tials. In other words, we are led to regard $A_\mu(x)$ as a physical variable. This means that we must be able to define the physical difference between two quantum states which differ only by gauge transformation. It will be shown in a future paper that in a system containing an undefined number of charged particles (i.e., a superposition of states of different total charge), a new Hermitian operator, essentially an angle variable, can be introduced, which is conjugate to the charge density and which may give a meaning to the gauge. Such states have actually been used in connection with recent theories of superconductivity and superfluidity[12] and we shall show their relation to this problem in more detail.

ACKNOWLEDGMENTS

We are indebted to Professor M. H. L. Pryce for many helpful discussions. We wish to thank Dr. Chambers for many discussions connected with the experimental side of the problem.

[12] See, for example, C. G. Kuper, *Advances in Physics*, edited by N. F. Mott (Taylor and Francis, Ltd., London, 1959), Vol. 8, p. 25, Sec. 3, Par. 3.

PHYSICAL REVIEW VOLUME 115, NUMBER 3 AUGUST 1, 1959

Theory of Multiple Scattering: Second Born Approximation and Corrections to Molière's Work

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(Received March 11, 1959)

The formula given by Molière for the scattering cross section of a charged particle by an atom, on which has been based the formula for the "screening angle" χ_α in his theory of multiple scattering, has been examined and found to contain an inconsistent approximation in all orders of the parameter $\alpha_1 = zZ/137\beta$ except the lowest (the first Born approximation). In the present work, the correct expression of Dalitz is used for the single-scattering cross section of a relativistic Dirac particle by a screened atomic field up to the second Born approximation. It is found that the effect of the deviation from the first Born approximation on the screening angle is much smaller than Molière's expression for this quantity would lead one to believe. This is so because the deviation from the first Born approximation is very small at the small angles that go into the definition of the screening angle. In Molière's work, all the effect of the deviation from the first Born approximation on the distribution function $f(\theta)$ for multiple scattering is contained in the quantity B which depends only on χ_α. In the present work, it is shown that in a consistent treatment of terms of various orders in α_1, there exist additional terms of order $zZ/137$ in the distribution function. These terms, which represent the second Born approximation, become important at large angles. Calculations have been carried out for the scattering of 15.6-Mev electrons by Au and Be. The $1/e$ widths of the distribution function obtained are in good agreement with the experimental result of Hanson *et al.*, whereas Molière's theory gives too great a width compared with the experimental value in the case of Be.

I. INTRODUCTION

THE theory of scattering of fast charged particles by atoms is of importance for the analysis of such experimental results as the scattering of high-energy mesons and electrons in going through sheets of matter. An "exact" theory of multiple scattering has been given by Goudsmit and Saunderson.[1] Its application to a specific scattering problem invokes the knowledge of the law of single scattering by an isolated atom. In a paper in 1947, Molière[2] gives a (nonrelativistic) formula for the scattering of a fast charged particle by a screened Coulomb field, in which an approximation higher than the usual first Born approximation is attempted. In a second paper Molière[3] gives a theory of multiple scattering which has later been shown by Bethe[4] to be obtainable from the theory of Goudsmit and Saunderson by making certain approximations. For the single-scattering law to be used in the theory of multiple scattering, Molière uses the result he obtained in his earlier paper.[2]

Hanson *et al.*[5] have measured the scattering of 15.6-Mev electrons by gold and beryllium foils and compared their experimental results with those calculated according to Molière's theory. The calculated "$1/e$ width" of the distribution has been found to be in excellent agreement with the observed value in the case of gold, but is somewhat too large in the case of beryllium.

In the case of the scattering of μ mesons (in cosmic rays) by matter, the rather scanty data[6] (for large scattering angles) seem to be in agreement with Molière's theory. Here, for high enough energies of the

* National Research Council Postdoctorate Fellows.
[1] S. A. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940), and **58**, 36 (1940).
[2] G. Molière, Z. Naturforsch. **2a**, 133 (1947).
[3] G. Molière, Z. Naturforsch. **3a**, 78 (1948).
[4] H. A. Bethe, Phys. Rev. **89**, 1256 (1953).
[5] Hanson, Lanzl, Lyman, and Scott, Phys. Rev. **84**, 634 (1951).
[6] George, Redding, and Trent, Proc. Phys. Soc. (London) **A66**, 533 (1953); I. B. McDiarmid, Phil. Mag. **45**, 933 (1954); **46**, 177 (1955).

μ mesons when the de Broglie wavelength is comparable with nuclear dimensions, the scattering theory must be developed for a charge distribution in the nucleus instead of for a point charge. Many such calculations,[7] some of which are based on the relativistic theory, have been carried out in recent years. When the law of single scattering is modified to take into account the finite size of the charge distribution in the nuclei, the theoretical distribution for multiple scattering becomes too small, at large angles, by a factor of order 10, compared with the admittedly scanty observed data.[8] This has led some authors to suggest that this anomalous scattering might indicate some interactions other than Coulombic between fast μ mesons and the atomic nuclei. Such a suggestion also encounters difficulties. Any modification of the potential between μ mesons and nuclei must be in accord with the experiments on the interaction of stopped μ mesons with nuclei.[9] Also the poor statistics of the present data does not provide enough ground for taking such a suggestion seriously.

Owing to such discrepancies, however, it seems of importance to re-examine the theory of Molière on which many such comparisons with experimental data have been based. It is the purpose of the present work to do so and to give the distribution function for multiple scattering which contains a correct second Born approximation for the single scattering in the screened Coulomb field.

In Secs. II–V, it is shown that the result for the single-scattering cross section given by Molière is incorrect. In Sec. VI, Dalitz's[10] relativistic formula, derived in the second Born approximation, for the scattering of a spin-$\frac{1}{2}$ particle in an exponentially screened Coulomb field has been used to derive Molière's distribution function $f(\theta,t)$ for multiple scattering. We have followed the procedure of Bethe[4] starting with the exact expression of the distribution function as given by Goudsmit and Saunderson.[1] The calculations are carried out correctly to the next order of $zZ/137$; the screening angle parameter χ_α and the parameter B are redefined. The expansion of the distribution function obtained by us is then carried out in powers of $1/B$. The numerical values of the distribution function up to terms of order $1/B$ are tabulated in Table III for Au and Be. For the sake of comparison Bethe's values are also included in the table.

[7] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956), and references quoted therein.

[8] However, the situation in this respect is somewhat clarified by the experiments of Fukui, Kitamura, and Watase. According to them, while one experimental setup gives results in agreement with the theory of Coulomb scattering from extended nuclei, the other setup is in agreement with Molière's theory for a point nucleus. We are indebted to Professor J. Rainwater for bringing this work to our attention.

[9] Conversi, Pancini, and Piccioni, Phys. Rev. **71**, 209 (1947); V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).

[10] R. H. Dalitz, Proc. Roy Soc. (London) **A206**, 509 (1951).

II. EXACT THEORY OF SCATTERING BY A CENTRAL FIELD

To facilitate the discussion of Molière's theory we shall give the exact theory (Faxen-Holtsmark) of scattering and show how Molière's various results are obtainable as approximations to the exact theory. The exact theory is well known. For a particle of mass m, momentum $\mathbf{p}=\hbar\mathbf{k}$, scattered by a field $V(r)$, the scattered amplitude in a direction χ from the direction of $\mathbf{p}$ is given by

$$f(\chi)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\chi),\tag{1}$$

where the phase shifts δ_l are given by

$$\sin\delta_l=-\frac{2m}{\hbar^2}\int^{\infty}\left(\frac{\pi kr}{2}\right)^{\frac{1}{2}}J_{l+\frac{1}{2}}(kr)V(r)\psi_l(r)dr.\tag{2}$$

Here $\psi_l(r)$ is the radial wave function of the scattered particle which behaves asymptotically for large distances r as

$$\psi_l(r)\sim\sin\left(kr-\frac{l\pi}{2}+\delta_l\right)\Big/k.\tag{3}$$

The function $\psi_l(r)$, being a solution of the Schrödinger equation with the potential $V(r)$, depends on $V(r)$. If ze, Ze are the charges of the incident and scattering particle, respectively, i.e.,

$$\lim_{r\to 0} rV(r)=zZe^2,\tag{4}$$

and v is the velocity of the incident particles, it is convenient to introduce the parameter α_1,

$$\alpha_1=zZe^2/\hbar v=zZ/137\beta\equiv\alpha/\beta,\tag{5}$$

β being v/c. For $\alpha_1\ll 1$, one may expand δ_l in (2) in powers of α_1,

$$\delta_l=\alpha_1\delta_l^{(0)}+\alpha_1^2\delta_l^{(1)}+\alpha_1^3\delta_l^{(2)}+\cdots.\tag{6}$$

