine coupling for nuclei with spin.

Hence, in summary, the purpose of the present paper is to bring the study of radiative muon capture up to date by developing the consequences of the full $V-A$ theory, including induced couplings and other relativistic effects. The influence of these terms on the photon spectrum and circular polarization is computed with particular emphasis on the radiative Godfrey reaction $\mu^-+{}_6C^{12}\rightarrow {}_5B^{12}+\nu+\gamma$.

With this introduction, we may now turn to Sec. II in which the details of the formalism are presented.

II. FORMALISM

In this section we develop the formalism needed to compute nuclear radiative muon capture on the $V-A$ two-component theory.

The Hamiltonian for muon capture is written in the Feynman–Gell-Mann form as[13]

$$H=H_0+H_{\mathrm{st}}+8^{\frac{1}{2}}GJ_\mu J_\mu{}^\dagger. \tag{1}$$

Here H_0 is the free Hamiltonian, H_{st} is the strong interaction Hamiltonian, including electromagnetic couplings. The weak interaction is written as a current J_μ interacting with itself with the universal unrenormalized strength G. J_μ may be split into a vector and an axial vector part:

$$J_\mu=J_\mu{}^V+J_\mu{}^A.$$

[15] A. Fujii and H. Primakoff (to be published).

[16] Bernstein, Lee, Yang, and Primakoff, Phys. Rev. **111**, 313 (1958).

We shall assume, with Feynman and Gell-Mann, that $J_\mu{}^V$ obeys the differential conservation law $\partial_\mu J_\mu{}^V=0$. For muon capture the part of $J_\mu{}^V$ which is relevant is

$$J_{\mu-}{}^V=\bar{\psi}_N\gamma_\mu\tau^{(-)}\psi_N+i[\phi_\pi T^{(-)}\partial_\mu\phi_\pi{}^*-\phi_\pi{}^*T^{(-)}\partial_\mu\phi_\pi]+\bar{\psi}_\nu\gamma_\mu\psi_\mu. \tag{2}$$

Thus written, $J_{\mu-}{}^V$ changes protons into neutrons. The pion current term guarantees the conservation law $\partial_\mu J_\mu{}^V=0$. We shall ignore possible strange-particle effects on the current.

The axial vector current does not obey a conservation law, and we shall not specify it explicitly. However, if we define the G conjugation operator in the usual way,

$$G=Ce^{i\pi I_2}, \tag{3}$$

where C is the charge conjugation operation and $\mathbf{I}$ the total isotopic spin vector, then $J_{\mu-}{}^V$ as written above satisfies

$$GJ_{\mu-}{}^VG^{-1}=J_{\mu-}{}^V. \tag{4}$$

We shall now assume that $J_{\mu-}{}^A$ is composed of terms satisfying[17]

$$GJ_{\mu-}{}^AG^{-1}=-J_{\mu-}{}^A. \tag{5}$$

The conventional axial vector current $\bar{\psi}_N(-i\gamma_\mu\gamma_5)\tau^{(-)}\psi_N$ has this property. This restriction on the currents, plus Lorentz invariance, implies that the matrix elements of the currents between physical nucleons take the form

$$\langle n|J_\mu{}^A|p\rangle=\bar{u}_n[A^{(\mu)}(q^2)(-i\gamma_\mu\gamma_5)+B^{(\mu)}(q^2)q_\mu\gamma_5]u_p, \tag{6}$$

and

$$\langle n|J_\mu{}^V|p\rangle=\bar{u}_n[C^{(\mu)}(q^2)\gamma_\mu-iD^{(\mu)}(q^2)\sigma_{\mu\nu}q_\nu]u_p. \tag{7}$$

Here $q_\mu=(p-n)_\mu$. The four functions $A^{(\mu)}$, $B^{(\mu)}$, $C^{(\mu)}$, and $D^{(\mu)}$ are discussed below. It is the G-conjugation symmetry which eliminates the other two possible covariants in the matrix element of the currents, which are allowed by Lorentz invariance; i.e., q_μ for the vector and $\gamma_5\sigma_{\mu\nu}q_\nu$ for the axial vector.

As has been discussed in the literature,[12,15] in the universal Fermi theory $A^{(\mu)}(0)$ is identified with the axial vector coupling constant of β decay. However, in radiative muon capture $q_\lambda=\nu_\lambda+K_\lambda-\mu_\lambda$, so that q^2 can be the order of m^2, where m is the muon mass. In any case there are strong theoretical arguments[12] to show that $A^{(\mu)}$ is a slowly varying function of q^2. In fact Goldberger and Treiman[12] have indicated that

$$\frac{A^{(\mu)}(q^2)}{A^{(\mu)}(0)}\simeq 1-\frac{1}{\pi}\frac{q^2}{4M_p{}^2}\simeq 1. \tag{8}$$

On the strength of this we shall neglect the momentum dependence of $A^{(\mu)}$ and identify $GA^{(\mu)}$ with the renormalized axial vector coupling constant of beta decay, $g_A{}^{(\beta)}$.

[17] For a discussion of other possible theories see S. Weinberg, Phys. Rev. (to be published).

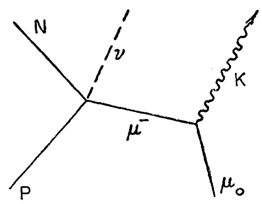

FIG. 2. The principal Feynman diagram for radiative muon capture by a proton. K is the photon momentum and μ_0 and μ' are the initial and intermediate muon momenta.

The quantity $mB^{(\mu)}(q^2)G$ is shown to be an induced pseudoscalar coupling constant in ordinary muon capture where the momentum difference q_λ may be transferred to the lepton matrix element and the Dirac equation used to reduce it to pseudoscalar form. It has been shown[12] that $mB^{(\mu)}G \equiv g_P^{(\mu)} \simeq \pm 8 g_A^{(\beta)}$. The plus sign goes with the dispersion-theoretic calculation of g_P and either sign is possible if $B^{(\mu)}$ is estimated by using the empirical $\pi \rightarrow \mu + \nu$ decay rate. Evidence from the observed rate of the Godfrey reaction,[9,15]

$$\mu^- + {}_6\mathrm{C}^{12} \rightarrow {}_5\mathrm{B}^{12} + \nu,$$

seems to favor the dispersion-theoretic sign.

The combination $GC^{(\mu)}(q^2)$ is the vector coupling constant. Since the vector current is taken to be conserved,

$$C^{(\mu)}(q^2)/C^{(\mu)}(0) = F_1^V(q^2)/F_1^V(0). \qquad (9)$$

Here $F_1^V(q^2)$ is the isotopic vector part of the charge form factor of the proton as measured in electron-proton scattering.[18] If we suppose that the *total* charge form factor of the proton $F_1^{(p)} = F_1^S + F_1^V$ is approximately $2F_1^V$, i.e., if we make use of the fact that the neutron has approximately zero charge radius, then the variation in the vector coupling constant as a function of momentum transfer can be obtained from the formula

$$F_1^V(q^2)/F_1^V(0) \simeq 1 - \tfrac{1}{6} q^2 \langle r_p^2 \rangle. \qquad (10)$$

Even for $q \sim m$, Eq. (10) indicates that $F_1^V(q^2)/F_1^V(0)$ departs from unity by only three percent. Hence we shall ignore its variation over the photon spectrum and take $GC^{(\mu)} = g_V^{(\beta)}$, the observed β-decay coupling constant. The conservation of the vector current implies that $C^{(\mu)} \simeq 1$.

