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Recoil Momentum Distribution in Electron Pair Production*

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Electron pair production by a very energetic photon in the field of a particle of arbitrary mass (in particular in the fields of an electron and a nucleus) is studied following the work of Borsellino. The distribution of recoil momenta q is calculated for q of order of the electron mass and it is shown that the recoil distribution is independent of the mass of the recoil particle if appropriate variables are used. It is also explicitly shown that the mass of the recoil particle does not make any difference in the recoil distribution for very small q (of order $q_{\min}$). The total cross section must therefore be independent of the mass of the recoil particle in the high-energy limit, as previously stated by Borsellino. The Wheeler-Lamb result for pair production in the field of a bound electron is also justified. The results also describe the electromagnetic production of any fermion pair if certain restrictions are satisfied.

I

THE theory of electron pair production by a photon in the field of an electron has been studied in most detail by Votruba[1] and Borsellino,[2,3] by using Dirac's positron theory in Born approximation. Feynman diagrams of the process are given in Fig. 1 and four more diagrams, which are obtained by exchanging the two electrons in the final state, must be added. The processes corresponding to diagrams (c), (d) and their exchange diagrams are referred to as $\gamma-e$ interactions.[4] Votruba's calculation is complete in that it involves all possible processes. His final expression, however, is so long and complicated that it is difficult to handle; consequently, in order to carry out a general analytic integration, approximations which may introduce errors are required. In particular, Votruba finds that the distribution of recoil momenta q over the region q of order unity[5] is difficult to obtain and, therefore, in evaluating the total cross section, he does not include the contribution from this region correctly.

Borsellino developed his theory for a particle of arbitrary mass M in whose field the electron pair is produced. Consequently he neglected the $\gamma-e$ interaction and exchange terms in his calculation. The errors[6] caused by this procedure are presumably negligible at high photon energies because the probability of large momentum transfer, where the effect of $\gamma-e$ interaction and exchange is important, is negligibly small. Borsellino's calculation should therefore be nearly correct at high incident photon energies, except for the unimportant case when the recoil momentum is of the same order as that of the incident photon.

In this paper, the recoil distribution function for high incident photon energies is obtained from the previous calculation of Borsellino in a simple and tractable form. The recoil distribution function for electron pair production in the field of an electron is

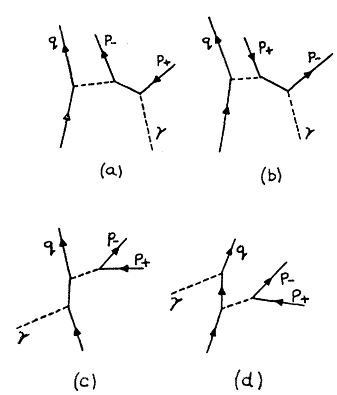

Fig. 1. Feynman diagrams for electron pair production by a photon in the field of an electron.

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[1] V. Votruba, Bull. intern. acad. Tcheque sci. **49**, 19 (1948).
[2] A. Borsellino, Nuovo cimento **4**, 12 (1947).
[3] A. Borsellino, Rev. univ. nacl. Tucumán, **A.6**, 7 (1947).
[4] J. Joseph and F. Rohrlich, Revs. Modern Phys. **30**, 354 (1958).
[5] We use the electron mass as a unit, and also set $\hbar=c=1$ throughout.
[6] For a detailed discussion of the $\gamma-e$ interaction and exchange effects, see reference 4.

compared with the corresponding distribution for that in the field of a particle of arbitrary mass M (in particular when M is equal to the mass of a nucleus), and it is seen that the two agree exactly if appropriate variables are used. It is also explicitly shown that the mass of the recoil particle does not make any difference in the recoil distribution for very small q (of order $q_{\min}$). When the recoil electron is originally bound in an atom, our result agrees with that of Wheeler and Lamb.[7] The results also describe the electromagnetic production of any Fermion pair if certain restrictions are satisfied.

