Dispersive Effects in Electron-Nucleus Interactions*

NICHOLAS A. KRALL† AND E. E. SALPETER
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received February 16, 1959)

Elastic scattering of electrons from nuclei is influenced by the possibility of virtual transitions to excited nuclear states in intermediate states. Such dispersive corrections to electron-deuteron elastic scattering are calculated in second order Born approximation for incident electron energies from 200 Mev to 500 Mev for various values of the momentum transfer. The static second-order Born formulas are also evaluated. Similar, but less accurate, calculations are carried out for He^4, C^{12}, and some heavier elements. These results are used to find small corrections to nuclear radii obtained from an analysis of the Stanford experiments.

Dispersive effects on the energy level shifts in bound s states of hydrogenic atoms are also calculated, using second-order perturbation theory for the deviation of the nuclear charge distribution from a point charge. These small shifts are evaluated for deuterium and He^4 and, less accurately, for other nuclei with $Z<50$.

I. INTRODUCTION

PROBLEMS involving Coulomb interactions between electrons and nuclei have reached a stage of considerable experimental sophistication. Electron scattering has been developed into an increasingly delicate instrument by the Stanford researchers,[1] and the energy of low-lying bound states in certain atoms has been determined with amazing accuracy following the original work of Lamb.[2]

The interpretation of these experiments has been based on a picture of the nucleus as a static charge cloud.[1] The Stanford workers have determined electron scattering from such a model with almost infinite accuracy, using a phase-shift analysis and numerical techniques, and the "nuclear size" correction to the Lamb shift has also used a static picture of the nucleus.[3]

In view of the importance of electron scattering in directly measuring the deuteron wave function and the charge distribution of heavier elements, and in view of the continuing interest in the Lamb shift, it is of interest to examine these phenomena—elastic scattering and Lamb shift—for the effects of virtual nuclear states other than the ground state.

The present work examines accurately electron-nucleus elastic scattering in second Born approximation, and the shift of $2S$ levels due to nuclear structure in second order perturbation theory. In both cases the formulation contains a sum over possible intermediate states of the nucleus. When this intermediate state is not the ground state, we have the contribution from effects ignored in state pictures of the nucleus. These are termed dispersive effects, and have previously been the subject only of qualitative evaluation.[4] Here we will discuss, in Sec. II, a method for dealing accurately with the sum over intermediate states in a semi-numerical fashion, and use this method to derive cross sections for electron-deuteron scattering in first and second Born approximation at energies from 200 Mev to 500 Mev for large momentum change, and for electron scattering from other elements (He^4, C^{12}, Ca^{40}, Bi^{209}) at fewer combinations of energy and angle. A similar method for the bound state problem is discussed and used in Sec. III to evaluate the shift in $2S$ levels due to nuclear size and structure for all nuclei up to $Z=50$.

A discussion of the results from electron scattering, their interpretation in terms of nuclear sizes, and their significance for earlier calculations of the bound state problem will be presented in Sec. IV.

II. ELECTRON SCATTERING

The Born approximation for Coulomb scattering from a point charge, to second order, has been clearly and correctly expounded by Dalitz,[5] and the formalism in his paper can be extended[6] to include scattering from an extended charge source or from a real nucleus containing nucleons which may occupy virtual states other than the ground state during the interaction. In terms of incoming and scattered electrons of momentum $\mathbf{p}_1$ and $\mathbf{p}_2$, intermediate states of plane wave electrons of momentum $\mathbf{p}$, the electron mass m, and intermediate nuclear states ψ_n, the amplitude for electron scattering

* Supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.
† Now at the John Jay Hopkins Laboratory for Pure and Applied Science, General Atomic Corporation, San Diego, California.
[1] R. Hofstadter, Ann. Rev. Nuclear Sci. **7**, 231 (1957). This excellent review also contains a complete list of references to earlier experimental and theoretical work on electron scattering.
[2] Triebwasser, Dayhoff, and Lamb, Phys. Rev. **89**, 98 (1953); Dayhoff, Triebwasser, and Lamb, Phys. Rev. **89**, 106 (1953).
[3] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press, Inc., New York, 1957), Sec. 20.
[4] L. I. Schiff, Phys. Rev. **98**, 756 (1955); H. S. Valk and B. J. Malenka, Phys. Rev. **104**, 800 (1956).
[5] R. Dalitz, Proc. Roy. Soc. London **206**, 509 (1951).
[6] N. A. Krall, Ph.D. thesis, Cornell University, 1959 (unpublished). This work contains complete details concerning methods and results indicated here.