Substituting for δ_l from Eq. (6) in Eq. (1) and expanding $e^{2i\delta_l}$, one gets[11]

$$\begin{aligned}f(\chi)=(1/k)\textstyle\sum_l(2l+1)\{&\alpha_1\delta_l^{(0)}+\alpha_1^2\delta_l^{(1)}\\&+i(\alpha_1\delta_l^{(0)})^2+\alpha_1^3\delta_l^{(2)}+2i\alpha_1^3\delta_l^{(0)}\delta_l^{(1)}\\&-\tfrac{4}{3}(\alpha_1\delta_l^{(0)})^3+\cdots\}P_l(\cos\chi),\end{aligned}\tag{7}$$

and

$$\begin{aligned}|f(\chi)|^2=|f(\chi)_{\text{Born}}|^2&+2f(\chi)_{\text{Born}}\\&\times[(\alpha_1^2/k)\textstyle\sum_l(2l+1)\delta_l^{(1)}P_l(\cos\chi)]+\cdots.\end{aligned}\tag{8}$$

The first term on the right in Eq. (8), of order α_1^2, is the usual Born approximation which is obtained from the exact theory by making two *consistent* approximations of

$$\text{(i) replacing } e^{2i\delta_l}-1 \text{ by } 2i\delta_l,\tag{9}$$

[11] T. Y. Wu, Phys. Rev. **73**, 934 (1948).

and

(ii) replacing δ_l in Eq. (2) by

$$\alpha_1\delta_l^{(0)} = -\frac{\pi m}{\hbar^2}\int_0^\infty [J_{l+\frac{1}{2}}(kr)]^2 rV(r)dr. \tag{10}$$

For a screened Coulomb field,

$$V(r) = -\frac{zZe^2}{r}e^{-\lambda r}, \tag{11}$$

one obtains

$$\alpha_1\delta_l^{(0)} = \pi k\alpha_1\int_0^\infty [J_{l+\frac{1}{2}}(kr)]^2 e^{-\lambda r}dr, \tag{12}$$

and the familiar result

$$f(\chi)_{\mathrm{Born}} = \frac{m}{2\pi\hbar^2}\int e^{\frac{i}{\hbar}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}V(r)d^3r$$
$$= 2zZe^2m/\hbar^2(q^2+\lambda^2), \quad q = 2k\sin(\chi/2). \tag{13}$$

For the Coulomb field, one obtains, on letting $\lambda \to 0$, the *exact* result which is also the Rutherford formula

$$|f(\chi)_R|^2 = |f(\chi)_{\mathrm{Born}}|^2$$
$$= \left(\frac{zZe^2m}{2k^2\hbar^2}\right)^2 \frac{1}{\sin^4(\chi/2)} = \left(\frac{2\alpha_1}{k}\right)^2\frac{1}{[2\sin(\chi/2)]^4}. \tag{14}$$

The second term in Eq. (8) gives the second Born approximation and is of order α_1^3 since $f(\chi)_{\mathrm{Born}}$ is of order α_1.

III. MOLIÈRE'S APPROXIMATION

From the exact theory above, it is very simple to obtain the results of Molière as approximations. Thus on making in Eq. (1) the replacements

(i) $\quad P_l(\cos\chi)$ by $(\chi/\sin\chi)^{\frac{1}{2}}J_0[(l+\frac{1}{2})\chi]$, (15)

(ii) $\quad \sum_l (2l+1)$ by $2\int l dl$, (16)

of the exact theory, and introducing the parameter ρ by

$$l\hbar = \rho\hbar k, \tag{17}$$

one obtains directly from Eq. (1) Molière's formulas [Eqs. (4.5) and (4.6) of Molière's paper[2]],

$$f(\chi)_M \simeq \frac{1}{2ik}\left(\frac{\chi}{\sin\chi}\right)^{\frac{1}{2}}\int_0^\infty 2l dl J_0(l\chi)(e^{i\Phi_l}-1)$$
$$= \frac{k}{i}\left(\frac{\chi}{\sin\chi}\right)^{\frac{1}{2}}\int_0^\infty \rho d\rho J_0(\rho k\chi)(e^{i\Phi(\rho)}-1), \tag{18}$$

where Φ_l is Molière's notation for $2\delta_l$. In Molière's paper, wherever calculations are explicitly carried out [see Eqs. (2.2) and (6.2), (6.4) of Molière's paper[2]], for the potential with exponential screening the following approximation for Φ_l has always been made, namely,

$$\alpha_1\Phi(\rho) = \frac{2zZe^2m}{\hbar^2 k}\int_\rho^\infty \frac{e^{-\lambda r}}{(r^2-\rho^2)^{\frac{1}{2}}}dr$$
$$= 2\alpha_1\int_\rho^\infty \frac{e^{-\lambda r}}{(r^2-\rho^2)^{\frac{1}{2}}}dr, \tag{19}$$

or

$$\Phi(\rho) = 2K_0(\lambda\rho), \tag{20}$$

where $K_0(\lambda\rho)$ is a Bessel function. That the expression (19) for $\Phi_l = \alpha_1\Phi(\rho)$ is in actual fact the *first* Born approximation, Eq. (12), can be seen either from Eqs. (2.1)–(2.2) of Molière's paper,[2] or from a comparison with Eq. (12) which gives

$$\alpha_1\delta_l^{(0)} = \alpha_1\pi k\frac{1}{\pi k}Q_l\left(1+\frac{\lambda^2}{2k^2}\right), \tag{21}$$

where Q_l is the Legendre function of the second kind. Since ρ is a continuous variable and l discrete, the equivalence of in (20) of Molière and $\delta_l^{(0)}$ is best seen by noting

$$\lim_{l\to\infty} Q_l(\cosh(y/l)) = K_0(y),$$

and

$$\lim_{l\to\infty} Q_l\left(1+\frac{\lambda^2}{2k^2}\right) = K_0(\lambda l/k) = K_0(\lambda\rho),$$

so that

$$\Phi_l = \alpha_1\delta_l^{(0)}. \tag{22}$$

The expression (18) obviously differs from the Rutherford formula, and Molière proposes to make in Eq. (18) the replacement

$$\frac{\chi}{\sin\chi} \to \left(\frac{\chi}{2\sin(\chi/2)}\right)^4, \tag{23}$$

so that the "adjusted" $f(\chi)_M$ leads to his expression in Eq. (4.6),

$$|f(\chi)_M|^2 = |f(\chi)_R|^2\frac{(k\chi)^4}{4\alpha_1^2}$$
$$\times\left|\int_0^\infty \rho d\rho J_0(k\lambda\rho)(e^{i\Phi(\rho)}-1)\right|^2. \tag{24}$$

For the potential of a Thomas-Fermi atom,

$$V(r) = -\frac{zZe^2}{r}\omega(r\lambda_0), \quad \lambda_0 = Z^{\frac{1}{3}}/0.885a_0, \tag{25}$$

where a_0 = Bohr radius = 0.5292×10^{-8} cm, the quantity $\Phi(\rho)$ in Eqs. (19) and (20) becomes

$$\Phi(\rho) = 2\int_\rho^\infty \frac{\omega(r\lambda_0)}{(r^2-\rho^2)^{\frac{1}{2}}}dr, \tag{26}$$

which of course is still the *first* Born approximation as shown above. On introducing

$$\chi_0=\hbar\lambda_0/p=\lambda_0/k, \quad y=\rho\lambda_0=l\lambda_0/k, \tag{27}$$

Eqs. (26) and (24) become

$$\Phi(y)=2\int_y^\infty \frac{\omega(r')dr'}{(r'^2-y^2)^{\frac{1}{2}}}, \tag{26'}$$

$$|f(\chi)_M|^2=|f(\chi)_R|^2\frac{1}{4\alpha_1^2}\left(\frac{\chi}{\chi_0}\right)^4 \times\left|\int_0^\infty ydyJ_0\left(\frac{\chi}{\chi_0}y\right)(e^{i\alpha_1\Phi(y)}-1)\right|^2, \tag{28}$$

which is Eq. (6.6) of Molière's[2] paper and is the one used in obtaining his final formula, Eq. (8.4), given in the Summary of his paper.

It is clear from this simplified treatment of Molière's[2] derivation that the result, Eq. (28), contains the approximations (15) and (16) together with the artifice (23) as a compensating approximation. If Eq. (19) or (26) is used, which are the first Born approximations, then (28) can yield a consistent first Born approximation but not the higher Born approximations.