II

We will express the distribution of recoil momenta q for electron pair production in the field of a particle of mass M by an incident photon of energy k as

$$\frac{d\sigma(k,q,M)}{dq}=\alpha r_0^2\frac{d\Omega(k,q,M)}{dq}=\alpha r_0^2\Gamma(k,q,M). \qquad (1)$$

Where α is the fine structure constant and r_0 is the classical electron radius. The general expression of $\Gamma(k,q,M)$ as given in reference 3 is very lengthy and complicated. It will, however, be shown in the following that it can be greatly simplified at high incident photon energies.

The minimum value of q for a given energy of the pair electrons follows from the definition of q, plus the assumption that both pair electrons go in the forward direction:

$$q_{\min}(f)=k-p_--p_+=W(q_{\min})-M+k/(2E_-E_+) \qquad (2)$$
$$\equiv k/(2E_-E_+)=W(q_{\min})-M+1/2kf(1-f),$$

where $f=E_+/k$ and $W(q_{\min})=(M^2+q_{\min}^2)^{\frac{1}{2}}$ is the energy of the recoil particle. Both E_+ and E_-, which are the energies of the created positron and electron with momenta p_+ and p_-, respectively, have been assumed large compared with unity. The smallest value of $q_{\min}$ is obtained by setting $E_+=E_-=k/2$; for other energy distributions, $q_{\min}$ is larger;

$$q_{\min}(f=\tfrac{1}{2})\equiv q_{\min}=W(q_{\min})-M+2/k. \qquad (3)$$

The kinetic energy of the recoil particle, $W(q_{\min})-M$, is always negligible compared with the momentum $q_{\min}$, provided $M\geqslant 1$ (and $k\gg 1$) which is necessarily true for production of electron pairs. If, however, a heavy pair (e.g., muons) is produced in the field of a lighter particle (e.g., an electron) and if we now denote by M the ratio of the mass of the field particle to that of the pair particle, then $M<1$; in this case the relation $W(q_{\min})-M\ll q_{\min}$ is fulfilled only if

$$kM\gg 1 \qquad (4)$$

in addition[8] to $k\gg 1$. If this is fulfilled, or if $M\geqslant 1$, then (3) simplifies to

$$q_{\min}=2/k, \qquad (5)$$

and (2) also simplifies correspondingly.

(a) For $k\gg 1$, $q\sim q_{\min}$.

After a reasonably straightforward but rather lengthy calculation, we may reduce the distribution function to the following expression:

$$\Gamma(k,q,M)=\frac{4}{q}\Bigg\{\frac{2}{3}\left(1-\frac{2}{kq}\right)^{\frac{1}{2}}\left(\frac{7}{6}+\frac{25}{6kq}-\frac{2}{(kq)^2}\right)$$
$$-\left(\frac{2}{kq}-\frac{1-\ln(2kq)}{(kq)^2}+\frac{4}{3(kq)^3}\right)\ln\frac{1+(1-2/kq)^{\frac{1}{2}}}{1-(1-2/kq)^{\frac{1}{2}}} \qquad (6)$$
$$-\frac{2}{(kq)^2}\left[L\left(\frac{1+(1-2/kq)^{\frac{1}{2}}}{2}\right)-L\left(\frac{1-(1-2/kq)^{\frac{1}{2}}}{2}\right)\right]\Bigg\},$$

where $L(y)=\int_0^y[\ln(1-x)]/x dx$ denotes the Spence function.[9] To obtain formula (6), we have partly used expansions of Borsellino's formulae which are valid only for $\frac{1}{2}kq-1\gg 1/Mk$. Since $Mk\gg 1$ and since (6) vanishes at $kq=2$ as it should, this restriction is rather unimportant.

It is interesting to note that the recoil mass M does not occur in expression (6). This is a formal proof of the theorem that the mass of the recoil particle does not make any difference for $q\ll 1$. The physical reason for this is that, for $q\ll 1$ (of order $q_{\min}$), the field in which the pair is produced behaves as if it were a static one. This theorem permits us to use Bethe's formula[10] for very small q (or order $q_{\min}$) for electron pair production in the field of an electron, even though it was derived for the nuclear case.