On the Feynman–Gell-Mann theory the function $D^{(\mu)}$ is directly related to $F_2(q^2)$, the magnetic form factor of electron-nucleon scattering. Precisely speaking,

$$\frac{D^{(\mu)}(q^2)}{C^{(\mu)}(q^2)} = \frac{F_2^{(p)}(q^2) - F_2^{(n)}(q^2)}{F_1^{(p)}(q^2) - F_1^{(n)}(q^2)} \cong \frac{\mu_p - \mu_n}{2M}. \qquad (11)$$

In the last step we have replaced the experimental form factors by their value at zero momentum transfer.[18]

If the induced couplings are, for the moment, neglected then the Feynman diagram in Fig. 2 represents the dominant contribution to radiative K capture in the primitive process $\mu^- + p \rightarrow n + \nu + \gamma$. We shall neglect, throughout this work, contributions to the photon emission due to the radiation by the proton. The larger energy denominator makes these contributions an order of magnitude smaller and of the same order as the uncertainties in the nuclear matrix elements. When induced couplings are included, then other diagrams involving charged pions, which are responsible for the presence of the induced couplings, must be included. We shall discuss these in detail in the next two sections. In the present section we shall develop the consequences of the $V-A$ theory with no induced couplings, but including relativistic corrections. This means we shall evaluate the matrix element associated with Fig. 2. Further, we shall make the assumption that nuclear K capture is representable as the K capture of a muon by a collection of noninteracting physical protons. In other words we shall neglect meson exchange contributions to muon capture. Estimates in the literature indicate that these are also effects of about 10%.[19]

If α and β are two directions of photon polarization then, under the assumptions discussed above, it is easy to show that the transition probability summed over muon and neutrino spins can be written as

$$\sum_{s_\nu, s_\mu} M_\alpha M_\beta^\dagger = \frac{e^2}{32K^3 m^2 \nu} \frac{1}{2I_i+1} \frac{1}{\pi a^3} \sum_{m_i, m_f} \sum_{i,j=1}^{2} \sum_{\lambda,\sigma=1}^{4} C_i C_j^* N_{\lambda\sigma}^{ij}$$
$$\times \mathrm{Tr}[(1+\gamma_4)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_\beta^* \gamma K \gamma_4 O_\sigma^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda^i \gamma K \boldsymbol{\epsilon}_\alpha \cdot \boldsymbol{\gamma}]. \qquad (12)$$

The new notation introduced here is the following: ϵ_α is the photon polarization vector in the direction α; ϵ_α may be complex if the polarization is not linear; $C_1 = g_V^{(\beta)}$, $C_2 = g_A^{(\beta)}$, $O_\lambda^1 = -i\gamma_4\gamma_\lambda\gamma_5$, $O_\lambda^2 = \gamma_4\gamma_\lambda$, I_i is the total initial nuclear spin, m_i and m_f are the components of the nuclear spins over which we are summing, and a is the muon Bohr radius. We have replaced the muon wave function by its value at the origin and have used the free-particle Dirac Green's function for the muon in the intermediate state. The nuclear matrix elements are contained in the quantity $N_{\lambda\sigma}^{ij}$; that is,

$$N_{\lambda\sigma}^{ij} = M_\lambda^i M_\sigma^{j*}, \qquad (13)$$

where

$$M_\lambda^i = \int \cdots \int \psi_{\mathrm{nuc}}^{f*} \{ \sum_{i=1}^{A} \delta(\mathbf{r} - \mathbf{r}_i) \boldsymbol{\tau}_i^{(-)}$$
$$\times \exp[-i(\boldsymbol{\nu} + \mathbf{K}) \cdot \mathbf{r}] J_\lambda^{i\,\mathrm{eff}} \} \psi_{\mathrm{nuc}}^i d\mathbf{r}_1 \cdots d\mathbf{r}_A d\mathbf{r}. \qquad (14)$$

The $J_\lambda^{i\,\mathrm{eff}}$ are the effective vector and axial vector

[18] See, for example, J. Bernstein and M. Goldberger, Revs. Modern Phys. **30**, 465 (1958), for a discussion of these form factors.

[19] Blin-Stoyle, Gupta, and Primakoff (to be published).

currents. These are defined below and have been derived by transcribing the results of free proton capture under the assumption that the nuclear protons capture as if they were independent. We have retained in the J's the leading relativistic corrections which are of order ν/M or K/M and we have dropped terms involving P_i/M, where P_i is the initial proton momentum. The induced couplings will have a similar form but they will be larger since the induced coupling constants are greater than the vector and axial vector coupling constants. Under these assumptions,

$$\mathbf{J}^{1\,\mathrm{eff}}=\boldsymbol{\sigma},\quad J_4^{1\,\mathrm{eff}}=i\boldsymbol{\sigma}\cdot(\boldsymbol{\nu}+\mathbf{K})/2M, \tag{15a}$$

and

$$\mathbf{J}^{2\,\mathrm{eff}}=i\boldsymbol{\sigma}\cdot(\mathbf{K}+\boldsymbol{\nu})\boldsymbol{\sigma}/2M,\quad J_4^{2\,\mathrm{eff}}=1. \tag{15b}$$

Using Eq. (12) we may study four quantities:

(1) the γ energy spectrum and the capture rate;
(2) the γ, ν angular correlation:
(3) the circular polarization of the γ's;
(4) for polarized muons, the angular distribution of the γ's relative to the muon spin direction.

The first two quantities are evidently independent of parity conservation or nonconservation in the capture process, and the second two vanish if parity is conserved. In fact there is at present no *experimental* evidence that parity is not conserved in muon capture and so the observation of 3 or 4 is of some interest even apart from subtle details.

We shall begin the general discussion by proving that a measurement of 3 is equivalent to a measurement of 4 under the assumptions which lead to Eq. (12).

To this end we define a quantity $\beta(K)$, the circular polarization of photons of energy K, as follows:

$$\beta(K)=[N_R(K)-N_L(K)]/[N_R(K)+N_L(K)]. \tag{16}$$

$N_R(K)$ is the number of right circularly polarized photons with energy K and likewise N_L is the number of left circularly polarized quanta. We may compute N_R from Eq. (12) by setting $\epsilon_\alpha=\epsilon_\beta=\epsilon_R=(\epsilon_1+i\epsilon_2)/\sqrt{2}$. The lepton trace in Eq. (12) can be rewritten by making use of the identity

$$\begin{aligned}\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_R\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_R^*=\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_R^*\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_R=1-\boldsymbol{\sigma}\cdot\mathbf{K}/K,\\ \boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_L\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_L^*=\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_L^*\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_L=1+\boldsymbol{\sigma}\cdot\mathbf{K}/K,\end{aligned} \tag{17}$$

where $\mathbf{K}/K$ is the unit photon momentum vector. Thus

$$\begin{aligned}&\mathrm{Tr}[(1+\gamma_4)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_R^*\gamma K\gamma_4 O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i\gamma_K\epsilon_R\cdot\boldsymbol{\gamma}]\\ &=8K^2\,\mathrm{Tr}\left[\left(\frac{1-\boldsymbol{\sigma}\cdot\mathbf{K}/K}{2}\right)\left(\frac{-i\gamma K\gamma_4}{2K}\right)\times O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i\right],\\ &\mathrm{Tr}[(1+\gamma_4)\boldsymbol{\gamma}\cdot\boldsymbol{\varepsilon}_L^*\gamma K\gamma_4 O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i\gamma_K\epsilon_L\cdot\boldsymbol{\gamma}]\\ &=8K^2\,\mathrm{Tr}\left[\left(\frac{1+\boldsymbol{\sigma}\cdot\mathbf{K}/K}{2}\right)\left(\frac{-i\gamma K\gamma_4}{2K}\right)\times O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i\right].\end{aligned} \tag{18}$$