IV. MOLIÈRE'S FORMULA FOR SINGLE SCATTERING (THOMAS-FERMI FIELD)

In applying Eq. (28) to the scattering of a charged particle by an atom with the screening due to atomic electrons taken into account, Molière represents the Thomas-Fermi function $\omega(r\lambda_0)$ in Eq. (25) by

$$\omega(r')=\omega(r\lambda_0)=\sum_{i=1}^{3} a_i e^{-b_i r'}, \tag{29}$$

where a_i and b_i are constants. On using Eq. (26′), one obtains, as in Eq. (20),

$$\Phi(y)=2\sum_{i=1}^{3} a_i K_0(b_i y), \tag{30}$$

where [$K_0(x)$ being denoted as $L_0(x)$ in Molière's Eq. (7.2)]

$$K_0(x)=-I_0(x)\ln(x/2) +\sum_{m=0}^{\infty}(x/2)^{2m}\frac{1}{(m!)^2}\psi(m+1). \tag{31}$$

Here $I_0(x), \psi(m+1)$ are the Bessel function and logarithmic derivative of the gamma function,[12] respectively.

The crucial steps taken by Molière are then the following:

(i) $\Phi(y)$ in Eq. (30) is expanded, after using the numerical values of a_i and b_i in Eq. (29),

$$\Phi(y)=0.516-2\ln y-0.81y^2-2.21y^2\ln y+\cdots. \tag{32}$$

[12] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1948).

(ii) Substituting for $\Phi(y)$ from Eq. (32) into Eq. (28), neglecting 1 in comparison to the exponential term, and setting $x=y\chi/\chi_0$, $f(\chi)_M$ in Eq. (28) is obtained from [see (37) below]

$$\frac{|f(\chi)_M|^2}{|f(\chi)_R|^2}=\frac{1}{2\alpha_1^2}\left|\int_0^\infty xdxJ_0(x)\left[x^{-2i\alpha_1}+\frac{i\alpha_1x^{2(1-i\alpha_1)}}{(\chi/\chi_0)^2}\times\left(-0.81-2.21\ln\frac{x}{(\chi/\chi_0)}\right)\right]\right|^2; \tag{33}$$

this leads, upon integration, to the final result

$$\frac{|f(\chi)_M|^2}{|f(\chi)_R|^2}=1-\frac{8.85}{(\chi/\chi_0)^2}\left[1+\alpha_1^2\times 2.303\log_{10}\left(\frac{0.00072\chi^4}{(0.13+\frac{1}{3}\alpha_1^2+\alpha_1^4)\chi_0^4}\right)\right], \tag{34}$$

which forms the basis of his theory of multiple scattering.

V. REMARKS ON MOLIÈRE'S RESULT, EQ. (34)

(A) We shall now show that the above method and hence the resulting Eq. (34) for the ratio (f_M/f_R) is incorrect except to the order involving the zeroth power of α_1.[13] In using the *first* Born approximation, Eq. (26) or Eq. (26′), for the calculation of $\Phi(y)=2\alpha_1\delta_l^{(0)}$ in Eq. (30), the scattered amplitude $f(\chi)_M$ is essentially

$$f(\chi)_M=\frac{1}{k}\sum_l(2l+1)[\alpha_1\delta_l^{(0)}+i(\alpha_1\delta_l^{(0)})^2 -\tfrac{4}{3}(\alpha_1\delta_l^{(0)})^3+\cdots]P_l(\cos\chi), \tag{35}$$

which is obtained from Eq. (7) by dropping all the higher order phase shifts $\delta_l^{(1)}$, $\delta_l^{(2)}$, etc. It is for this reason that in Eq. (34), no term of order α_1 appears, which would have been there if $\delta_l^{(1)}+i(\delta_l^{(0)})^2$, which are both of the same order in α_1, were used as in Eq. (7), instead of $i(\delta_l^{(0)})^2$ in Eq. (35). For the same reason, the term of order α_1^2 in Eq. (34) is incorrect since

$$\delta_l^{(2)}+2i\delta_l^{(0)}\delta_l^{(1)}-\tfrac{4}{3}(\delta_l^{(0)})^3$$

must be used instead of $-\frac{4}{3}(\delta_l^{(0)})^3$. Thus the calculation in Eqs. (30), (33), and (34) involves an inconsistent approximation in which not all the terms of the same order in α_1 are included in each order except the first. Reference to Eqs. (7) and (8) shows that the correct formula for $|f(\chi)|^2/|f(\chi)_R|^2$ contains a term of order α_1 coming from $\delta_l^{(1)}$ in Eq. (7).

[13] The discussion in Sec. *V* appears in the mimeographed notes of lectures given by T. Y. Wu at the Joint Conference of the Canadian Mathematical Congress and the Theoretical Physics Division of the Canadian Association of Physicists, held at the University of Alberta in August, 1957. That the expression (34) cannot be correct in its dependence on α_1 is most easily seen by comparing it with the correct expression (40) for $\sigma_D(\chi)$ given by Dalitz in which one passes to the nonrelativistic case by making $\beta\rightarrow 0$.

The statement (at the end of Sec. 6 of Molière's paper) that the scattering is independent of the sign of the potential is also seen to be correct only if one neglects the higher-order $\delta_l^{(1)}$ in Eq. (8).

(B) The detailed calculations in Eqs. (32)–(34) are objectionable for the following reason. The Bessel function $K_0(x)$ in Eqs. (30) and (31) has the following asymptotic behaviors,

$$K_0(x) \to -\ln x \to \infty \quad \text{as } x \to 0,$$
$$K_0(x) \to e^{-x} \to 0 \quad \text{as } x \to \infty; \tag{36}$$

when the expression (32) containing terms only up to y^2 is used for $\Phi(y)$, it is seen to be incorrect for large y. Since the integration in Eqs. (28) or (33) is over the range

$$0 \leqslant y \leqslant \infty,$$

the resulting Eq. (34) arising from the integration in Eq. (33) cannot be correct.

Equation (33) involves the further expansion of

$$\begin{aligned} e^{i\alpha_1\Phi(y)} &= \exp\{-i\alpha_1[0.516-2\ln y \\ &\qquad -y^2(0.81+2.21\ln y)]\} \\ &= Ay^{-2i\alpha_1}\exp[-i\alpha_1 y^2(0.81+2.21\ln y)] \\ &= Ay^{-2i\alpha_1}[1-i\alpha_1 y^2(0.81+2.21\ln y)]. \end{aligned} \tag{37}$$

A consistent calculation at this point would require an expansion of $y^{-2i\alpha_1}$, and the retaining of terms up to order α_1. If this is done, then one would have obtained a term of order α_1^2 in Eq. (34) [which is however incorrect since terms of the same order from $\delta_l^{(2)}$, $\delta_l^{(1)}$ have been forgotten, as discussed in (A) above]. The use of the full expression $y^{-2i\alpha_1}$ in Eq. (37) implies Eq. (35) for this part [i.e., with (32) for $\Phi(y)$] of $\exp[i\alpha_1\Phi(y)]$. Thus, on retaining $y^{-2i\alpha_1}$ completely, while retaining terms only up to α_1 in $\exp[-i\alpha_1 y^2 \times(0.81+2.21\ln y)]$ in Eq. (37), one obtains Eq. (34) which is incorrect in the higher orders in α_1 in the logarithm.

From this it is clear that the formula, Eq. (34), is incorrect to all orders of α_1 except the zeroth and this correct part

$$\frac{|f(\chi)|^2}{|f(\chi)_R|^2} = 1 - \frac{8.85}{(\chi/\chi_0)^2} \tag{38}$$

can be obtained very simply by making use of the familiar result, Eq. (13),

$$f(\chi) = f(\chi)_R \times q^2 \sum_i \frac{a_i}{q^2+(b_i\lambda_0)^2}, \tag{39}$$

which passes again to the Rutherford $f(\chi)_R$ if b_i in Eq. (39) approaches zero, i.e., for an unscreened Coulomb field. For the set of a_i, b_i of Molière,

$$a_1=0.1, \quad a_2=0.55, \quad a_3=0.35,$$
$$b_1=6, \quad b_2=1.2, \quad b_3=0.3,$$
$$\lambda_0 \text{ as given in (25)},$$

one obtains by direct calculation

$$\frac{q^2}{\lambda_0^2}\sum_i \frac{a_i}{b_i^2+(q/\lambda_0)^2} \simeq 1 - \frac{4.425}{(q/\lambda_0)^2}, \quad \frac{\lambda_0}{q} < 1,$$

so that by using Eq. (39) one obtains Eq. (38) for $\chi > \chi_0$.