The factor

$$(1-2/kq)^{\frac{1}{2}} \qquad (7)$$

appears in every term of Eq. (6). This arises from the fact, mentioned in (2), that the minimum q depends on f, the fraction of energy in the positron, and attains the value (5) only for $f=\frac{1}{2}$. For a given q, the maximum and minimum f permissible are

$$f_{1,2}=\tfrac{1}{2}\pm\tfrac{1}{2}\left(1-\frac{2}{kq}\right)^{\frac{1}{2}}, \qquad (8)$$

so that (7) simply represents the permissible interval of f.

The formula (6) agrees exactly with that given in

[7] J. A. Wheeler and W. E. Lamb, Phys. Rev. **55**, 858 (1939).

[8] $kM>2$ is actually a necessary condition for pair production in the field of a light particle of $M\ll 1$, because $q_{\min}+M-W(q_{\min})<M$ whatever the value of $q_{\min}$. Therefore by high incident photon energy we mean $kM\gg 1$ in addition to $k\gg 1$.

[9] K. Mitchell, Phil. Mag. **40**, 351 (1949).

[10] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).

reference 4 which is derived from Votruba's result (reference 1) for the same region.[11] This is expected, because the cintribution from the $\gamma-e$ interaction and exchange terms is negligible in this region. $\Gamma(k,q,M)$ for $q\ll 1$ depends on q through kq which is a characteristic of the distribution.

(b) For $k\gg 1$, $q\ll k$, $kq\gg 1$.

In this case $\Gamma(k,q,M)$ can conveniently be expressed as a series of descending powers of k, viz.,

$$\Gamma(k,q,M)=\Gamma_0(k,q,M)+k^{-1}\Gamma_1(k,q,M)+\cdots; \tag{9}$$

$\Gamma_0(k,q,M)$ is given by

$$\Gamma_0(k,q,M)=\frac{2M}{3kW}\frac{q}{[M(W-M)]^2}\left\{\frac{R(k,q,M)^{\frac{1}{2}}}{q-W+M}+\frac{k[1-2M(W-M)]N(k,q,M)}{[M(W-M)]^{\frac{1}{2}}[M(W-M)+2]^{\frac{1}{2}}}\right\}, \tag{10}$$

where

$$W=(M^2+q^2)^{\frac{1}{2}},$$

$$R(k,q,M)=\{k(q-W+M)-M(W-M)\}\{k(q-W+M)-M(W-M)-2\}, \tag{11}$$

$$N(k,q,M)=\ln\frac{k(q-W+M)\{M(W-M)+1\}-M(W-M)\{M(W-M)-2\}-\{M(W-M)\}^{\frac{1}{2}}\{M(W-M)+2\}^{\frac{1}{2}}R(k,q,M)^{\frac{1}{2}}}{k(q-W+M)}.$$

The expressions (10) and (11) can be greatly simplified, as is shown in Appendix A, with the result

$$\Gamma_0(k,\xi[q,M])=\frac{2}{3}\frac{1}{\xi^2}\left\{1+\frac{(2\xi-1)}{\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}}\ln[1+\xi+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}]\right\}, \tag{12}$$

where

$$\xi\equiv M(W-M), \tag{13}$$

and a general notation

$$\Gamma(k,\eta,M)=d\Omega(k,\eta,M)/d\eta, \tag{14}$$

for any observable quantity $\eta=q, W, \xi, \cdots$ etc., is used. Formula (12) will also be valid for the electromagnetic production of any Fermion pair provided the conditions $k\gg 1$, $q\ll k$, and $kq\gg 1$ and, in addition, $kM\gg 1$ are satisfied. The unit of mass in this case, as previously mentioned, is the mass of one particle of the produced pair, and M is the ratio of the mass of the recoil particle to that of the pair particle.