Written in this form, first suggested by Cutkosky,[5] Eq. (18) is identical to the lepton trace for the beta decay of a nucleus with emission of a zero-mass positron with a definite helicity characterized by the projection operators $\frac{1}{2}(1\pm\boldsymbol{\sigma}\cdot\mathbf{K}/K)$. Thus we have the theorem that the circular polarization of a photon emitted in radiative K capture is identical to the helicity of a zero-mass positron emitted in beta decay. It is instructive to compare the lepton traces involved in N_R-N_L and N_R+N_L. Using Eq. (18) and the identity $\boldsymbol{\sigma}=-i\gamma_4\boldsymbol{\gamma}\gamma_5$, we have

$$N_R+N_L\sim\mathrm{Tr}[(-i\gamma K\gamma_4)O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i],$$

while

$$N_R-N_L\sim\mathrm{Tr}[\gamma_5(-i\gamma K\gamma_4)O_\sigma{}^{j\dagger}(1+\gamma_5)(-i\gamma\nu)\gamma_4 O_\lambda{}^i].$$

The two traces are identical except for a factor of γ_5 in N_R-N_L. Hence for a given covariant, $\beta=\pm1$ depending on whether γ_5 commutes or anticommutes with $O_\sigma{}^i$. This is a familiar result from the usual discussion of helicity in beta decay and establishes the fact that on the two-component $V-A$ theory, without induced couplings, the photons are 100% right circularly polarized in radiative K capture. The induced couplings must be examined more closely and we shall do so in the next sections. It is clear that photons with opposite circular polarization cannot interfere in the energy spectrum.

We shall now show that the circular polarization determines the angular distribution, relative to the muon spin direction, of photons emitted in the radiative muon capture of polarized muons.[20] The essential point in the argument is that during the photon emission by the muon the proton behaves like a spectator as far as angular momentum balance goes. Thus the photon angular distribution relative to the muon spin will be of the form $1+\beta\cos\theta$ and for a 100% polarized muon we shall show that this β (which is here unity) is just the circular polarization discussed above.

We shall consider a photon which is 100% right circularly polarized and suppose that $\theta=\pi$. This situation is shown in Fig. 3. The replacement of the muon wave function by its value at the origin is equivalent to neglecting the muon momentum in the K shell. Thus the intermediate muon will emerge at $\theta=0°$ in order to conserve energy-momentum. However, this configura-

[20] The line of argument which we follow was suggested to us by R. E. Cutkosky in a private communication for which we are grateful.

tion can never conserve angular momentum in the intermediate state since the initial angular momentum of $\frac{1}{2}$ in the positive z direction cannot be matched after the photon emission. Thus the photon angular distribution must be $1+\cos\theta$ and in general the correlation coefficient is the circular polarization β. It is clear that this argument is not more general than the assumptions which lead to Eq. (12).

Of course, as the muon cascades into the K shell, it becomes depolarized even if it was originally 100% polarized by the $\pi\rightarrow\mu+\nu$ decay. Therefore the *observed* angular distribution will have the form $(1+P\beta\cos\theta)$, where P is the residual muon polarization. In practice one need not rely on theoretical estimates of P since the electron asymmetry in the competing process $\mu\rightarrow e+\nu+\bar{\nu}$ measures P directly. Empirically P is the order of 10 to 20% for spin-zero nuclei and is close to zero for nuclei with spin.

We may now proceed by means of Eq. (12) to a computation of the photon spectrum. This is given by

$$N(x)dx=\frac{e^2}{32\pi^4}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}\frac{1}{2I_i+1}(1-x)^2x\,dx \times\sum_{m_im_f}\sum_{i,j=1}^{2}\sum_{\lambda,\sigma=1}^{4}\int\int C_iC_j^* \times N_{\lambda\sigma}^{ij}\tfrac{1}{4}\,\mathrm{Tr}\left[\frac{(-i\gamma K)}{K}\gamma_4O_\sigma^{j\dagger}(1+\gamma_5) \times\frac{(-i\gamma\nu)}{\nu}\gamma_4O_\lambda^{i}\right]d\Omega_\nu d\Omega_\gamma. \quad (19)$$

The notation is as before except that we have introduced the variable $x=K/\epsilon_f$, where ϵ_f is the photon end-point energy and is given by $\epsilon_f\cong m[1+(E_{\mathrm{nuc},i}-E_{\mathrm{nuc},f})/m]$, the E_{nuc} being the nuclear energies. We have neglected the muon K-shell binding energy. The neutrino momentum is given by $\nu=\epsilon_f(1-x)$. For orientation we shall, for the moment, ignore the relativistic corrections to the nuclear matrix elements and consider the nonrelativistic terms, which will be the leading terms in the theory. Making this assumption, we have

$$N(x)dx=\frac{e^2}{32\pi^4}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}(1-x)^2xdx\frac{1}{2I_i+1}\sum_{m_im_f} \times\int\int\left\{|g_V|^2\left[1+\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}\right]|M_4{}^2|^2 +|g_A|^2\left[\mathbf{M}^1\cdot\frac{\mathbf{K}}{K}\mathbf{M}^{1*}\cdot\frac{\boldsymbol{\nu}}{\nu}+\mathbf{M}^{1*}\cdot\frac{\mathbf{K}}{K}\mathbf{M}^1\cdot\frac{\boldsymbol{\nu}}{\nu}\right] +|g_A|^2\left[1-\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}\right]\mathbf{M}^1\cdot\mathbf{M}^{1*}\right\}d\Omega_\nu d\Omega_\gamma, \quad (20)$$

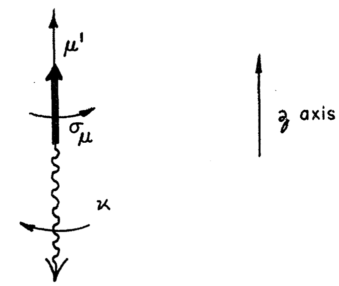

FIG. 3. The emission of a 100% right circularly polarized photon is shown for $\theta=\pi$.

where, as a reminder,

$$M_4{}^2=\int\cdots\int\psi_{\mathrm{nuc}}{}^{f*}\{\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)\tau_i{}^{(-)} \times\exp[-i(\boldsymbol{\nu}+\mathbf{K})\cdot\mathbf{r}]\}\psi_{\mathrm{nuc}}{}^{i}d\mathbf{r}_1\cdots d\mathbf{r}_Ad\mathbf{r},$$

and

$$\mathbf{M}^1=\int\cdots\int\psi_{\mathrm{nuc}}{}^{f*}\{\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)\tau_i{}^{(-)} \times\exp[-i(\boldsymbol{\nu}+\mathbf{K})\cdot\mathbf{r}]\boldsymbol{\sigma}_i\}\psi_{\mathrm{nuc}}{}^{i}d\mathbf{r}_1\cdots d\mathbf{r}_Ad\mathbf{r}.$$

It is clear that any $V-A$ interference term would involve the vectors $\mathbf{K}$ and $\boldsymbol{\nu}$ in the combination $\mathbf{K}\times\boldsymbol{\nu}$. But if effects in the nucleon velocities are ignored, the matrix element $\mathbf{M}^1$ can only depend on the vector $\mathbf{K}+\boldsymbol{\nu}$ times a suitable pseudoscalar function, since the initial and final spins are summed over. Hence there are no $V-A$ interference terms in this approximation although there are such terms when nucleon velocity effects are included, as we show below.