(C) The correct expression for $|f(\chi)|^2$ up to higher orders of α_1 can be obtained by the method of *Wu*[11] and of Jost and Pais.[14] In fact $|f(\chi)|^2$ has been calculated for the potential (11) in the relativistic theory by Dalitz.[10] A direct calculation of $|f(\chi)|^2$ for the Thomas-Fermi potential (25), or the "actual" potential $V(r)$, would be more complicated, but as we shall see in Sec. VII below, such a calculation is not necessary, and one can extend to any screened field $V(r)$ the result obtained by using the exponential screening. For high-energy particles scattered through large angles (where the agreement between the observed μ-meson scatterings and the theoretical values seem to be in question), the effect of screening by the atomic electrons can be neglected, but then the charge distribution over the finite size of the nucleus must be taken into account. Many calculations have recently been made along this line.[7] These correct results instead of Eq. (34) for the single scattering must be employed in the exact theory of multiple scattering,[1] and for the comparison with the experimental data.

VI. DISTRIBUTION FUNCTION, TO SECOND BORN APPROXIMATION, FOR SINGLE SCATTERING

In the previous sections it has been shown that the expression for the single-scattering cross section given by Molière is correct only to the first Born approximation. In this section we rederive an expression for the distribution function $f(\theta,t)$ for multiple scattering using the relativistic formula for the scattering cross section derived by Dalitz[10] for the exponentially screened potential (11). We do not, in this paper, take into account the complications introduced by the charge distribution over the finite size of the nucleus, but shall simply treat the nuclei as point charges.

The formula for the single-scattering cross section given by Dalitz for the scattering of a particle with charge ze, mass m, momentum p, energy E, on a point nucleus of charge Ze (screened exponentially) is

$$\begin{aligned} \sigma_D(\chi) = \frac{4\alpha^2\hbar^2E^2}{c^2[\hbar^2\lambda^2+4p^2\sin^2(\chi/2)]^2}\Big\{&[1-\beta^2\sin^2(\chi/2)] \\ &\times\left[1-[\hbar^2\lambda^2+4p^2\sin^2(\chi/2)]\frac{\alpha}{\beta}\frac{1}{\pi^2}\operatorname{Re}(I+J)\right] \\ &-(1-\beta^2)\frac{\alpha}{\beta}[\hbar^2\lambda^2+4p^2\sin^2(\chi/2)]\frac{1}{\pi^2}\operatorname{Re}(I-J)\Big\}, \end{aligned} \tag{40}$$

[14] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

where

$$\frac{1}{\pi^2}\operatorname{Re}(I\pm J)=-\frac{1}{\sin(\chi/2)\{\hbar^4\lambda^4+4p^2[\hbar^2\lambda^2+p^2\sin^2(\chi/2)]\}^{\frac{1}{2}}}\left[1\pm\frac{\hbar^2\lambda^2+2p^2}{2p^2\cos^2(\chi/2)}\right]$$

$$\times\tan^{-1}\left(\frac{\hbar\lambda p\sin(\chi/2)}{\{\hbar^4\lambda^4+4p^2[\hbar^2\lambda^2+p^2\sin^2(\chi/2)]\}^{\frac{1}{2}}}\right)\pm\frac{1}{2p^2\cos^2(\chi/2)}\left[\tan^{-1}\left(\frac{2p}{\hbar\lambda}\right)-\frac{1}{\sin(\chi/2)}\tan^{-1}\left(\frac{p\sin(\chi/2)}{\hbar\lambda}\right)\right]. \quad (41)$$

Here λ is the screening parameter introduced in Eq. (11) and which we now take to be $\mu(Z^{\frac{1}{3}}/0.885a_0)$ (a_0=Bohr radius=0.5292×10^{-8} cm) in this paper; μ is a constant of order unity (see Sec. VII).

In deriving the multiple-scattering distribution, we follow the simplified method of Bethe.[4] According to Goudsmit and Saunderson,[1] the angular distribution, for any angle small or large, is given *exactly* by

$$f(\theta,t)=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})P_l(\cos\theta)$$
$$\times\exp\left\{-Nt\int_0^{\pi}\sin\chi d\chi\sigma_D(\chi)[1-P_l(\cos\chi)]\right\}, \quad (42)$$

where $f(\theta,t)\sin\theta d\theta$ is the actual number of scattered particles between θ and $\theta+d\theta$. Here t is the actual distance travelled along the path of the scattered particle; however, we shall make the approximation of taking t to be the foil thickness. N is the number of scattering atoms per cm^3. Now our purpose is to simplify the right-hand side of Eq. (42) and find an expression for $f(\theta,t)$ using the cross section $\sigma_D(\chi)$ given by Dalitz. We shall write

$$Nt\sigma_D(\chi)=2\chi_c^2\frac{1}{4(1-\cos\chi)^2}q(\chi), \quad (43)$$

where we have introduced[4]

$$\chi_c^2=4\pi Nte^4Z(Z+1)z^2/(pc\beta)^2,$$

and

$$q(\chi)=\left(\frac{4p^2\sin^2(\chi/2)}{\hbar^2\lambda^2+4p^2\sin^2(\chi/2)}\right)^2$$
$$\times\left\{1-\beta^2\sin^2(\chi/2)-[1-\beta^2\sin^2(\chi/2)]\right.$$
$$\times[\hbar^2\lambda^2+4p^2\sin^2(\chi/2)]\frac{\alpha}{\beta}\frac{1}{\pi^2}\operatorname{Re}(I+J)$$
$$\left.-(1-\beta^2)\frac{\alpha}{\beta}[\hbar^2\lambda^2+4p^2\sin^2(\chi/2)]\frac{1}{\pi^2}\operatorname{Re}(I-J)\right\}. \quad (44)$$

In order to evaluate the integral on the right-hand side of Eq. (42), following Bethe, we break up the integral into two parts, from 0 to k and from k to π, where we choose k such that

$$\chi_0\ll k\ll 1/l, \quad (45)$$

where, as in (27),

$$\chi_0=\hbar\lambda/p. \quad (46)$$

Let us write the exponential in Eq. (42) as

$$\exp[-Q_l'-Q_l''],$$

where

$$Q_l'=Nt\int_0^k\sigma_D(\chi)\sin\chi d\chi[1-P_l(\cos\chi)]$$

and

$$Q_l''=Nt\int_k^{\pi}\sigma_D(\chi)\sin\chi d\chi[1-P_l(\cos\chi)]. \quad (47)$$

Consider the evaluation of Q_l' first. If we write $Nt\sigma_D(\chi)$ in the form given in Eq. (43), we see that $q(\chi)$ will occur in the integrand. Substituting the values of $\operatorname{Re}(I\pm J)$ into Eq. (44), we can write $q(\chi)$ in a somewhat different form:

$$q(\chi)=\left[\frac{2(1-\cos\chi)}{\chi_0^2+2(1-\cos\chi)}\right]^2\left\{1-\beta^2\sin^2(\chi/2)+2\frac{\alpha}{\beta}\left[\left(\frac{\chi_0}{\sin(\chi/2)}\right)^2+4\right]X\tan^{-1}(\chi_0X)+\alpha\beta[\chi_0^2+4\sin^2(\chi/2)]\right.$$
$$\left.\times\left(\left[\frac{1}{2}\left(\frac{\chi_0}{\sin(\chi/2)}\right)^2-1\right]X\tan^{-1}(\chi_0X)-\frac{1}{2}\left[\tan^{-1}\left(\frac{2}{\chi_0}\right)-\frac{1}{\sin(\chi/2)}\tan^{-1}\left(\frac{\sin(\chi/2)}{\chi_0}\right)\right]\right)\right\}, \quad (48)$$

where

$$X=\sin(\chi/2)\{\chi_0^4+4[\chi_0^2+\sin^2(\chi/2)]\}^{-\frac{1}{2}}.$$

In the region where χ is small, $0<\chi<k$, we can replace $\sin(\chi/2)$ by $\chi/2$ itself and the arctan by its argument, thus obtaining

$$q(\chi)=\left(\frac{\chi^2}{\chi_0^2+\chi^2}\right)^2\left\{1-\frac{\beta^2\chi^2}{4}+2\frac{\alpha}{\beta}\frac{\chi_0(\chi_0^2+\chi^2)}{\chi_0^4+4\chi_0^2+\chi^2}\right.$$
$$+\alpha\beta(\chi_0^2+\chi^2)\left[\frac{\chi_0}{\chi_0^4+4\chi_0^2+\chi^2}\left(\frac{\chi_0^2}{2}-\frac{\chi^2}{4}\right)\right.$$
$$\left.\left.-\frac{1}{2}\left(\tan^{-1}\frac{2}{\chi_0}-\frac{2}{\chi}\tan^{-1}\frac{\chi}{2\chi_0}\right)\right]\right\}. \quad (49)$$