When $M\gg 1$, as is the case for electron pair production in the field of a nucleus, we have

$$\xi=q^2/2,$$

which is independent of M. This case was previously treated by Bethe, whose formula[12] for $q\gg q_{\min}$ reduces to

$$\Gamma_B(k,q,M)=\frac{8}{3}\frac{1}{q}\left\{\frac{\ln[\rho^{\frac{1}{2}}+(\rho+1)^{\frac{1}{2}}]}{\rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}}+\frac{1}{q^2}\left(1-\frac{\ln[\rho^{\frac{1}{2}}+(\rho+1)^{\frac{1}{2}}]}{\rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}}\right)\right\}, \tag{15}$$

where $\rho=q^2/4$. Using $\rho=\xi/2$, one can easily show that Bethe's formula (15) is identical with our (12).

The recoil distribution (12) in terms of the variable ξ is independent of the mass of the recoil particle.[13] This is a very remarkable and rather unexpected result which should be useful experimentally.

III

In the following discussion, we will restrict ourselves to the case of electron pair production in the field of an electron. Here we have $\xi=W-1$ and $\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}=q$, and thus for (12), we can write

$$\Gamma_0(k,q,1)=\frac{2}{3}\frac{q}{W(W-1)^2}\left[1+\frac{2W-3}{q}\ln(W+q)\right]. \tag{16}$$

When W is much larger than 1, (16) can be simplified by a further approximation to give

$$\Gamma_0(k,q,1)=(2/3q^2)[1+2\ln(2q)]. \tag{17}$$

The first correction factor $\Gamma_1(k,q,1)$ is reduced to the simpler approximate expression in Appendix B. It is clear that $\Gamma_0(k,q,1)$ as given in (16) is of order unity since it is independent of k and since q is of order 1. In the same way $\Gamma_1(k,q,1)$ is seen to be of order $\ln k$ (+ a constant of order 1).

Since $k\gg 1$ and the recoil momentum $q\sim 1$ is much smaller than the momenta of the created pair, the effects arising from the $\gamma-e$ interaction and exchange terms should not be large; they may partially compensate each other because they seem to have opposite signs. As discussed in reference 4, the contribution of the $\gamma-e$ interaction terms is expected to be of relative order $q/k\sim 1/k$, while the contribution of the exchange

[11] Formula (6) may also be derived from the article of Jost, Luttinger, and Slotnick, Phys. Rev. **80**, 189 (1950) for the same region.

[12] Compare Eqs. (32), (38), (41), and (44) of reference 10 as applied to pair production.

[13] This fact, that the recoil distribution in terms of the variable ξ is independent of the mass of the recoil particle, is equally valid for the case of $q\sim q_{\min}$. In this case we merely put $q=(2\xi)^{\frac{1}{2}}$ in Eq. (6).

terms should be of order q/E_- which, integrated over the (nearly uniform) energy spectrum of the negative pair electron, yields a result of order $k^{-1}\ln k$. We may therefore expect that (16) gives a good representation of the distribution of recoil momenta q in the region where q is of order unity, the error probably being of relative order $k^{-1}\ln k$.

The distribution function $\Gamma_0(k,q,1)$ as given in (16) is plotted as a function of q in Fig. 2. The curves corrected for the term $k^{-1}\Gamma_1(k,q,1)$ in (9), using the formulas as given in Appendix B, are given in the same figure for $k=100$, 400, and 1000. However, the corrections made there are not complete, because further modification for the $\gamma-e$ interaction and exchange terms should also be made.

Since the recoil distribution in terms of the variable ξ is the same for an electron as for a nucleus, the total cross section for pair production will be the same in both cases, if we (a) assume $k\gg 1$ and (b) neglect screening. This result was first obtained by Borsellino and should be correct except for terms of order $k^{-1}\ln k$, as pointed out by Borsellino.

When q is very small ($q\ll 1$ but $kq\gg 1$), both (6) and (16) reduce to

$$\Gamma_0(k,q,1)=28/9q\equiv\Gamma_{00}(q), \tag{18}$$

as was shown by Bethe.[10] The result of the integration[14] of (16) over q from an arbitrary but small $q=q_0$ ($q_{\min}\ll q_0\ll 1$) to infinity is

$$\int_{q_0}^{\infty}\Gamma_0(k,q,1)dq=\frac{2}{3}\left\{\frac{14}{3}\ln\left(\frac{1}{q_0}\right)+\frac{41}{9}\right\}. \tag{19}$$