Now to obtain an explicit photon spectrum from Eq. (20), the nuclear matrix elements must be freed of their dependence on the photon momenta. This is done by expanding the plane wave $\exp[-i(\boldsymbol{\nu}+\mathbf{K})\cdot\mathbf{r}]$ into the spherical harmonic series

$$\exp[-i(\boldsymbol{\nu}+\mathbf{K})\cdot\mathbf{r}]=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(i)^lj_l(|\boldsymbol{\nu}+\mathbf{K}|r) \times Y_{lm}(\boldsymbol{\nu}+\mathbf{K})Y_{lm}(\mathbf{r}). \quad (21)$$

The j's are the spherical Bessel functions and the Y's are spherical harmonics. This is not the usual beta-decay expansion into degrees of forbiddenness, which would be very slowly convergent since $|\boldsymbol{\nu}+\mathbf{K}|r\geqslant1$ for all but the very lightest nuclei. To get a rough idea that Eq. (21) is a reasonably convergent expansion for muon capture we can replace $j_l(x)$ by its value for small arguments,

$$j_l(x)\underset{x\rightarrow0}{\longrightarrow}\frac{x^l}{(2l+1)!!},$$

where $(2l+1)!!\equiv1\times3\times5\times\cdots\times(2l+1)$. From this we see, that although $x\geqslant1$, the double factorials cause rather rapid convergence of the series, especially since angular momentum and parity considerations require that only every other order can contribute to a given transition. For very light nuclei $A \leqslant 10$, it is a good approximation to use this asymptotic formula to evaluate spectra. In this case the product of any two matrix elements can be expressed as

$$M_\lambda{}^i M_\sigma{}^{j*} = 16\pi^2 \sum_{l,l'=0}^{\infty} \sum_{m,m'} (i)^{l-l'} \frac{|\mathbf{v}+\mathbf{K}|^{l+l'}}{(2l+1)!!(2l'+1)!!} \times Y_{lm}(\mathbf{v}+\mathbf{K}) Y^*_{l'm'}(\mathbf{v}+\mathbf{K}) m_\lambda{}^i(lm) m_\sigma{}^{j*}(l'm'),$$

with

$$m_\lambda{}^i(lm) = \int \cdots \int \psi_{\text{nuc}}{}^{f*} \Big[\sum_{i=1}^{A} \tau_i{}^{(-)} J_\lambda{}^{i\,\text{eff}} r_i{}^l \times Y_{lm}{}^*(\mathbf{r}_i)\Big] \psi_{\text{nuc}}{}^i d\mathbf{r}_1 \cdots, d\mathbf{r}_A.$$

In this way the photon momentum dependence has been removed from the nuclear matrix elements and the usual angular momentum techniques can be used to evaluate any given spectrum.

For somewhat heavier nuclei, like carbon, introducing the asymptotic expansion for $j_l(x)$ involves an error of as much as 40%. On the other hand, we *can* drop higher terms in the Bessel function series without much error. Therefore we may expect the spectrum to be well represented for such nuclei by

$$N(x)dx = \frac{e^2}{32\pi^4} \frac{\epsilon_f{}^4}{m^2} \frac{1}{\pi a^3} (1+x)^2 x\, dx 1 (2I_f+1) \times \int\int \Big\{ |g_V|^2 |\textstyle\int j_0|^2 + |g_A|^2 |\textstyle\int j_0 \boldsymbol{\sigma}|^2 + \frac{\mathbf{K}}{K} \cdot \frac{\mathbf{v}}{v} [|g_V|^2 |\textstyle\int j_0|^2 - \frac{1}{3} |g_A|^2 |\textstyle\int j_0 \boldsymbol{\sigma}|^2] \Big\} d\Omega_\nu d\Omega_\gamma. \quad (22)$$

In Eq. (22) we have introduced the obvious shorthand,

$$\int j_0 O = \int \cdots \int \psi_{\text{nuc}}{}^{f*} \Big[\sum_{i=1}^{A} \tau_i{}^{(-)} \delta(\mathbf{r}-\mathbf{r}_i) \times j_0(|\mathbf{v}+\mathbf{K}|r) O_i\Big] \psi_{\text{nuc}}{}^i d\mathbf{r}_1 \cdots, d\mathbf{r}_A d\mathbf{r},$$

where O is any spin operator. To obtain Eq. (22) we have replaced the sum over initial and final spins by a spatial average.

The explicit formulas for the spectra are buried in the $\int j_0 O$ since the argument of j_0, $|\mathbf{v}+\mathbf{K}|r$, contains the angle between $\mathbf{v}$ and $\mathbf{K}$ which must be integrated over. For carbon it is quite accurate to take only the first two terms in the expansion of j_0, i.e., we write

$$j_0(|\mathbf{v}+\mathbf{K}|r) \cong 1 - \tfrac{1}{6} |\mathbf{v}+\mathbf{K}|^2 r^2. \quad (23)$$

This has been shown in considerable detail by Fujii and Primakoff[15] who find that

$$\frac{\epsilon_f{}^2}{6} \frac{|\int \boldsymbol{\sigma} r^2|}{|\int \boldsymbol{\sigma}|} \simeq 0.19$$

for the transition between the ground state of ${}_6C^{12}$ and ${}_5B^{12}$. Both electron scattering data and nuclear shell model calculations give essentially the same answer. Of course, the principal effect of these terms is in interference with the terms in $\int \boldsymbol{\sigma}$. These interference terms are as large as 40% for carbon. For heavy nuclei any expansion of j_0 is very dangerous and one must make use of a different technique.[21] So long as Eq. (23) is a good approximation the theory will contain the nuclear matrix elements in the ratio

$$R_A = \tfrac{1}{6}\epsilon_f{}^2 \left|\int \boldsymbol{\sigma} r^2\right| \Big/ \left|\int \boldsymbol{\sigma}\right|$$

for the axial vector, and

$$R_V = \tfrac{1}{6}\epsilon_f{}^2 \left|\int r^2\right| \Big/ \left|\int 1\right|$$

for the vector. The angular integrations may be done, having introduced Eq. (23) into Eq. (22), and the resulting spectrum reads

$$N(x)dx = \frac{e^2}{2\pi^2} \frac{\epsilon_f{}^4}{m^2} \frac{1}{\pi a^3} (1-x)^2 x\, dx 1 (2I_f+1) \times \{|g_V|^2 |\textstyle\int 1|^2 [1 - \frac{2}{3} R_V (4x+3-4x^2)] + |g_A|^2 |\textstyle\int \boldsymbol{\sigma}|^2 \times [1 + (2/9) R_A (20x - 20x^2 - 9)]\}. \quad (24)$$