Also we may use for χ small the relation

$$P_l\left(1-\frac{\chi^2}{2}+\frac{\chi^4}{24}\right)$$
$$\simeq 1-\frac{l(l+1)}{4}\chi^2\left[1+\left(\frac{1}{6}-\frac{l(l+1)}{4}\right)\frac{\chi^2}{4}\right]. \quad (50)$$

Using (43) and the simplified expressions (49) and (50) in Eq. (46), we have

$$Q_l'=\tfrac{1}{2}\chi_c^2 l(l+1)$$
$$\times\int_0^k\frac{d\chi}{\chi}q(\chi)\left[1+\left(\frac{1}{6}-\frac{l(l+1)}{4}\right)\frac{\chi^2}{4}\right]. \quad (51)$$

The integrals occurring in Eq. (51) are all elementary. We shall evaluate them up to terms of order k^2. Since $\chi_0=\hbar\lambda/p$ is of the order of 10^{-2} to 10^{-3} for electrons of a few Mev energy, we shall keep only terms of order χ_0 and $\chi_0\ln\chi_0$ in the final result and neglect χ_0^2 and other higher-order terms.[15] Then we obtain

$$Q_l'=\tfrac{1}{2}\chi_c^2 l(l+1)\left\{\ln\left(\frac{k}{\chi_0}\right)-\tfrac{1}{2}+L\frac{k^2}{8}-\frac{\beta^2k^2}{8}\right.$$
$$+\frac{2\alpha\chi_0}{\beta}\left[\ln\left(\frac{k}{\chi_0}\right)+L\frac{k^2}{8}-\tfrac{4}{3}\ln 2\right]-\pi\alpha\beta\frac{k^2}{8}$$
$$\left.+\alpha\beta\int_0^k\frac{d\chi}{\chi}\frac{\chi^4}{\chi_0^2+\chi^2}(1+\tfrac{1}{4}L\chi^2)\frac{1}{\chi}\tan^{-1}\left(\frac{\chi}{2\chi_0}\right)\right],$$

where L stands for $[\frac{1}{6}-\frac{1}{4}l(l+1)]$ in this expression. The last integral occurring in Q_l', can be written in such a way as to separate out its strongly k-dependent terms.

It is easy to see that

$$\int_0^k d\chi\frac{\chi^2}{\chi_0^2+\chi^2}(1+\tfrac{1}{4}L\chi^2)\tan^{-1}\left(\frac{\chi}{2\chi_0}\right)$$
$$=k\tan^{-1}\left(\frac{k}{2\chi_0}\right)-2\chi_0\ln\left(\frac{k}{2\chi_0}\right)$$
$$-2\chi_0\int_0^{k/2\chi_0}\frac{dx}{1+4x^2}\tan^{-1}x-\chi_0Lk^2$$
$$+\text{higher order terms.} \quad (52)$$

The integral occurring on the right-hand side of Eq. (52) has $k/2\chi_0$ as its upper limit. Now $k/2\chi_0$ is a large quantity and most of the value of the integral comes from the lower limit only. Thus we can replace the upper limit in this integral by ∞, with practically no error. Thus we have

$$Q_l'=\tfrac{1}{2}\chi_c^2 l(l+1)\left\{\ln\left(\frac{k}{\chi_0}\right)-\tfrac{1}{2}+L\frac{k^2}{8}-(\beta^2+\pi\alpha\beta)\frac{k^2}{8}\right.$$
$$+\frac{2\alpha}{\beta}\chi_0\left[\ln\left(\frac{k}{\chi_0}\right)+L\frac{k^2}{8}-\tfrac{4}{3}\ln 2\right]$$
$$+\alpha\beta\left[\frac{\pi}{2}k-2\chi_0-2\chi_0\ln\left(\frac{k}{2\chi_0}\right)-\chi_0L\frac{k^2}{4}\right.$$
$$\left.\left.-2\chi_0\int_0^\infty\frac{dx}{1+4x^2}\tan^{-1}x\right]\right\}. \quad (53)$$

Now let us proceed to the evaluation of Q_l''. In the range $k<\chi<\pi$, the expression (48) for $q(\chi)$ can be simplified by expanding it in powers of $[\chi_0/2\sin(\chi/2)]$ since this is a small quantity now. It is easy to show that to terms of order χ_0,

$$q(\chi)=1-\beta^2\sin^2(\chi/2)$$
$$+\pi\alpha\beta[\sin(\chi/2)-\sin^2(\chi/2)]+2(\alpha/\beta)\chi_0-2\alpha\beta\chi_0,$$

and from (43), (47),

$$Q_l''=\tfrac{1}{2}\chi_c^2\int_k^\pi d\chi\frac{\sin\chi}{(1-\cos\chi)^2}$$
$$\times\left[1-\beta^2\sin^2(\chi/2)+\pi\alpha\beta[\sin(\chi/2)-\sin^2(\chi/2)]\right.$$
$$\left.+\frac{2\alpha\chi_0}{\beta}(1-\beta^2)\right][1-P_l(\cos\chi)].$$

The integrals occurring in this expression, except the one involving $\pi\alpha\beta\sin(\chi/2)$, can be easily evaluated without any further approximation for the integrand[16] by the method developed by Bethe.[4] The integral involving the term $\pi\alpha\beta\sin(\chi/2)$ in the integrand can be easily evaluated by using the generating function for the

[15] It is for this reason we cannot go to the limit $\beta\rightarrow 0$ in the expressions that follow. If we want to include the limiting case of $\beta\rightarrow 0$ also, we should keep χ_0^4 in X and also the other dominant terms in the evaluation of Q_l' and Q_l''.

[16] The only restriction we have here again is that $k<1/l$.

Legendre polynomials. Thus we find, to terms of order k^2.

$$\int_k^\pi d\chi \frac{\sin\chi}{(1-\cos\chi)^2}[1-P_l(\cos\chi)] = \frac{l(l+1)}{2}\left[2\ln\left(\frac{2}{k}\right)-2\left(\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{l}\right)-L\frac{k^2}{4}+\cdots\right],$$

$$\int_k^\pi d\chi \sin\chi\frac{\sin^2(\chi/2)}{(1-\cos\chi)^2}[1-P_l(\cos\chi)] = \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\frac{1}{l}\right)-\frac{l(l+1)}{8}k^2+\cdots,$$

and

$$\int_k^\pi d\chi \sin\chi\frac{\sin(\chi/2)}{(1-\cos\chi)^2}[1-P_l(\cos\chi)] = 2l-\frac{l(l+1)}{2}k+\cdots.$$

Assembling the terms, and introducing $\Psi(l)+C=1+\frac{1}{2}+\frac{1}{3}+\cdots+(1/l)$, where $\Psi(l)$ is the logarithmic derivative of the Γ function and C is Euler's constant $=0.577\cdots$, we find

$$\begin{aligned}Q_l''=\tfrac{1}{2}\chi_0^2\Bigg\{&l(l+1)\left[\ln\left(\frac{2}{k}\right)-L\frac{k^2}{8}+1-\Psi(l)-C\right.\\&+(\beta^2+\pi\alpha\beta)\frac{k^2}{8}-\pi\alpha\beta\frac{k}{2}\\&\left.+2\alpha\chi_0\frac{(1-\beta^2)}{\beta}\left(\ln\frac{2}{k}-L\frac{k^2}{8}+1-\Psi(l)-C\right)\right]\\&-(\beta^2+\pi\alpha\beta)[\Psi(l)+C]+2\pi\alpha\beta l\Bigg\}.\end{aligned}$$

Combining Q_l' and Q_l'', we easily see that all the k-dependent terms cancel up to the order to which we have retained them.[17] In fact we have

$$\begin{aligned}Q_l'+Q_l''=\tfrac{1}{2}\chi_c^2\Bigg\{&l(l+1)\left[\ln\frac{2}{\chi_0}+\tfrac{1}{2}-\Psi(l)-C\right.\\&+\frac{2\alpha\chi_0}{\beta}\left(\ln\frac{2}{\chi_0}-\tfrac{4}{3}\ln 2\right)\\&-2\chi_0\alpha\beta+2\chi_0\alpha\beta\ln\chi_0\\&-2\chi_0\alpha\beta\int_0^\infty dx\frac{\tan^{-1}x}{1+4x^2}\\&\left.+\frac{2\alpha\chi_0}{\beta}(1-\beta^2)[1-\Psi(l)-C]\right]\\&-(\beta^2+\pi\alpha\beta)[\Psi(l)+C]+2\pi\alpha\beta l\Bigg\}.\qquad(54)\end{aligned}$$