with momentum change $\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1$ is, with $\mathbf{p}\equiv p_\mu\gamma^\mu$,

$$M=-ie^2\delta(\epsilon_2-\epsilon_1)8\pi^2\gamma^0 u_2\Big\{F(q)/q^2-e^2(2\pi^2)^{-1}\int d^4p \times\delta(p_4-\epsilon_1)(\mathbf{p}_2-\mathbf{p})^{-2}(\mathbf{p}-\mathbf{p}_1)^{-2}\sum_n\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n} \times\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{n0}(E_n-E_0+\not{p}-m)^{-1}\gamma^0\Big\}u_1. \quad (1)$$

The matrix elements $\langle e^{i\mathbf{p}\cdot\mathbf{R}}\rangle_{0n}$ are taken between the nucleon ground state ψ_0 and the state ψ_n, and the scattering has been derived, for simplicity, from the potential due to a single nucleon whose coordinate is R, $V=-e^2/|\mathbf{r}-\mathbf{R}|$. The restriction to one nucleon is lifted for complex nuclei by summing over protons, $V=-\sum_i e^2|\mathbf{r}-\mathbf{R}_i|$; for the deuteron the one-nucleon formula, is, of course, the proper formulation. The expression (1) is exact in the first two terms of the Born expansion retained, and the usual notation $F(q)\equiv\langle e^{i\mathbf{q}\cdot\mathbf{R}}\rangle_{00}$ has been used to define the nuclear form factor.

The problem in evaluating (1) centers on evaluating the sum over nuclear states. A simple means of dealing with the sum would be to ignore E_n-E_0 in the denominator, and use closure:

$$\sum_0^\infty(E_n-E_0+\not{p}-m)^{-1}\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n}\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{n0} \cong(\not{p}-m)^{-1}\langle e^{i(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{00}. \quad (2)$$

This approximation has been used often before,[4] and is based on the following considerations. If there is approximate momentum balance in the intermediate state, the nucleon momentum will be near $\mathbf{p}$, then $E_n\sim p^2/M$, where M is the nucleon mass, compared to the electron energy ϵ of about p_0c; then $E_n/\epsilon_0\sim p_0c/Mc^2$, a small number for electrons of 200 Mev/c. This implies that E_n-E_0 may be ignored in the denominator for most electron intermediate states. However, when $p=p_0$, the electron part of the denominator vanishes, and it is clear that neglecting E_n-E_0 in the denominator is a bad approximation there. The integral over p depends strongly on the region p near p_0; so while it is clear why (2) might be used, for simplicity, to give a first approximation, this approximation gives only a very qualitative result. The procedure used in the present quantitative treatment is to write

$$\sum_0^\infty(E_n-E_0+\not{p}-m)^{-1}\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n}\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{n0} =\sum_0^\infty[(\not{p}-m)^{-1}-(E_n-E_0)(\not{p}-m)^{-1}(E_n-E_0+\not{p}-m)]\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n}\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{n0} =(\not{p}-m)^{-1}F(q)+\text{remainder}. \quad (3)$$

The remainder is then evaluated, using appropriate approximations, and will, of course, give its biggest contribution from the region p near p_0. To see best the approximations used to get the remainder, we restate the problem in terms of the cross sections, after spin sums have been taken, obtainable in a well-known fashion from the amplitude M,

$$d\sigma/d\Omega=[\cos^2(\theta/2)]e^4F^2q^{-2}+[\csc(\theta/2)-1]e^6\pi F^2q^{-2} -2e^6m^2F\pi^{-2}q^{-2}R, \quad (4)$$

where the first term is the first Born approximation result, the second term is the contribution from the crude approximation represented by (2), and the third term is the remainder—the difference between the exact second Born statement and the crude approximation stated in (2). Writing $E_{0n}\equiv E_n-E_0$, we have

$$R=(-p_0/2m)\oint d^3p(\mathbf{p}_2-\mathbf{p})^{-2}(\mathbf{p}-\mathbf{p}_1)^{-2}\sum_{n=1}^\infty(p_0^2-p^2)^{-1} \times(p_0^2-p^2-E_{0n}^2+2mE_{0n})^{-1}\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n} \times\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{n0}\{E_{0n}(2m-E_{0n})(\mathbf{p}_1+\mathbf{p}_2) \cdot(\mathbf{p}_1+\mathbf{p}_2+2\mathbf{p})(m)^{-1}+4E_{0n}(p_0^2-p^2)\}. \quad (5)$$

$$R\equiv(-p_0/2m)\int f(p)dp; \qquad \oint\equiv\text{Cauchy principal value}. \quad (6)$$