In the Godfrey transition, which is $0 \to 1$ (no), there is no vector contribution in the approximation where relativistic effects are neglected. For this transition $|\int \boldsymbol{\sigma}|^2$ may be obtained empirically from the rate of the reaction ${}_5B^{12} \to e^- + \bar{\nu} + {}_6C^{12}$ which is given by the equation

$$w^{(\beta)} = \frac{1}{2\pi^3} |\textstyle\int \boldsymbol{\sigma}|^2 f |g_A{}^{(\beta)}|^2 m_e{}^5,$$

where

$$f = \int_1^{E_{\max}} F(Z,E)(E_{\max}-E)^2 (E^2-1)^{\frac{1}{2}} dE$$

f is 5.625×10^5 for this transition.[15] $w^{(\beta)}$ is known to be 33.15 sec^{-1} and hence $|\int \boldsymbol{\sigma}|^2$ is known. In getting $|\int \boldsymbol{\sigma}|^2$ from the beta decay rate we take advantage of the fact that corrections to the allowed approximation

[21] See H. Primakoff, Revs. Modern Phys. (to be published), for a discussion of the closure method as applied to muon capture in heavy nuclei.

are small. The absolute scale of the spectrum is probably not of particular interest as opposed to the relative shape which depends on R_A and must be taken from theory.

According to Eq. (22), the gamma-neutrino angular correlation is given in the "allowed" approximation by

$$1+\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}\left\{\frac{|g_V|^2|\int 1|^2-\frac{1}{3}|g_A|^2|\int\boldsymbol{\sigma}|^2}{|g_V|^2|\int 1|^2+|g_A|^2|\int\boldsymbol{\sigma}|^2}\right\}. \qquad (25)$$

Since this seems a somewhat academic quantity we shall not refine the theory for it further. We may mention that a measurement of the correlation between the decay beta and the inner bremsstrahlung gamma would be of interest in the radiative Godfrey experiment, but we shall not work out the theory here.

It is important to know the relative rates of radiative to nonradiative muon capture. To this end we need a formula for the ordinary capture rate. This has been discussed frequently in the literature[15,21] and is given, in the notation already introduced, by

$$\frac{\epsilon_f^2}{8\pi^2}\frac{1}{\pi a^3}\frac{1}{2I_i+1}\sum_{m_i,m_f}\int\sum_{i,j=1}^{2}\sum_{\lambda,\sigma=1}^{4}N_{\lambda\sigma}{}^{ij}\tfrac{1}{4}$$
$$\times\mathrm{Tr}\left[(1+\gamma_4)O_\sigma{}^{j\dagger}\left(\frac{-i\gamma\nu}{\nu}\right)(1+\gamma_5)O_\lambda{}^{i}\right]d\Omega_\nu. \qquad (26)$$

As above, the leading terms in the rate are given by

$$W^{(\mu)}=\frac{\epsilon_f^2}{2\pi}\frac{1}{\pi a^3}1(2I_f+1)\{|g_V|^2|\int j_0|^2+|g_A|^2|\int j_0\boldsymbol{\sigma}|^2\}. \qquad (27)$$

To get a rough idea of the ratio of the radiative to the nonradiative rates, we can replace j_0 by unity in both Eq. (22) and Eq. (27) and take the ratio. In this case the nuclear matrix elements cancel and the integral over the photon spectrum is immediate. The ratio comes out to be

$$\frac{\alpha}{12\pi}\left(\frac{\epsilon_f}{m}\right)^2\simeq 2\times 10^{-4}.$$

This answer is well known from the theory of radiative electron K capture.[1] Empirically the rate for ordinary muon capture in the Godfrey experiments is between 0.7×10^4 and 0.9×10^4 sec^{-1}.[9] Therefore we anticipate an absolute radiative capture rate here of about one a second. In view of the very intense muon beams now available[7] and of the large number of very-high-energy quanta in the spectrum, this is very likely a usable capture rate.

With the machinery so far assembled it is straightforward and only somewhat tedious to compute the leading relativistic corrections to the spectrum given by Eq. (20), again neglecting the induced couplings. We give these corrections, along with the main terms, in the approximation that leads to Eq. (22). Thus

$$N(x)dx=\frac{e^2}{32\pi^4}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}(1-x)^2xdx$$
$$\times 1(2I_f+1)\int\int\left\{|g_V|^2|\int j_0|^2\left[1+\frac{\epsilon_f}{M}\right]\right.$$
$$+|g_A|^2|\int j_0\sigma|^2\left[1+\frac{1}{3}\frac{\epsilon_f}{M}\right]$$
$$+\frac{2}{3}\frac{\epsilon_f}{M}(g_Ag_V{}^*+g_Vg_A{}^*)(x-\tfrac{1}{2})|\int j_0\boldsymbol{\sigma}|^2$$
$$+\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}[|g_V|^2|\int j_0|^2-\tfrac{1}{3}|g_A|^2$$
$$\left.\times|\int j_0\boldsymbol{\sigma}|^2]\right\}d\Omega_\nu d\Omega_\gamma. \qquad (28)$$

For ${}_6C^{12}$, where we can expand j_0 as before, we have the formula

$$N(x)dx=\frac{e^2}{2\pi^2}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}(1-x)^2xdx$$
$$\times 1(2I_f+1)\left\{|\int\sigma|^2\left[|g_A|^2\left(1+\frac{1}{3}\frac{\epsilon_f}{M}\right.\right.\right.$$
$$\left.+\frac{2}{9}R_A[20x-20x^2-9]\right)$$
$$\left.\left.+\frac{2}{3}\frac{\epsilon_f}{M}(g_Ag_V{}^*+g_Vg_A{}^*)(x-\tfrac{1}{2})\right]\right\}. \qquad (29)$$

An interesting new feature brought in by the relativistic corrections are the $V-A$ interference terms in the spectrum. We shall see in Sec. IV that the Gell-Mann weak magnetic terms are of this character and are the dominant relativistic corrections.

It is very instructive to specialize Eq. (28) to the case of radiative muon capture by a free proton and then to compare the answer with the apparently unrelated electron spectrum in the beta decay of the free neutron also including relativistic effects. For consistency, in this calculation, we have included contributions to the spectrum which arise when one corrects the neutrino momentum for nuclear recoil. Such terms are neglected by writing $\nu/\epsilon_f=(1-x)$. This approximation is entirely reasonable for a nucleus as heavy as carbon. The leading corrections, for the protons, are given by

$$\frac{\nu^2}{\epsilon_f^2}=(1-x)^2\left[1-2x\frac{\epsilon_f}{M}\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}-\frac{\epsilon_f}{M}\left(\frac{x^2+(1-x)^2}{(1+x)}\right)\right].$$

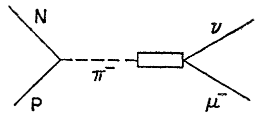

FIG. 4. This diagram illustrates the origin of the induced pseudoscalar in muon capture.