We may write this in the form

$$\int_{q_0}^{\infty}\Gamma_0(k,q,1)dq=\int_{q_0}^{1}\Gamma_{00}(q)dq+\frac{82}{27}. \tag{20}$$

The total cross section can thus be obtained by assuming the simple formula (18) to be valid up to $q=1$, and then adding 82/27 for the contribution of larger q. Votruba,[1,4] who was not able to get a valid expression for q of order 1 or larger, estimated the total cross section to be simply given by the first term in (20) which is in error by the constant 82/27. Joseph and Rohrlich, in reference 4, accept Votruba's total cross section even though they realize that he did not treat the important region $q\sim 1$ correctly.

Finally we consider the initial binding of the recoil electron in an atom. It was pointed out by Wheeler and Lamb[7] that this binding affects only small q, just as screening does for the pair production in the field of a nucleus. This is because, for large q (or order 1), the recoil electron receives sufficient energy to be considered free. Now Wheeler and Lamb assumed that for large q (or order 1), the cross section was the same as for pair production in the field of a proton. Since we have proved in this paper that this assumption is correct, the Wheeler-Lamb calculation is thereby fully justified.

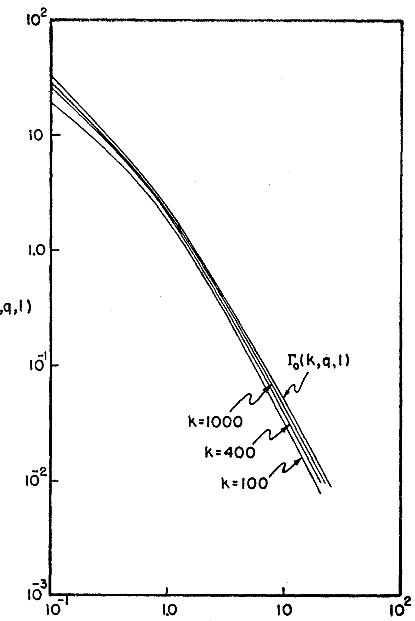

FIG. 2. The recoil momentum distribution function $\Gamma_0(k,q,1)$ and the corrected values due to the lower order terms for $k=100$, 400, and 1000, plotted as functions of the recoil momentum q.

ACKNOWLEDGMENTS

Recent experiments[15] for the recoil momentum distribution and the total cross section in triplet production show good agreement[16] with our results. We thank Professor G. Cocconi, E. Hart, and E. Malamud for informing us of their experimental results.

APPENDIX A

A close look at the expressions (10) and (11) shows that $\Gamma_0(k,q,M)$ generally depends on the quantity $M(W-M)$. We will therefore introduce a variable

$$\xi\equiv M(W-M),\quad\text{where}\quad W=(M^2+q^2)^{\frac{1}{2}}. \tag{13}$$

Therefore

$$W-M=\xi/M\quad\text{and}\quad Mq=\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}. \tag{A.1}$$

Since $WdW=qdq$,

$$\frac{d\xi}{M}=dW=\frac{qdq}{W}. \tag{A.2}$$

[14] The integral of $\Gamma_0(k,q,1)$ of (16) is given as

$$\int\Gamma_0(k,q,1)dq=\frac{1}{9}\left[\frac{4}{q(W-1)}-\frac{13q}{W-1}-\frac{q}{W+1}\right]\ln(W+q)+\frac{2}{9}\left[7\ln(W-1)-\frac{2}{W-1}\right].$$

[15] Hart, Cocconi, Cocconi, and Sellen, following paper [Phys. Rev. **115**, 678 (1959)]; E. Malamud, this issue [Phys. Rev. **115**, 687 (1959)].

[16] See in particular Figs. (3) and (4) of Hart *et al.*, reference 15.