As indicated in Eq. (6), we shall carry out the integration over the absolute value, p, of the electron momentum after the other operations. It should be noted that in the cross section (4) we have only retained terms of order e^6, the interference terms between the first and second Born amplitudes, besides the pure first-order term in e^4. The terms in e^8 which we have omitted, the square of the second Born amplitude, should be smaller than our terms in e^6 by a factor of the order of $Z/137$ except at angles where the form factor F is unusually small.

To evaluate R we divide the integration over p into three regions: (*i*) p large enough ($p\geq1.05p_0$) so that we can ignore E_{0n} in the denominator, (ii) p small enough ($p\leq0.95p_0$) that we can also ignore E_{0n} in the denominator, and (iii) $p_0^2-p^2$ small enough that it may be ignored in the denominator, retaining only the terms in E_{0n}. Using this division, the operator representation $E_n\langle A\rangle_{0n}=\langle AH_0\rangle_{0n}$, and the sum rule

$$\sum_0^\infty f(E_n)\langle A\rangle_{0n}\langle B\rangle_{n0}=\langle Af(H_0)B\rangle_{00},$$

we can write the integrand of (6) in the three regions

of p as follows:

$|p-p_0|>0.05p_0$:

$$f(p)=\int d\Omega\, p^2(p_0^2-p^2)^{-2}(\mathbf{p}_2-\mathbf{p})^{-2}(\mathbf{p}-\mathbf{p}_1)^{-2}$$
$$\times\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\{(H_0-E_0)(2m-H_0+E_0)(\mathbf{p}_2+\mathbf{p}_1)$$
$$\cdot(\mathbf{p}_2+\mathbf{p}_1+2\mathbf{p})m^{-1}+4(H_0-E_0)(p_0^2-p^2)\}$$
$$\times e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{00}; \quad (7)$$

$p\cong p_0$:

$$f(p)=\Delta^{-1}\oint_{p_0(1-\Delta/2)}^{p_0(1+\Delta/2)} dp\int d\Omega\, p^2(\mathbf{p}_2+\mathbf{p}_1)$$
$$\cdot(\mathbf{p}_2+\mathbf{p}_1+2\mathbf{p})(p_0^2-p^2)^{-1}(p_2-p)^{-2}(p-p_1)^{-2}m^{-1}$$
$$\times[\langle e^{i\mathbf{q}\cdot\mathbf{R}}\rangle_{00}-\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{00}\langle e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}\rangle_{00}]. \quad (8)$$

The reason the regions in which we may neglect E_{0n} in the denominator extend so close to $p=p_0$ is as mentioned earlier, that the important nuclear intermediate states are those where approximate momentum balance is achieved, and $E_{0n}\sim(p_0c)^2/Mc^2\sim 40$ Mev for $p_0\sim 200$ Mev/c. Then $p^2-p_0^2$ is five times as big as $(E_n-E_0)^2$ even for $p=1.05p_0$. The matrix elements $\langle e^{i(\mathbf{p}_2-\mathbf{p})\cdot\mathbf{R}}\rangle_{0n}$ were also examined explicitly, and confirmed the intuitive feeling that the important nucleon states[7] were actually those with $E_n<50$ Mev. The matrix elements in (7) are solved by simply deducing the commutation relations between $e^{i(\mathbf{p}-\mathbf{p}_1)\cdot\mathbf{R}}$ and $H_0=-\nabla^2(R)/2M+V(R)$; the resulting matrix element is then an expectation value over the ground-state wave function of the nucleus considered. The angular integrations in this first integral are involved but standard; then in this region $f(p)$ reduces to a combination of analytic expressions multiplying matrix elements such as $\langle e^{i\mathbf{q}\cdot\mathbf{R}}2i\cos\theta dV/dR\rangle_{00}$, which depend on the nuclear ground-state wave function and on the assumed nuclear potential, and which may be evaluated analytically in simple cases, such as a Hulthén deuteron, or by high-speed electronic computer in more complicated cases.