Therefore in the reaction $\mu^-+p\rightarrow n+\nu+\gamma$, the photon spectrum is given by

$$N(x)dx=\frac{e^2}{4\pi^2}\frac{1}{\pi a^3}\frac{\epsilon_f^4}{m^2}(1-x)^2xdx$$
$$\times\biggl\{(|g_V|^2+3|g_A|^2)\biggl[1-\frac{\epsilon_f}{M}\biggl(\frac{x^2+(1-x)^2}{(1-x)}\biggr)\biggr]$$
$$+(|g_A|^2+|g_V|^2)\frac{\epsilon_f}{M}-\frac{2}{3}x\frac{\epsilon_f}{M}(|g_V|^2-|g_A|^2)$$
$$+2(g_Vg_A^*+g_Ag_V^*)\frac{\epsilon_f}{M}(x-\tfrac{1}{2})\biggr\},\qquad(30a)$$

while the electron spectrum in the beta decay of the neutron is given by

$$N(E_K)dE_K$$
$$=\frac{1}{2\pi^3}KE_K\epsilon_f^2\biggl(1-\frac{E_K}{\epsilon_f}\biggr)^2dE_K\biggl\{(|g_V|^2+3|g_A|^2)$$
$$\times\biggl[1-\frac{\epsilon_f}{M}\biggl(\frac{K^2}{\epsilon_f^2}+\biggl(1-\frac{E_K}{\epsilon_f}\biggr)^2\biggr)\bigg/\biggl(1-\frac{E_K}{\epsilon_f}\biggr)\biggr]$$
$$-\frac{2}{3}\frac{K^2}{E_KM}(|g_V|^2-|g_A|^2)+(|g_A|^2+|g_V|^2)$$
$$\times\frac{\epsilon_f}{M}\biggl(1-\frac{m_e^2}{\epsilon_fE_K}\biggr)-2(g_Vg_A^*+g_Ag_V^*)$$
$$\times\frac{\epsilon_f}{M}\biggl(\frac{E_K}{\epsilon_f}-\frac{1}{2}-\frac{1}{2}\frac{m_e^2}{\epsilon_fE_K}\biggr)\biggr\},\qquad(30b)$$

with $E_K=(m_e^2+K^2)^{\frac{1}{2}}$.

It is clear from Eq. (30b) that the relative electron spectrum in the beta decay of the neutron is identical to the photon spectrum in the process $\mu^-+p\rightarrow n+\nu+\gamma$ if we set $m_e=0$ and change the sign of the last term. This is as it should be since we have previously shown that radiative K capture and nuclear *positron* beta decay for zero-mass positrons have identical leptonic traces. The change of sign in the $V-A$ interference term for the *electron* beta decay of the neutron is a consequence of the well-known fact that under charge conjugation

$$\bar{\psi}^c\gamma_\mu\psi^c=-\bar{\psi}\gamma_\mu\psi,$$

while

$$\bar{\psi}^c(-i\gamma_\mu\gamma_5)\psi^c=\bar{\psi}(-i\gamma_\mu\gamma_5)\psi.$$

This is as far as we shall carry the theory without induced couplings. In the next section we consider the induced pseudoscalar and in Sec. IV we shall discuss the Gell-Mann weak magnetic term.

III. THE INDUCED PSEUDOSCALAR

In the previous section, contact between strong and weak interactions was made only indirectly through the nuclear matrix elements. In this section and the next, the strong interactions enter directly and hence there is the additional element of uncertainty in the conclusions that we draw which arises when one deals with strong interactions by Feynman diagrams and perturbation theory. However, we believe that the theory which we give isolates the dominant effects.

The induced pseudoscalar in muon capture is generated by the process depicted in Fig. 4.[12] A calculation of this diagram shows, in the notation of Sec. II, that $mB^{(\mu)}(m^2)G\equiv g_P^{(\mu)}\simeq\pm 8g_A^{(\beta)}$. The plus sign goes with a calculation of the black box in Fig. 4 that includes only nucleon-antinucleon pairs, for example the dispersion-theory computation of Goldberger and Treiman,[12] and the minus sign is a possibility if the empirical $\pi\rightarrow\mu+\nu$ lifetime, which depends on the square of the relevant matrix element, is used to evaluate the loop, as Wolfenstein has done.[12] Preliminary data on the nonradiative Godfrey process,[9] when compared with the detailed theory of Fujii and Primakoff,[15] seem to rule out the minus sign as a possibility. In the theory which we develop below, the pseudoscalar is incoherent with the other contributions of the $V-A$ theory in radiative capture effects so that this sign ambiguity is irrelevant.

Let us begin by replacing the nonlocalities of the loop in Fig. 4 by an effective local Lagrangian L_π, i.e., we shrink the loop to a point. Then

$$L_\pi=g_A\frac{f_A(q^2)}{m_\pi}\frac{\partial\phi_\pi}{\partial x_\lambda}\bar{\psi}_\nu(-i\gamma_\lambda\gamma_5)\psi_\mu.\qquad(31)$$

To the extent that this procedure is justifiable we have two contributing diagrams to radiative muon capture via the induced pseudoscalar as shown in Fig. 5. The second of these diagrams—the "catastrophic" term

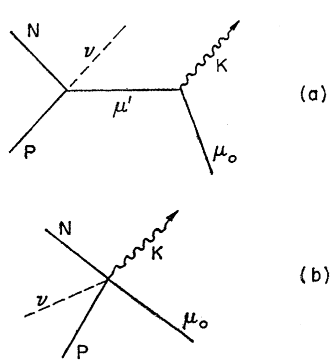

FIG. 5. These diagrams are the "local" approximation to the induced pseudoscalar contribution to radiative muon capture. Fig. 5(b) is the "catastrophic" term.

—comes from making the replacement $(\partial/\partial x_\lambda)\to(\partial/\partial x_\lambda)-eA_\lambda$ in Eq. (31). Such a procedure has been made familiar by the analysis of Treiman and Wyld[22] of radiative pion decay of real, as opposed to virtual, pions. The lepton matrix elements corresponding to these diagrams are, after setting $q=K+\nu-\mu_0$, where μ_0 is the energy-momentum of the initial muon, and using the Dirac equation to eliminate the neutrino momentum,

$$\bar{u}_\nu\left[-i\gamma(K-\mu_0)\gamma_5\frac{1}{i\gamma(\mu_0-K)+m}\gamma\cdot\boldsymbol{\varepsilon}-\gamma_5\gamma\cdot\boldsymbol{\varepsilon}\right]u_\mu. \quad (32)$$

The second term in Eq. (32) is the "catastrophic" term. Equation (32) may easily be rewritten as

$$\bar{u}_\nu\left(-m\gamma_5\frac{1}{i\gamma(\mu_0-K)+m}\gamma\cdot\boldsymbol{\varepsilon}\right)u_\mu. \quad (33)$$

Hence, in the local approximation, the induced pseudoscalar term reduces to a local four-fermion "intrinsic" γ_5 coupling with the coupling constant g_P defined above. It is clear from the remarks of Sec. II that this coupling *by itself* produces internal bremsstrahlung quanta that are 100% left circularly polarized. Therefore all effects computed with Eq. (33) add incoherently to the rest of the theory. The corrections to the local theory are shown in Fig. 6. It is easy, after Treiman and Wyld, to exhibit the general structure in momentum space of these graphs. The above authors show that for real pion radiative decay, when the electron mass is neglected, the entire set of diagrams can be reduced to the local γ_5 form. This is no longer so for virtual pion decay into muons since we can use neither $\epsilon\cdot P_\pi=0$ nor $m\cong 0$, which are essential conditions for their proof. Nonetheless, it can be seen that the nonlocal contributions will be at most of order m/M compared to the leading terms. It may be worthwhile, at some future time, to try a more detailed theory for these, but in this first estimate, we shall simply drop them. On the other hand the catastrophic term must be kept since it is of the same order of magnitude as the muon emission term, and is necessary for the gauge invariance of the calculation.