In this expression, we are now in a position to introduce the screening angle χ_α which will take care of all the contribution from the small angles. If we let

$$\ln\frac{1}{\chi_\alpha}=\ln\frac{1}{\chi_0}-\frac{2\alpha}{\beta}\chi_0(\ln\chi_0+\tfrac{1}{3}\ln 2)-2\alpha\beta\chi_0\left[1+\int_0^\infty dx\frac{\tan^{-1}x}{1+4x^2}-\ln\chi_0\right],\qquad(55)$$

then $Q_l'+Q_l''$ becomes

$$\begin{aligned}Q_l'+Q_l''=\tfrac{1}{2}\chi_c^2\Bigg\{&l(l+1)\left[\ln\frac{2}{\chi_\alpha}-\tfrac{1}{2}\right.\\&\left.+[1-\Psi(l)-C]\left(1+2\frac{\alpha}{\beta}\chi_0(1-\beta^2)\right)\right]\\&-(\beta^2+\pi\alpha\beta)[\Psi(l)+C]+2\pi\alpha\beta l\Bigg\}.\qquad(56a)\end{aligned}$$

From Eq. (55), we find

$$\chi_\alpha=\chi_0\left\{1+2\alpha\chi_0\left[\frac{1}{\beta}\ln\chi_0+\frac{0.2310}{\beta}+\beta\left(1+\int_0^\infty dx\frac{\tan^{-1}x}{1+4x^2}-\ln\chi_0\right)\right]\right\}.\qquad(55a)$$

Numerical integration of the integral occurring in this expression leads to the value 0.4480. Thus,

$$\chi_\alpha^2=\chi_0^2\left\{1+4\alpha\chi_0\left[\left(\frac{1-\beta^2}{\beta}\right)\ln\chi_0+\frac{0.2310}{\beta}+1.4480\beta\right]\right\},\qquad(55b)$$

where

$$\chi_0=(\hbar/p)(\mu Z^{\frac{1}{3}}/0.885a_0),\quad \mu=0(1).$$

This relation between χ_α and χ_0 is true to the first order in $\alpha=zZ/137$. We see that the term proportional to α arises due to the deviation from the first Born approximation. In the screening angle χ_α, α occurs in the form $\alpha\chi_0$, which makes this entire correction very small in comparison with unity. This is as it should be, because at very small angles the deviations from first Born approximation are very small and thus χ_α and χ_0 should not differ very much from one another. Molière[2] gets, instead of (55a),

$$\chi_\alpha^2=\chi_0^2[1.13+3.76\alpha^2/\beta^2],^{18}\qquad(55c)$$

[17] This expression for $Q_l'+Q_l''$ is valid for $l\chi_0<1$ where $Q_l'+Q_l''>0$. Of course when l is large, the upper limit in the integral occurring in the square bracket of Eq. (55) can no longer be infinity but a suitable finite upper limit has to be substituted. The numerical value of the integral will will then be somewhat less than 0.4480 [see Eq. (55b)].

[18] The coefficient 1.13 results from the use of the sum of three exponentials (29), (29a) to represent the Thomas-Fermi potential. In (55b) we obtain 1 instead of his 1.13 because we are using a single exponential only to represent the screened atomic potential (see Sec. VII). The χ_0 in Eqs. (55b) and (55c), however, are not the same.

TABLE I. Comparison of Molière's and our correction term to (χ_α^2/χ_0^2).

	$Z=4$		$Z=79$	
β	Molière $3.76\ \alpha^2/\beta^2$	From Eq. (55b) $4\alpha\chi_0[\cdots]$ $\mu=1$	Molière $3.76\ \alpha^2/\beta^2$	From Eq. (55b) $4\alpha\chi_0[\cdots]$ $\mu=1$
0.7	6.54×10^{-3}	-2.81×10^{-3}	2.617	-8.96×10^{-2}
0.8	5.01×10^{-3}	-0.76×10^{-3}	2.004	-1.32×10^{-2}
0.9	3.96×10^{-3}	$+0.36\times10^{-3}$	1.5832	$+2.77\times10^{-2}$
0.99	3.27×10^{-3}	$+0.33\times10^{-3}$	1.3084	$+1.82\times10^{-2}$
0.999	3.21×10^{-3}	$+0.114\times10^{-3}$	1.2513	$+0.61\times10^{-2}$

in which

$$\chi_0=(\hbar/p)(Z^{\frac{1}{3}}/0.885a_0).$$

(55c) has been obtained from (34) and is claimed by Molière to be valid for all values of α, i.e., to all orders of deviation from the first Born approximation. Equation (34), however, has been shown in Sec. V to be incorrect, and it is now obvious that (55c) is not correct even to the second Born approximation. In Table I, we compare Molière's correction term with ours as a function of β and Z of the scatterer and it is seen that Molière's correction term (α^2/β^2) is wrong even in the order of magnitude for large α. His large coefficient of this term can perhaps be traced to the use of the expression (32) up to the y^2 terms in the evaluation of (28). See discussions under (B), Sec. V, above.

Now in Eq. (56a) let us introduce χ_α' by the relation

$$\ln\frac{2}{\chi_\alpha}=\ln\frac{2}{\chi_\alpha'}-\tfrac{1}{2}+C-\frac{2\alpha\chi_0}{\beta}(1-\beta^2)(1-C);$$

then we can write

$$Q_l'+Q_l''=\tfrac{1}{2}\chi_c^2\{l(l+1)[\ln(2/\chi_\alpha')-\xi\Psi(l)]+2\pi\alpha\beta l-(\beta^2+\pi\alpha\beta)[\Psi(l)+C]\},$$

where

$$\xi=1+(2\alpha\chi_0/\beta)(1-\beta^2). \qquad (56b)$$

We know that for large l, asymptotically,

$$\Psi(l)\simeq\ln(l+\tfrac{1}{2}).$$

Let us introduce the parameter b by the relation

$$b=\xi\ln(\chi_c^2/4)-\ln(\chi_\alpha'^2/4), \qquad (57)$$

and let

$$y=(l+\tfrac{1}{2})\chi_c.$$

Now the distribution function $f(\theta,t)$ is given by

$$f(\theta,t)=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})P_l(\cos\theta)\exp[-Q_l'-Q_l''].$$

According to the Euler-Maclaurin summation formula,

$$\sum_{l=0}^{\infty}g(l+\tfrac{1}{2})\simeq\frac{1}{\chi_c}\left[\int_0^\infty dy\,g(y)+\frac{1}{24}g'(0)+\cdots\right].$$

Also, we have approximately

$$P_l(\cos\theta)\simeq J_0((l+\tfrac{1}{2})\theta)=J_0(\theta y/\chi_c).$$

Thus $f(\theta,t)$ can be expressed as

$$f(\theta,t)=\frac{1}{\chi_c^2}\int_0^\infty g(y)dy, \qquad (58)$$

where $g(y)$ is given by

$$g(y)=yJ_0\left(\frac{\theta}{\chi_c}y\right)\exp\left[-\tfrac{1}{4}y^2\left\{b-\xi\ln\frac{y^2}{4}\right\}\right]\times\exp\left\{\tfrac{1}{2}\chi_c^2\left[\tfrac{1}{8}\left(b-\xi\ln\frac{y^2}{4}\right)-\pi\alpha\beta\left(2\frac{y}{\chi_c}-1\right)+(\beta^2+\pi\alpha\beta)\left(C+\ln\frac{y}{\chi_c}\right)\right]\right\}. \qquad (59)$$

Following Molière and Bethe,[4] we make the transformation

$$b=B-\xi\ln B, \qquad (60)$$

and introduce the variable $u=B^{\frac{1}{2}}y$; then we obtain

$$f(\theta,t)=\frac{1}{\chi_c^2B}\int_0^\infty g(u)du, \qquad (61)$$

with $g(u)$ given by

$$g(u)=uJ_0\left(\frac{\theta}{\chi_c\sqrt{B}}u\right)\exp\left[\frac{u^2}{4B}\left\{-B+\xi\ln\frac{u^2}{4}\right\}\right]\times\exp\left\{\tfrac{1}{2}\chi_c^2\left[\tfrac{1}{8}\left(B-\xi\ln\frac{u^2}{4}\right)-\pi\alpha\beta\left(\frac{2u}{\chi_c\sqrt{B}}-1\right)+(\beta^2+\pi\alpha\beta)\left(C+\ln\frac{u}{\chi_c\sqrt{B}}\right)\right]\right\}. \qquad (62)$$