In the central region $p\sim p_0$, the matrix elements for the selected nuclear ground state give complicated functions of $(\mathbf{p}-\mathbf{p}_1)$ and $(\mathbf{p}_2-\mathbf{p})$, and the angular integration $d\Omega$ was done on the IBM 650 computer for all cases considered. These regions were joined by a smooth curve and R was evaluated by numerically integrating under the curve.

The major error in this procedure comes in joining the regions $|p-p_0|>0.05p_0$ with the region $p\sim p_0$.

[7] For the case $p<p_0$, the denominator of (6) vanishes at large E_n. The principal value of the sum over this singularity gave a contribution an order of magnitude smaller than the sum over the regions where E_n could be safely ignored. The integrations over such singularities for the second Born amplitude also lead to terms involving residues at each pole but these terms are purely imaginary and do not contribute to the first-second Born interference terms of order e^6 which we retain in our Eq. (4) for the cross section.

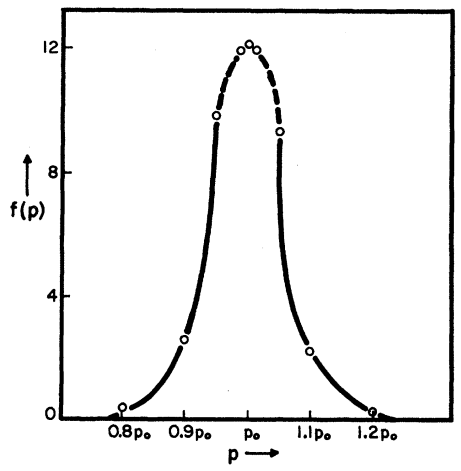

FIG. 1. $f(p)$ in arbitrary units *vs* p, from $R=(-\epsilon/2m)\int f(p)dp$, showing the joining of regions $|p-p_0|>0.05p_0$ and the region $p\approx p_0$, for electron-deuteron scattering, with $p_0=400$ Mev/c, and $q=2.2$ f^{-2}.

Figure 1 gives a typical demonstration of what $f(p)$ generally looks like. It is clear that the big contribution to R comes from electron intermediate momenta near p_0, especially $|p-p_0|<1.2p_0$. An error of about 10% comes from this joining of the two regions. This remainder itself is generally larger than, or at least comparable in size to, the crude approximation (2).

It is worth noting here that the remainder, containing containing E_{0n} as a factor in the numerator, contains only dispersive effects (intermediate states other than the ground state), while the crude approximation contains a static part ($n=0$) as well. To obtain the part of the second Born approximation due only to dispersion (the part ignored in a static charge picture of the nucleus), we write the second Born approximation to scattering from a static charge and subtract it from the total second Born formulation just described:

$$(d\sigma/d\Omega)_{\text{static charge}}=[\cos^2(\theta/2)]e^4F^2q^{-2}-2e^6m^2F\pi^{-2}q^{-2}S,$$

$$S=\int d^3p\, F(\mathbf{p}_2-\mathbf{p})F(\mathbf{p}-\mathbf{p}_1)(p_0^2-p^2+i\epsilon)^{-1}$$
$$\times(\mathbf{p}_2-\mathbf{p})^{-2}(\mathbf{p}-\mathbf{p}_1)^{-2}[2p_0{}^3m^{-2}\cos^2(\theta/2)$$
$$+p_0m^{-2}\mathbf{p}\cdot(\mathbf{p}_2+\mathbf{p}_1)]. \quad (9)$$

This integral is very similar to the one encountered in $f(p)$ for $p\sim p_0$, and was evaluated numerically in the same fashion on the IBM 650.