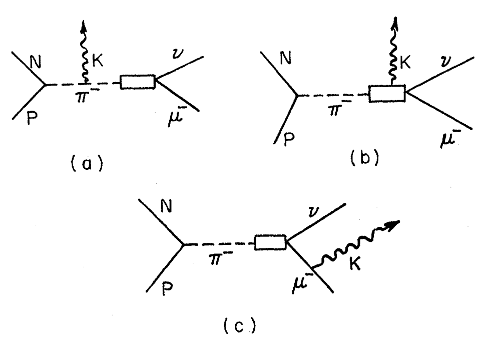

FIG. 6. These diagrams are typical nonlocal contributions to radiative muon capture via the induced pseudoscalar.

[22] S. Treiman and W. Wyld, Phys. Rev. **101**, 1552 (1956). We are very grateful to C. N. Yang for questioning the gauge invariance of a preliminary calculation of the induced pseudoscalar and pointing out the relevance of the paper quoted above.

As a parenthetical remark we note that, by the above arguments, γ rays emitted in the real process $\pi\to\mu+\nu+\gamma$ must be 100% left circularly polarized if the universal Fermi theory of this decay is accepted. This will also be true of γ's in the process $K\to\mu+\nu+\gamma$. Unfortunately the sense of γ circular polarization is the same for a scalar or a pseudoscalar K particle so that γ circular polarization experiments will not distinguish between these. At the end of the next section we give a formula for $\beta(x)$, the circular polarization as a function of photon momentum which is to be expected in the radiative Godfrey experiment on the basis of the theory developed in this paper. Making the approximation

$$\gamma_5\to(\boldsymbol{\sigma}\cdot\mathbf{K}+\boldsymbol{\nu})/2M$$

in the nuclear matrix elements and using Eq. (33) for the lepton matrix, we may derive the following expression for the photon spectrum due to the induced pseudoscalar:

$$N_P(x)dx=\frac{e^2}{32\pi^4}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}\frac{1}{2I_i+1}(1-x)^2\,xd\frac{|g_P|^2}{4M^2}\times\sum_{m_im_f}\int\int\Bigg\{\Bigg|\int\cdots\int\psi_{\text{nuc}}{}^{f*}\times\Big[\sum_{i=1}^{A}\tau_i{}^{(-)}\delta(\mathbf{r}-\mathbf{r}_i)\boldsymbol{\sigma}_i\exp(-i(\boldsymbol{\nu}+\mathbf{K})\cdot\mathbf{r})\Big]\times\psi_{\text{nuc}}{}^{i}dr_1\cdots dr_A d\mathbf{r}\times(\mathbf{K}+\boldsymbol{\nu})\Bigg|^2\times\left[1-\frac{\mathbf{K}}{K}\cdot\frac{\boldsymbol{\nu}}{\nu}\right]\Bigg\}d\Omega_\nu d\Omega_\gamma. \quad (34)$$

Since Eq. (34) represents a correction to the principal contributions to the spectrum we can, with sufficient accuracy, set the plane wave equal to unity, i.e., make the allowed approximation, and derive, after integration over the neutrino and photon angles,

$$N_P(x)dx=\frac{e^2}{32\pi^4}\frac{\epsilon_f^4}{m^2}\frac{1}{\pi a^3}1(2I_f+1)(1-x)^2x\,dx\frac{16\pi^2}{9}\frac{\epsilon_f^2}{4M^2}\times|g_P|^2|\textstyle\int\sigma|^2\{8x^2-8x+3\}. \quad (35)$$

The scale of these corrections is evidently $(m^2/4M^2)\times|g_P|^2\simeq 20\%$. We shall now turn to the weak magnetic terms.

IV. WEAK MAGNETISM

As we have remarked in Sec. II, the presence in Eq. (2) of the term

$$i[\phi_\pi T^{(-)}\partial_\mu\phi_\pi{}^*-\phi_\pi{}^*T^{(-)}\partial_\mu\phi_\pi]$$

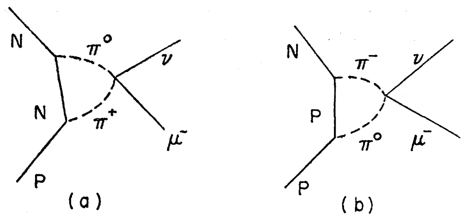

FIG. 7. These diagrams show the "weak magnetic" contributions to muon capture.

enables us to impose the conservation condition $\partial_\mu J_\mu{}^V=0$ on the vector current. It is clear that this term, when multiplied into the suitable lepton matrix element, allows the direct decay mode $\pi^-\rightarrow\pi^0+e^-+\bar{\nu}$ with a known rate that has been given by Feynman and Gell-Mann.[13] In muon capture we may have the virtual events $\pi^++\mu^-\rightarrow\pi^0+\nu$ and $\pi^0+\mu^-\rightarrow\pi^-+\nu$ intervene in the capture process as is shown in Fig. 7. The totality of such diagrams yields the function $D^{(\mu)}(q^2)$, discussed before. We do not, however, have to resort to any perturbation-theoretic arguments to evaluate $D^{(\mu)}$. Rather we can take advantage of the fact that $J_{\mu-}{}^V$ and $J_{\mu z}{}^V$, the isotopic vector current operator of electromagnetic theory, are isotopic rotations of each other and thereby use empirical electromagnetic data to evaluate $D^{(\mu)}$. This was done in obtaining Eq. (11) of Sec. II. In calculating the weak magnetic contributions to radiative muon capture we must again consider both muon emission terms, such as Fig. 8(a), and catastrophic and nonlocal terms such as Figs. 8(c) and 8(e). One can convince oneself that, owing to charge independence,

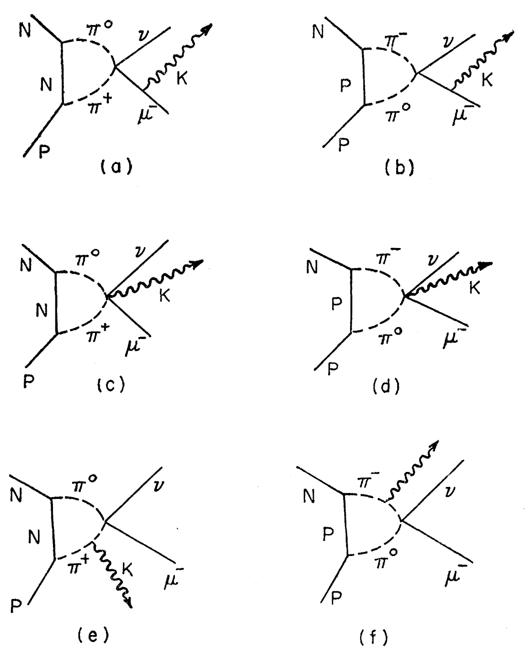

FIG. 8. These diagrams show possible "weak magnetic" contributions to radiative muon capture. The "catastrophic" terms 8(c) and 8(d) cancel each other.