The expression for $g(u)$ can be rearranged and simplified to the form[19]

$$g(u)=K\exp\{[1+\tfrac{1}{2}\chi_c^2(\beta^2+\pi\alpha\beta-\tfrac{1}{4}\xi)]\ln u\}\times J_0\left(\frac{\theta}{\chi_c\sqrt{B}}u\right)\exp(-u^2/4)\times\exp\left[\xi\frac{u^2}{4B}\ln\frac{u^2}{4}-\pi\alpha\beta\chi_c\frac{1}{\sqrt{B}}u\right], \qquad (63)$$

[19] For $l\chi_0>1$, $[-\frac{1}{4}u^2+\xi(u^2/4B)\ln(u^2/4)-\pi\alpha\beta\chi_c(1/\sqrt{B})u]$ in Eq. (63) becomes positive, so that $f(\theta,t)$ defined by Eq. (61) diverges. Strictly speaking, in view of our reference just preceding Eq. (54), the integral should have been extended only up to $l\sim1/\chi_0$ and so this divergence is spurious (see reference 4).

where K is a constant whose value is

$$K=\exp\frac{B\chi_c^2}{16}\left[1+\frac{8\pi\alpha\beta}{B}+2\xi\frac{\ln 2}{B}+8\frac{(\beta^2+\pi\alpha\beta)}{B}(C-\ln\chi_c\sqrt{B})\right]. \quad (64)$$

Making an expansion of the last exponential in Eq. (63) in powers of $B^{-\frac{1}{2}}$ rather than B^{-1} as previously done, we can write, after making some slight rearrangements,

$$\begin{aligned}f(\theta,t)=&\frac{K}{\chi_c^2B}\int_0^\infty du\exp\{[1+\tfrac{1}{2}\chi_c^2(\beta^2+\pi\alpha\beta-\tfrac{1}{4}\xi)]\ln u\}\\&\times J_0\left(\frac{\theta}{\chi_c\sqrt{B}}u\right)\exp\left(-\frac{u^2}{4}\right)\\&\times\left[1+\frac{1}{B}\left\{-\pi\alpha\beta\chi_c\sqrt{B}u+\xi\frac{u^2}{4}\ln\frac{u^2}{4}\right\}\right.\\&+\frac{1}{2!}\frac{1}{B^2}\left\{-2\pi\alpha\beta\chi_c\sqrt{B}u\xi\frac{u^2}{4}\ln\frac{u^2}{4}\right.\\&\left.\left.+\left(\xi\frac{u^2}{4}\ln\frac{u^2}{4}\right)^2\right\}+\cdots\right]. \quad (65)\end{aligned}$$

Let us compare the expression (65) with the corresponding expression obtained by Bethe [Eq. (25) in reference 4]. The term in β^2 in the exponent of u comes from the spin of the particle. The difference of ξ from 1 [see Eq. (56b)] and the terms in $\alpha\chi_c\sqrt{B}$ represent the contribution from the second Born approximation, which also appears in the dependence of B on $\alpha=zZ/137$ through Eqs. (60), (57), and (55b). In the work of Molière and Bethe, the spin term is neglected; $\xi=1$; the terms in $\alpha\chi_c\sqrt{B}$ are missing; and the relation (55b) is replaced by the incorrect relation (55c) of Molière. The two terms in $\alpha\chi_c\sqrt{B}$, being proportional to χ_c, are not as small as the correction $\alpha\chi_0$ in (55b), and for α not too small, become appreciable for scattering angles $\theta>B^{\frac{1}{2}}\chi_c$.

Table II. Experimental data[a] and calculated "1/e width" θ_ω for the multiple scattering of 15-Mev electrons by Au and Be foils.

	Au	Be
t Target thickness (mg/cm²)	37.20	491.3
E Average energy of electron in foil (Mev)	15.67	15.24
χ_c defined in Eq. following (43) (degree)	1.55	1.52
θ_ω observed (degree)	3.78	4.33
θ_ω, Molière's B, with $f^{(0)}$, $f^{(1)}$, $f^{(2)}$ (degree)	3.83(3.90)	4.56(4.60)
θ_ω, the present work, (65a), with $\mu=1.12$ in (68) (degree)	4.096	4.60
θ_ω, the present work, (65a), with $\mu=1.80$ in (68) (degree)	3.80	4.35

[a] See reference 5.

VII. COMPARISON WITH EXPERIMENTAL $f(\theta,t)$

To compare our result (65) with the experimental data and with the result of Molière, we have carried out a calculation for two sets of values of the parameters Z, χ_c, t that correspond to the experiments of Hanson, Lanzl, Lyman, and Scott[5] on the scattering of 15.7-Mev electrons by gold and beryllium foils. The data are shown in Table II.

Corresponding to these values of χ_c (see Table II), the exponent of u in the integrand in Eq. (65) can be taken as unity for the purposes of our present calculation. For these energies, $\beta=0.9995$, $\xi\simeq1$, and Eq. (65) can be written in the form

$$f(\theta,t)=\frac{K}{\chi_c^2B}\left[f^{(0)}+\frac{1}{B}(f^{(1)\prime}+f^{(1)})+\frac{1}{2!B^2}(f^{(2)\prime}+f^{(2)})+\cdots\right], \quad (65a)$$

where $f^{(0)}$, $f^{(1)}$, and $f^{(2)}$ have already been computed by Molière[3] and Bethe[4] and the $f^{(1)\prime}$ and $f^{(2)\prime}$ arise from the second Born approximation. We have been able to compute $f^{(1)\prime}$ from the values of the confluent hypergeometric function[12] up to $\theta/\chi_c\sqrt{B}=4$. Beyond this value of θ, the labor involved in summing the series for the hypergeometric function is excessive and we have not gone any farther. The integral $f^{(2)\prime}$ does not seem to be expressible in terms of known functions in simple forms, and its evaluation will have to await some help from computing machines.

From Eq. (65), it is seen that the distribution function $f(\theta,t)$ depends, through B in Eqs. (60) and (55), on the "screening angle" χ_α introduced in Eq. (55), which gives its dependence on α and on χ_0, the screening angle to the first Born approximation. For any screened field $V(r)$ whose scattering cross section is given by Eq. (43), the χ_α is given by[1]

$$\ln\frac{2}{\chi_\alpha}-\frac{1}{2}=\int_0^1\frac{q(y)}{y}dy,\quad y=\sin(\chi/2). \quad (66)$$

For the first Born approximation, we have

$$\ln\frac{2}{\chi_0}-\frac{1}{2}=\int_0^1\frac{q_B(y)}{y}dy, \quad (66a)$$

where $q_B(y)$ is the ratio of first Born approximation $\sigma_B(y)$ for the field $V(r)$ to the Rutherford $\sigma_R(y)$ for the unscreened field. For an exponential screening $e^{-\lambda r}$ as in Eq. (11), Eq. (66a) gives

$$\chi_0=\hbar\lambda/p, \quad (67)$$

and χ_α is given by Eq. (55). For the Thomas-Fermi field $-(zZe^2/r)\omega(\lambda_0 r)$ in Eq. (25), Eq. (66a) gives,[1,20] upon numerical integration,

$$\chi_0 \simeq 1.12(\hbar\lambda_0/p), \quad \lambda_0 = Z^{\frac{1}{3}}/0.885a_0. \tag{67a}$$

In our present work, we use the Dalitz formula for $\sigma_D(\chi)$ for an exponentially screened field, Eq. (11), and for this field, χ_0 is given by Eq. (67) where λ is yet to be adjusted to give the closest representation of the actual field $V(r)$. We may therefore write

$$\lambda = \mu\lambda_0, \tag{68}$$

where μ is a constant of order 1. It is clear that as long as we have not assigned any assumed value to μ, Eq. (68) amounts to only a change of notation and involves no new assumptions.

It is important, however, to note the following. As mentioned in the preceding paragraph, for the screened field in Eq. (11), χ_0 and χ_α are given by Eqs. (67) and (55), respectively. For the "actual" field $V(r)$, χ_0 and χ_α are given by the relations (66a) and (66), respectively, of which (67) and (55) are the respective special cases. Thus, by dissociating χ_0 in (55) from the special form (67) and writing it in the form (68), we may consider the result of the preceding section, calculated explicitly for the field (11), as having been extended, or generalized, to the field $V(r)$. This extension is a perfectly consistent one from the point of view of successive Born approximations in the problem of multiple scattering.