Deuteron—Electron Scattering

For the deuteron the Hulthén wave function and potential was used to evaluate $F(q)$ and the nuclear matrix elements appearing in $f(p)$. The calculation was done at 7 combinations of energy and angle, with results for σ_1, σ_2, $\sigma_{\text{dispersive}}$, σ_{static}, σ_{crude}, and $\sigma_{\text{remainder}}$

TABLE I. Results of the electron-deuteron scattering calculation, where $(d\sigma/d\Omega)_1$ = first Born approximation in cm²/sterad $(d\sigma/d\Omega)_2$ = second Born approximation = $(d\sigma/d\Omega)_{\text{crude}} + (d\sigma/d\Omega)_R = (d\sigma/d\Omega)_{\text{static}} + (d\sigma/d\Omega)_{\text{dispersive}}$. F is the deuteron form factor, and $d\sigma/d\Omega \equiv SF^2 \times 10^{-32}$ cm²/sterad.

p_0 (Mev/c)	q (f^{-1})	S_1	S_{crude}	S_R	S_2	S_{static}	$S_{\text{dispersive}}$
188	1.0	134	1.08	− 2.04	−0.96	0.13	−1.1
	1.35	29.2	0.67	− 0.061	0.054	−0.095	0.15
400	1.0	690	1.69	−11.3	−7.8	0.22	−8.0
	1.35	243	1.36	− 5.6	−4.3	−0.66	−3.6
	2.6	11.3	0.101	− 0.46	−0.36	0.12	−0.49
500	1.35	380	1.82	− 9.7	−7.8	−1.6	−6.2
	2.6	22.6	0.174	− 1.0	−1.0	0.05	−1.1

listed in Table I. The exact choice of model and wave function parameters must be of little importance in present application of this method, since this is such a small correction. A partial repeat of the calculation using a square-well nuclear potential indicates a change in σ_2 of about 10%.

Other Elements

The shell model was used to apply the formalism to complex nuclei, after extending it to include more than one nucleon. This extension was accomplished by the replacement, in formula (1),

$$\langle\exp(i\mathbf{A}\cdot\mathbf{R})\rangle_{0n}\langle\exp(i\mathbf{B}\cdot\mathbf{R})\rangle_{n0} \rightarrow \langle\textstyle\sum_j \exp(i\mathbf{A}\cdot\mathbf{R}_j)\rangle_{0n}\langle\sum_l \exp(i\mathbf{B}\cdot\mathbf{R}_l)\rangle_{n0}.$$

For states $n\neq 0$, if only single-particle excited states are considered,[8] the excited nucleon must be the same nucleon in both matrix elements, and

$$\langle\textstyle\sum_j \exp(i\mathbf{A}\cdot\mathbf{R}_j)\rangle_{0n}\langle\sum_l \exp(i\mathbf{B}\cdot\mathbf{R}_l)\rangle_{n0} = \sum_j\langle\exp(i\mathbf{A}\cdot\mathbf{R}_j)\rangle_{0n}\langle\exp(i\mathbf{B}\cdot\mathbf{R}_j)\rangle_{n0} = Z\langle\exp(i\mathbf{A}\cdot\mathbf{R})\rangle_{0n}\langle\exp(i\mathbf{B}\cdot\mathbf{R})\rangle_{n0}.$$

For the ground state, independent nucleons may appear in either matrix element, and a term appears here that was not present for the deuteron:

$$\sum_0^\infty (E_n-E_0+p-m)^{-1}\langle\sum_j \exp(i\mathbf{A}\cdot\mathbf{R}_j)\rangle_{0n}\langle\sum_l \exp(i\mathbf{B}\cdot\mathbf{R}_l\rangle_{n0} = Z\sum_0^\infty (E_n-E_0+p-m)^{-1}\langle\exp(i\mathbf{A}\cdot\mathbf{R})\rangle_{0n} \times\langle\exp(i\mathbf{B}\cdot\mathbf{R})\rangle_{n0}+Z(Z-1)(p-m)^{-1} \times\langle\exp(i\mathbf{A}\cdot\mathbf{R})\rangle_{00}\langle\exp(i\mathbf{B}\cdot\mathbf{R})\rangle_{00}. \quad (10)$$

The first term is identical with the deuteron term, and is handled identically, separating it into a crude approximation and a remainder. The second term is identical with the expression for the static second Born approximation, except for the $Z(Z-1)$ factor, as mentioned for the deuteron case. There are two differences between the complex nucleus calculation and the deuteron calculation. First the ground-state wave functions—and potential—are chosen differently, and secondly, the static second Born contribution exhibits a Z^2 dependance, compared with a Z dependance for the dispersive part. The latter feature, of course, is concealed in the deuteron calculation.