the catastrophic terms cancel each other: typically, the two terms in Fig. 8(c) and 8(d) cancel. As before we shall neglect the nonlocal terms like Fig. 8(e). Under these conditions the matrix element for the weak magnetic contribution to radiative K capture is written as

$$eg_V\frac{u_P-u_N}{2M}\left\{\int\cdots\int\psi_{\mathrm{nuc}}{}^{f*}\left[\sum_{i=1}^{A}\delta(\mathbf{r}-\mathbf{r}_i)\tau_i{}^{(-)}\right.\right.$$
$$\left.\left.\times\exp(-i(\mathbf{v}+\mathbf{K})\cdot\mathbf{r})(-i\sigma_{\lambda\nu}{}^{(i)}q_\nu)\right]\psi_{\mathrm{nuc}}{}^{i}d\mathbf{r}_1\cdots d\mathbf{r}_Ad\mathbf{r}\right\}$$
$$\times\left(\bar{u}_\nu\left[\gamma_\lambda\frac{1}{i\gamma(\mu_0-K)+m}\gamma\cdot\boldsymbol{\varepsilon}\right]u_\mu\right),\quad(36)$$

where, as usual, $q_\lambda=\nu_\lambda+K_\lambda-u_{0\lambda}$. The dominant con-

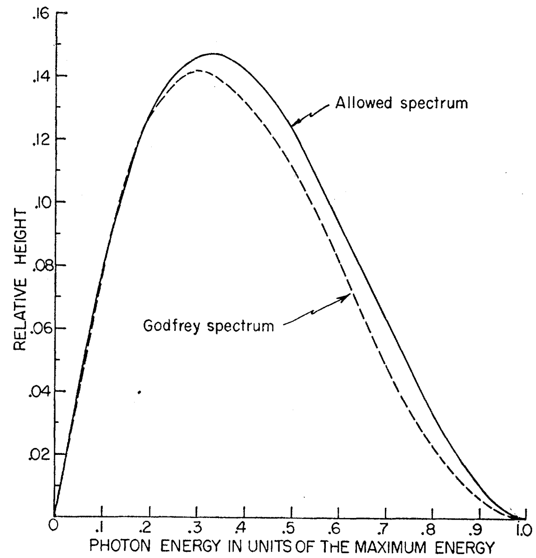

FIG. 9. The spectrum to be expected in the reaction $\mu^-+C^{12}\rightarrow B^{12}+\nu+\gamma$ on the full $V-A$ theory.

tribution to Eq. (36) is obtained by making the replacement

$$-i\sigma_{\mu\nu}q_\nu\rightarrow i\boldsymbol{\sigma}\times(\mathbf{v}+\mathbf{K})$$

in the nuclear matrix element. We have used energy-momentum conservation and neglected the K-shell muon momentum. The phase of the nuclear matrix element has been defined in such a way that when the matrix is specialized to a single proton and then isotopically rotated, the proton's anomalous magnetic moment comes out positive. It is easy to see that photons emitted by Eq. (36) *alone* are 100% polarized right, that is, in the same sense as the usual $V-A$ terms. In fact, as Eq. (37) below makes clear, the effect of Eq. (36) on the photon spectrum is to replace unity, the ordinary Dirac moment in units of e/M, by the full isotopic vector anomalous moment $1+\mu_p-\mu_n$ in the relativistic corrections to the allowed vector terms. The fact that the single-particle moments enter here is a consequence of our earlier assumption that nuclear muon capture can be described as the capture of a muon by an ensemble of noninteracting "dressed" protons. Since $\mu_p-\mu_n\simeq 3.7$, the weak magnetic $V-A$ interference term yields the dominant relativistic correction in radiative muon capture. Putting the entire theory together, we obtain for the photon spectrum in the approximation where only the allowed terms are kept in the relativistic corrections, and in which the first two terms in the expansion of j_0 are kept in the nonrelativistic quantities,

$$\begin{aligned} N(x)dx=&\frac{e^2}{2\pi^2}\frac{e_f^4}{m^2}\frac{1}{\pi a^3}(1-x)^2x\,dx1(2I_f+1)\\ &\times\left\{|g_V|^2|\textstyle\int 1|^2\left[1+(1+\mu_P-\mu_N)\frac{\epsilon_f}{M}\right.\right.\\ &\left.-\tfrac{2}{3}R_V(4x+3-4x^2)\right]+(g_Ag_V^*+g_Vg_A^*)\\ &\times|\textstyle\int\sigma|^2\tfrac{2}{3}(1+\mu_p-\mu_n)\frac{\epsilon_f}{M}(x-\tfrac{1}{2})\\ &+|g_A|^2|\textstyle\int\sigma|^2\left[1+\frac{1}{3}\frac{\epsilon_f}{M}+\frac{2}{9}R_A(20x-20x^2-9)\right]\\ &\left.+|g_P|^2|\textstyle\int\sigma|^2\frac{1}{9}\frac{e_f^2}{4M^2}[8x^2-8x+3]\right\}. \end{aligned} \tag{37}$$

In Fig. 9 we have plotted the relative photon spectrum which we expect in the radiative Godfrey experiment. Since this is an allowed Gamow-Teller transition, the terms in $|g_V|^2$ vanish. We have also plotted the allowed specimen, $(1-x)^2x$. The coupling constants g_A and g_V have been taken as real. The circular polarization as a function of photon frequency, $\beta(x)$, for the Godfrey experiment is given by

$$\begin{aligned} \beta(x)=&\left\{|g_A|^2\left[1+\frac{1}{3}\frac{\epsilon_f}{M}+\frac{2}{9}R_A(20x-20x^2-9)\right]\right.\\ &+(g_Ag_V^*+g_Vg_A^*)\tfrac{2}{3}(1+\mu_p-\mu_n)\frac{\epsilon_f}{M}(x-\tfrac{1}{2})\\ &\left.-|g_P|^2\frac{1}{9}\frac{\epsilon_f^2}{4M^2}(8x^2-8x+3)\right\}\\ &\times\left\{|g_A|^2\left[1+\frac{1}{3}\frac{\epsilon_f}{M}+\frac{2}{9}R_A(20x-20x^2-9)\right.\right.\\ &+(g_Ag_V^*+g_Vg_A^*)\tfrac{2}{3}(1+\mu_p-\mu_n)\frac{\epsilon_f}{M}(x-\tfrac{1}{2})\\ &\left.\left.+|g_P|^2\frac{1}{9}\frac{\epsilon_f^2}{4M^2}[8x^2-8x+3]\right\}^{-1}. \end{aligned} \tag{38}$$

Deviations from unity in β can only reflect the presence of the induced pseudoscalar. In fact, these come to about 20% at the high-energy end of the spectrum and may not be beyond present experimental techniques.

Then, in summary, corrections to the allowed terms in radiative muon capture are evidently of great theoretical interest and may, in the near future, be susceptible to measurement.[10]

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