Turning now to the actual numerical calculation, using Eq. (68) in (55b), we have, up to the second Born approximation,

$$\chi_\alpha^2 = \left(\mu\frac{\hbar\lambda_0}{p}\right)^2 \left[1 + 4\alpha\left(\mu\frac{\hbar\lambda_0}{p}\right) \times \left\{\frac{1-\beta^2}{\beta}\ln\left(\mu\frac{\hbar\lambda_0}{p}\right) + \frac{0.2310}{\beta} + 1.4480\beta\right\}\right]. \tag{69}$$

With $\mu = 1.12$ as in (67) corresponding to the Thomas-Fermi potential, the calculated "$1/e$ width" θ_ω of $f(\theta,t)$ from (65) does not agree very well with the observed value for either Au or Be, as shown in Table II. If, however, we choose for μ a value, namely

$$\mu \simeq 1.80,$$

such that the calculated θ_ω for Au agrees with the experimental value (see Table II), we then find that the same value $\mu = 1.80$ leads, for Be, to a value θ_ω also in good agreement with the experimental value (see Table II). It may be mentioned in connection with Table II that Hanson *et al.* in calculating θ_ω take account of the effect of the terms $f^{(1)}$, $f^{(2)}$ in the distribution function by fitting a Gaussian with a width slightly narrower than the width $\sqrt{B}\chi_c$ of the gaussian term $f^{(0)}$. This width according to them is $(B-1.2)^{\frac{1}{2}}\chi_c$. In our calculations we have calculated θ_ω by obtaining the full distribution function $f = f^{(0)} + (1/B)(f^{(1)\prime} + f^{(1)})$ and finding the value of θ at which the intensity drops to $1/e$ of its central value. We find that θ_ω calculated in our way are slightly broader than the θ_ω calculated according to Hanson *et al.* To give an idea of the magnitude of this difference, the θ_ω calculated according to our method are given in parenthesis along with Molière's value in row 5 of Table II. The other values of θ_ω quoted in Table II are calculated according to our method only.

In Table III are given the values of $f^{(0)}$, $f^{(1)}$, $f^{(1)\prime}$ of (65a) and (65) for $\mu = 1.80$, together with the values of $f^{(0)}$, $f^{(1)}$ given by Bethe[4] on the basis of Molière's treatment. It is seen that for scattering angles somewhat beyond the "Gaussian" region ($\theta/\sqrt{B}\chi_c \sim 3$ in Table III), the present result (65) predicts an $f(\theta)$ greater than that according to Molière and Bethe by about 20%. This greater $f(\theta)$ arises from the contribution from the second Born approximation. A more exact calculation, however, has to include the $f^{(2)\prime}$ which we have neglected in Table III.

If, as shown by Bethe,[4] the distribution function at large angles goes over into the cross section itself, the increase in the distribution function of 20% at angles $\theta \sim 15°$ is consistent with the increase of the cross section $\sigma_D(\theta)$ itself by about 20% over the first Born approximation value at these angles. In this connection it may be pointed out that the curves given in Mott and Massey's book,[21] for the cross section in the second Born approximation as a function of scattering angle, are from the calculations of Bartlett and Watson[22] based on the early incorrect formula of Mott, and give too large an increase over the first Born approximation cross section. The correct Dalitz formula[10] gives an increase of only about 20% at scattering angles $\theta \sim 15°$.

While the use of the same empirical value $\mu = 1.80$ in Eqs. (68) and (69) brings the theoretical θ_ω according

TABLE III. Numerical values of the distribution function.

$\theta/\sqrt{B}\chi_c$	$f^{(0)}$	$f^{(1)}$	$f^{(1)}$ ($\mu=1.80$)	$f^{(0)}+(1/B)\times(f^{(1)\prime}+f^{(1)})$ ($\mu=1.80$)	Bethe-Molière $f^{(0)}+(1/B)f^{(1)}$
0	2	0.8456	−0.4584	2.0555	2.116
0.2	1.9216	0.7038	−0.3744	1.9688	2.018
0.4	1.7214	0.3437	−0.2822	1.7302	1.768
0.6	1.4094	−0.0777	−0.2041	1.3690	1.400
0.8	1.0546	−0.3981	−0.1341	0.9783	1.050
1.0	0.7338	−0.5285	−0.0717	0.6478	0.661
1.2	0.4738	−0.4770	−0.0153	0.4033	0.408
1.4	0.2817	−0.3183	+0.0374	0.2415	0.238
1.6	0.1546	−0.1396	+0.0898	0.1475	0.136
1.8	0.0783	−0.0006	+0.1471	0.0993	0.078
2.0	0.0366	+0.0782	+0.0295	0.0520	0.047
3.0	0.00025	+0.0455	+0.0070	0.0080	0.006
4.0	2.3×10^{-7}	+0.0106	+0.0026	0.0019	0.0014

[20] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, Oxford, 1949), second edition, pp. 188–190, 196–198. The $\theta_{\min}$ there is defined by[1] the following relation instead of (66), ${}^1\!\int_0[q(y)/y]dy = \ln(2/\theta_{\min})$, so that $\theta_{\min} = e^{\frac{1}{2}}\chi_0$.

[21] Reference 20, p. 81.

[22] J. H. Bartlett and R. E. Watson, Proc. Am. Acad. Arts Sci. **74**, 53 (1950).

to Eq. (65) into satisfactory agreement with the observed θ_ω for both Au and Be, the question may be raised as to why the value $\mu \simeq 1.80$ seems to differ so much from the value $\mu \simeq 1.12$ in Eq. (67) calculated from the Thomas-Fermi potential (first Born approximation). A tentative answer is that the Thomas-Fermi potential, while satisfactory in a qualitative way in dealing with certain properties of the atom, may not be sufficiently good for the screening effect for very small scattering angles.[23] To see if this is the case, a calculation of χ_0 has been carried out for the Be atom for which the Fock field is available.[24]

On using for the wave functions of Fock and Petrashen[24]

$$R_{1s}(r)=N_1e^{-\alpha r},\quad R_{2s}(r)=N_2\left(1-\frac{\alpha+\beta}{3}r\right)e^{-\beta r},\quad (70)$$

the field $V(r)$ due to the $1s^2\ 2s^2$ configuration can be calculated analytically. From this $V(r)$, the scattered amplitude $f_B(\chi)$ (13) can be obtained, and finally the $q_B(\chi)$ and χ_0 of (66a) are obtained by integrations. It is found that

$$(\chi_0)_{\mathrm{H.F.}}=2.18(\hbar\lambda_0/p),\quad (71)$$

corresponding to $(\mu)_{\mathrm{H.F.}}=2.18$ in (68).
This value $\mu=2.18$ is in much better agreement with the empirical value $\mu=1.8$, and shows that the value $\mu=1.12$ from the Thomas-Fermi field is too low. The small difference between 2.18 and 1.8 may be due to (i) the approximate nature of the Hartree-Fock field itself, (ii) the further approximation (70) for the wave functions, and/or (iii) uncertainities in the empirical value $\mu=1.8$. Thus the situation of the present result of the theory of multiple scattering may be regarded as satisfactory.

We shall finally consider the comparison[5] between the results on Au and Be based on Molière's treatment and the experimental results of Hanson *et al.*[5] There the relation between χ_α and χ_0 is given by (55c) in which the term linear in α corresponding to the second Born approximation is missing and the coefficient of α^2 is incorrect by a few orders of magnitude. However, by some mysterious coincidence, the $3.76\alpha^2$ term in (55c) for the case of 15-Mev electrons scattered by Au, together with the term 1.13 [which according to Goudsmit and Saunderson,[1] Mott and Massey[20] should have been $(1.12)^2$, as given in (67)], gives for $\chi_\alpha{}^2$ the value $\chi_\alpha{}^2 \simeq 2.53\chi_0{}^2$ that leads to a calculated θ_ω in good agreement with the observed value (see Table II). This fortuitous agreement, however, does not obtain in the case of Be for which the α^2 term in (55c) is entirely negligible and $\chi_\alpha{}^2 \simeq 1.13\chi_0{}^2$ leading to a much too large calculated θ_ω (see Table II).

ACKNOWLEDGMENT

One of the authors, Wu, wishes to express his appreciation of the hospitality shown him by Professor R. Oppenheimer at the Institute for Advanced Study, Princeton.

[23] Mott and Massey, reference 20, p. 196, do remark that the use of Hartree field leads to a $\theta_{\min}$ differing slightly from the $\theta_{\min}$ from the Thomas-Fermi field. No numerical comparison, however, is given and the difference is dismissed as "unimportant". We are here just concerned with this difference.

[24] Fock and Marie Petrashen, Physik Z. Sowjetunion V. 8, 359 (1935).