In Table II we list cross sections calculated from this formalism, at several angles and energies for He^4 and C^{12}, and at fewer points for heavy nuclei Ca^{40} and Bi^{209}. The last two were included to exhibit the effect of the extra power of Z in the static part, namely that an accurate static charge calculation for heavy nuclei contains only completely negligible errors (fractions of a percent) due to ignoring dispersive effects. Table II lists σ_1, σ_2, $\sigma_{\text{dispersive}}$, and σ_{static} along with other cross sections. The term in $Z(Z-1)$, which did not appear in the deuteron calculation, is termed the cross term, σ_{ct}.

III. NUCLEAR STRUCTURE AND THE LAMB SHIFT

The problem of determining the energy levels of bound atomic S states is formulated in perturbation theory, using $V_0=-Ze^2/r$ as the unperturbed potential, and $V_1=\sum_i -e^2/|\mathbf{r}-\mathbf{R}_i|+Ze^2/r$ as the perturbation, with $\mathbf{R}_i$ the displacement of the ith proton, $\mathbf{r}$ the displacement of the electron, and Z the nuclear charge;

$$E=\epsilon_0+\Delta E_1+\Delta E_{2,\text{ static}}+\Delta E_{2,\text{ dispersive}}; \quad (11)$$

ϵ_0 is the energy level of a point-nucleus atom. The first-order energy shift, $\Delta E_1=\langle V_1\rangle_{00}\sim\epsilon_0(ZR_0/a_0)^2$, is evaluated in other places,[3] where R_0^2 is the mean squared radius of the nucleus and a_0 is the Bohr radius. $E_{2,\text{ static}}\sim\epsilon_0(ZR_0/a_0)^3$, as can be demonstrated[6] by studying the Schrödinger differential equation.[9] The smallness of the binding energy for the deuteron results in a larger value for $\Delta E_{2,\text{ dispersive}}$ than might have been expected.[10] From perturbation theory, then, writing ϕ_0 as the unperturbed Coulomb wave function and using plane waves $(2\pi)^{-\frac{3}{2}}e^{i\mathbf{p}\cdot\mathbf{R}}$ for the intermediate-state electrons (it will be shown that practically all the contributions to this effect comes from relativistic intermediate states, allowing the Coulomb force to be ignored in those states), we have

$$\Delta E_{2,\text{ dispersive}}=\int d^3r d^3r'\int d^3p\,\phi_0(r)\phi_0^*(r')(2\pi)^{-3} \times e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}\sum_1^\infty\langle e^2/|\mathbf{r}-\mathbf{R}|\rangle_{0n}\langle e^2/|\mathbf{r}'-\mathbf{R}|\rangle_{n0} \times[E_n-E_0+\epsilon(p)-\epsilon_0]^{-1}. \quad (12)$$

[8] Excited states corresponding to collective motions appear to have little effect on the results. This was checked by studying a liquid-drop picture adjusted to fit the giant resonances.

[9] More explicitly, $\Delta E_1=(4/3n^3)(ZR_0/a_0)^2\epsilon_0$ for the nS state in a hydrogenic atom. For a hollow-shell nucleus (constant potential inside), an explicit evaluation gives $\Delta E_{2,\text{ static}}=(2ZR_0/5a_0)\Delta E_1$ to lowest order in powers of R_0/a_0.

[10] Our thanks are due to Professor N. Kroll, who first pointed out in a discussion that $\Delta E_{2,\text{ dispersive}}$ might be large.

TABLE II. Result of electron-heavy nucleus scattering calculation, with $(d\sigma/d\Omega)_1$=first Born approximation, $(d\sigma/d\Omega)_2$=second Born approximation, $=(d\sigma/d\Omega)_{\text{static}}+(d\sigma/d\Omega)_{\text{dispersive}}=(d\sigma/d\Omega)_{\text{crude}}+(d\sigma/d\Omega)_{\text{remainder}}+(d\sigma/d\Omega)_{\text{cross term}}$, $(d\sigma/d\Omega)_{\text{cross term}}=[(Z-1)/Z]\times(d\sigma/d\Omega)_{\text{static}}$, and $d\sigma/d\Omega\equiv S\times10^{-30}$ cm²/sterad.