Substitution of (13) into (11) yields

$$R(k,\xi,M)=\{(k/M)(\xi^{\frac{1}{2}}[\xi+2M^2]^{\frac{1}{2}}-\xi)-\xi\}\{(k/M)(\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}-\xi)-\xi-2\},$$

$$N(k,\xi,M)=\ln\frac{(k/M)\{\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}-\xi\}(\xi+1)-\xi(\xi+2)-\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}R(k,\xi,M)^{\frac{1}{2}}}{(k/M)\{\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}-\xi\}}. \quad \text{(A.3)}$$

If we further introduce the abbreviation

$$S\equiv k(q-W+M)=(k/M)\{\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}-\xi\}, \quad \text{(A.4)}$$

substitution of (A.4) into (A.3) yields

$$R(k,\xi,M)=(S-\xi)(S-\xi-2), \quad \text{(A.5)}$$

$$N(k,\xi,M)=\ln\frac{S(\xi+1)-\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}R(k,\xi,M)^{\frac{1}{2}}-\xi(\xi+2)}{S}. \quad \text{(A.6)}$$

Expression (A.5) can be greatly simplified by noting that, under very general conditions,

$$S\gg\xi+2. \quad \text{(A.7)}$$

To prove this we consider the two cases (a) $\xi\ll M^2$, and (b) $\xi\gtrsim M^2$. The magnitude of M itself is unimportant for the present.

Case (a) $\xi\ll M^2$, therefore $q\ll M$. Then

$$Mq=\xi^{\frac{1}{2}}(\xi+2M^2)^{\frac{1}{2}}\gg\xi\simeq q^2/2, \quad \text{(A.8)}$$

Therefore

$$S\simeq kq, \quad \text{(A.9)}$$

and (A.7) is satisfied provided

$$q\ll k, \quad \text{(A.10)}$$

$$kq\gg 1. \quad \text{(A.11)}$$

(A.10) states simply that the recoil momentum is small compared with the total momentum as we have assumed throughout this paper. (A.11) implies that q is large compared with its minimum value, $q_{\min}=2/k$ as given in Eq. (5); smaller values of q were treated separately in Eq. (6).

Case (b) $\xi\gtrsim M^2$, therefore $q\gtrsim M$. Then

$$\xi\simeq qM, \quad S\simeq kM. \quad \text{(A.12)}$$

From this $S\gg\xi$ follows by (A.10), but to make $S\gg 2$ we must require also relation (4), i.e.,

$$kM\gg 1. \quad \text{(4)}$$

It was shown in Sec. (II) that this condition (4) must always be fulfilled, even if the field particle has less mass than the pair particles, as is the case for the production of a muon pair in the field of an electron.

If (A.10), (A.11), and (4) are satisfied, then (A.7) will be valid. We can then expand $R(k,\xi,M)^{\frac{1}{2}}$ in a power series, thus:

$$R(k,\xi,M)^{\frac{1}{2}}=S-(\xi+1)-1/2S-\cdots. \quad \text{(A.13)}$$

For most processes, this expression can be simplified by setting

$$R(k,\xi,M)=S^2. \quad \text{(A.14)}$$

For (A.6) we obtain, using the full (A.13):

$$Se^N=S(\xi+1)-\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}[S-(\xi+1)]-\xi(\xi+2)+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}/2S; \quad \text{(A.15)}$$

therefore

$$e^N=[1+\xi-\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}][1+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}/S]+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}/2S^2. \quad \text{(A.16)}$$

Now we have

$$\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}<\xi+1, \quad \text{(A.17)}$$

and therefore, using (A.7), we see that the second square bracket in (A.16) may be replaced by unity. The last term in (A.16) may also be neglected which is seen by dividing this last term by the first bracket, and then using (A.17) and (A.7). Then

$$\begin{aligned}N(k,\xi,M)&=\ln[1+\xi-\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}]\\&=-\ln[1+\xi+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}].\end{aligned} \quad \text{(A.18)}$$

Substituting (13), (A.2), (A.4), (A.14), and (A.18) into (10), we obtain

$$\Gamma_0(k,\xi(q,M))=\frac{2}{3\xi^2}\left\{1+\frac{2\xi-1}{\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}}\ln[1+\xi+\xi^{\frac{1}{2}}(\xi+2)^{\frac{1}{2}}]\right\}, \quad \text{(A.19)}$$

which is identical with (12).