	p_0	q(f^{-1})	S_1	S_{crude}	S_R	S_{static}	S_{disp}	S_2
He^4	188 Mev/c	0.62	33.7	0.18	−0.39	0.13	−0.27	−0.14
		1.34	0.26	2×10^{-3}	−0.0088	-9×10^{-4}	−0.006	−0.007
	400 Mev/c	1.39	1.90	0.01	−0.067	−0.011	−0.051	−0.062
		2.2	0.019	1.5×10^{-4}	−0.001	-6×10^{-4}	-5.6×10^{-4}	−0.0012
C^{12}	188 Mev/c	0.62	185	1.05	−2.0	3.8	−1.6	2.2
		0.98	7.7	0.06	−0.19	−0.11	−0.12	−0.23
		1.34	0.22	2×10^{-3}	−0.012	−0.026	−0.0055	−0.032
	400 Mev/c	1.39	1.2	0.006	−0.04	−0.11	−0.022	−0.13
		2.2	35×10^{-4}	2.6×10^{-4}	2×10^{-4}	3×10^{-4}	+0.0004	7×10^{-4}
Ca^{40}	188 Mev/c	0.62	855	4.8	−10	−51	−2.9	−54
		0.98	4.5	0.035	−0.11	−1.6	0.004	−1.6
		1.34	0.5	5×10^{-4}	0.0014	0.03	0.0004	0.03
Bi^{209}	188 Mev/c	0.62	1570	9.0	−14	−630	2.4	−632
		0.98	6	0.005	−0.023	−2.1	0.007	−2.1
		1.34	14×10^{-6}	0.1×10^{-6}	-2×10^{-6}	-6×10^{-4}	5.5×10^{-6}	$-6\times10^{-}$

As before, the expectation values $\langle V_1\rangle_{0n}$ are taken between the nuclear ground state ψ_0 and the nuclear excited state ψ_n. E_n refers to the nuclear energy, and ϵ refers to the electron energy. The sum is over all nuclear states excluding the ground state, thus including only effects due to dispersion in the sum.

The problem here takes on a more lucid aspect if the Coulomb functions $\phi_0(r)$ are written in momentum space. Then

$$\phi_0(r)=\int \exp(i\mathbf{s}\cdot\mathbf{r})\chi_0(s)d^3s,$$

$$\phi_0(r')=\int \exp(i\mathbf{l}\cdot\mathbf{r}')\chi_0(l)d^3l,$$

$$\int \exp(i\mathbf{Q}\cdot\mathbf{r})(-e^2/|\mathbf{r}-\mathbf{R}|)d^3r=4\pi\exp(i\mathbf{Q}\cdot\mathbf{R})Q^{-2}, \tag{13}$$

$$\Delta E_{2,\text{ disp}}=\int d^3p d^3s d^3l(2/\pi)e^4(\mathbf{p}+\mathbf{s})^{-2}(\mathbf{p}+\mathbf{l})^{-2}$$
$$\times\chi_0(s)\chi_0^*(l)\sum_1^\infty[E_n-E_0+\epsilon(p)-\epsilon_0]^{-1}$$
$$\times\langle\exp[i(\mathbf{p}+\mathbf{s})\cdot\mathbf{R}]\rangle_{0n}\langle\exp[-i(\mathbf{p}+\mathbf{l})\cdot\mathbf{R}]\rangle_{n0}.$$

Anticipating again that relativistic values of p, the intermediate-state electron momentum, are the important ones, we may clearly neglect s and l where they appear in combination with p, since $\chi_0(s)$ decreases very rapidly for s above Z times the Bohr momentum $\hbar/a_0\sim\alpha/2$ Mev/c, α being the fine structure constant. We may also note that $\int d^3s d^3l\,\chi_0(s)\chi_0^*(l)=|\phi_0(r)|_{r=0}^2\equiv|\phi(0)|^2$, and write finally

$$\Delta E_{2,\text{ disp}}=|\phi_0(0)|^2(2/\pi)e^4\int d^3p\,p^{-4}$$
$$\times\sum_1^\infty[E_n-E_0+\epsilon(p)-\epsilon_0]^{-1}\langle e^{i\mathbf{p}\cdot\mathbf{R}}\rangle_{0n}\langle e^{-i\mathbf{p}\cdot\mathbf{R}}\rangle_{n0}. \tag{14}$$