APPENDIX B

For the case of electron pair production in the field of an electron, we may write

$$\Gamma_1(k,q,1)=\frac{2q}{W(W-1)^2}\left\{\left((W-1)(W-5)+q(2-W)+\frac{W-1}{q-W+1}\right)\frac{R(k,q,1)^{\frac{1}{2}}}{k(q-W+1)}+\left((W-2)(q-W+1)+\frac{2(W-1)(q-W)}{q-W+1}\right)L(k,q,1)-2(W-1)I(k,q,1)\right\}, \quad \text{(B.1)}$$

where now

$$R(k,q,1)=\{k(q-W+1)-(W-1)\}\{k(q-W+1)-(W+1)\}$$

$$L(k,q,1)=\ln\frac{\{k(q-W+1)-(W-1)\}^{\frac{1}{2}}+\{k(q-W+1)-(W+1)\}^{\frac{1}{2}}}{\sqrt{2}},$$

$$N(k,q,1)=\ln\frac{k(q-W+1)W-q^2-qR(k,q,1)^{\frac{1}{2}}}{k(q-W+1)}, \tag{B.2}$$

$$I(k,q,1)=\int_{q_{\min}}^{q}F(k,q,1)dq,\quad F(k,q,1)=\frac{1}{W}\left\{\frac{N(k,q,1)}{2}+\frac{(W-q)L(k,q,1)}{q-W+1}\right\}.$$

When $k\gg1$ and $q\ll k$, with exception of the case where q is so small that $kq\sim2$, we can approximate $R(k,q,1)$ and $N(k,q,1)$ in the following way:

$$R(k,q,1)^{\frac{1}{2}}\simeq k(q-W+1)-W-\tfrac{1}{2}\frac{1}{k(q-W+1)-W}\simeq k(q-W+1), \tag{B.3}$$

$$N(k,q,1)\simeq\ln\left[(W-q)\left(1+\frac{q}{k(q-W+1)}\right)\right]\simeq\ln(W-q).$$

Further simplifications can be made in the following two cases:

(a) For $q\ll1$ but $kq\gg1$, we may as above approximate

$$L(k,q,1)\simeq\tfrac{1}{2}\ln[kq(2-q)]\simeq\tfrac{1}{2}\ln(2kq),$$
$$I(k,q,1)\simeq\tfrac{1}{4}\ln^2(2kq)-\tfrac{1}{12}\pi^2. \tag{B.4}$$

We also make similar approximations in the coefficients of $R(k,q,1)^{\frac{1}{2}}$, $L(k,q,1)$, and $I(k,q,1)$ in (B.1). Then substitution of (B.3) and (B.4) into (B.1) yields the approximate expression

$$\Gamma_1(k,q,1)\simeq\frac{4}{q}\left\{\left(\frac{3}{q}-\frac{7}{2}\right)-\left(\frac{2}{q}-1\right)\ln(2kq)-\tfrac{1}{2}\ln^2(2kq)+\frac{\pi^2}{6}\right\}. \tag{B.5}$$

(b) For $q\gg1$ but $q\ll k$, we can approximate

$$L(k,q,1)\simeq\tfrac{1}{2}\ln\{2k(q-W+1)\}\simeq\tfrac{1}{2}\ln(2k),$$
$$I(k,q,1)\simeq\tfrac{1}{4}\ln(2kq)\ln(k/2q)+\tfrac{1}{2}\ln(2k)-\tfrac{1}{12}t^2+1. \tag{B.6}$$

We also make similar approximations in the coefficients of $R(k,q,1)^{\frac{1}{2}}$, $L(k,q,1)$, and $I(k,q,1)$ in B(1). Then substitution of (B.3) and (B.6) into (B.1) yields the approximate expression

$$\Gamma_1(k,q,1)\simeq(1/q^2)\{(8-6q)+q(\tfrac{1}{3}\pi^2+4-\ln(2k)-\ln(2kq)\ln(k/2q))\}. \tag{B.7}$$

The correction term $k^{-1}\Gamma_1(k,q,1)$ is negative and its absolute value is a decreasing function of k.