To demonstrate the fact, stated earlier, that relativistic p is the important region, we note that (14) may be written as

$$\Delta E_{2,\text{ disp}}=|\phi(0)|^2 8e^4\int\frac{g(p)}{p}dp,$$
$$g(p)=\frac{1}{p}\sum_1^\infty[E_n-E_0+\epsilon(p)-\epsilon_0]^{-1}\langle e^{i\mathbf{p}\cdot\mathbf{R}}\rangle_{0n}\langle e^{-i\mathbf{p}\cdot\mathbf{R}}\rangle_{n0},$$

$p_0<(E_0/c)\sim2$ Mev/c:

$$g(p)\approx\frac{1}{p}\sum_1^\infty\frac{\langle1+i\mathbf{p}\cdot\mathbf{R}\rangle_{0n}\langle1-i\mathbf{p}\cdot\mathbf{R}\rangle_{n0}}{(E_n-E_0)}$$
$$=p\sum_n(E_n-E_0)^{-1}\langle R\cos\theta\rangle_{0n}^2, \tag{15}$$

$p>1/R_0\sim100$ Mev/c:

$$g(p)\approx p^{-2}[1-\langle e^{i\mathbf{p}\cdot\mathbf{R}}\rangle_{00}^2],$$

$1/R_0\gg p\gg E_0$:

$$g(p)\approx\frac{1}{p}\sum_1^\infty\frac{1}{p}\langle1+i\mathbf{p}\cdot\mathbf{R}\rangle_{0n}\langle1-i\mathbf{p}\cdot\mathbf{R}\rangle_{n0}=\tfrac{1}{3}R_0^2.$$

This indicates that $g(p)$ is very small at nonrelativistic values of p, rises to a constant value $R_0^2/3$ between the characteristic energies E_0 and $1/R_0$, and falls off rapidly for higher values. One approximation, then, used to get order-of-magnitude results for general nuclei, is

$$\Delta E_{2,\text{ disp}}=|\phi(0)|^2 8e^4\int_{E_0}^{1/R_0}\left(\frac{R_0^2}{3}\right)\left(\frac{1}{p}\right)dp$$
$$=8e^4|\phi(0)|^2\left(\frac{R_0^2}{3}\right)\ln\left(\frac{1/R_0}{E_0}\right). \tag{16}$$

For general nuclei this must be summed over all protons. This was done for general $Z\leqslant50$, and the results are plotted in Fig. 2.

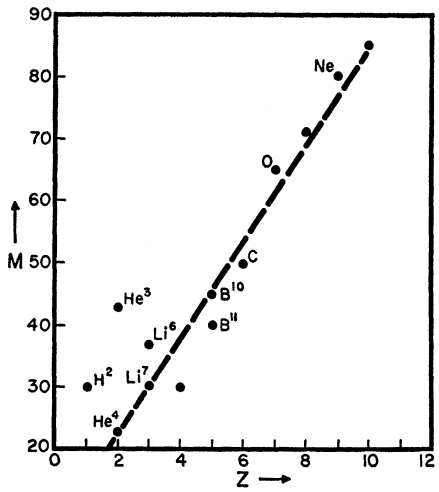

FIG. 2. $M \equiv 3\Delta E_{2,\,\mathrm{disp}}/|\phi(0)|^2 R_0^2 e^4$ *vs* Z for $Z \lesssim 10$. For elements Na to Ca, $M=(11Z-26)$; for elements Sc to Sn, $M=(12Z-56)$.

To obtain a more accurate value for the deuteron, we note that $\langle R\cos\theta\rangle_{0n} \equiv D_{0n}$ is a well-known matrix element,[11] and $g(p)=(1/p)\sum_1^\infty (E_n-E_0+p)^{-1}\langle R\cos\theta\rangle_{0n}^2 p^2$ can be evaluated explicitly to get $g(p)$ for regions $p \ll 100$ Mev/c. For $p \gg E_0/c \sim 2$ Mev/c, $g(p)=(1/p^2)\times[1-\langle e^{i\mathbf{p}\cdot\mathbf{R}}\rangle_{00}^2]$. Then $g(p)$ can be evaluated directly in the two overlapping regions of p, and $\Delta E_{2,\,\mathrm{disp}}$ obtained by a simple numerical integration. The result for this accurate calculation is, collecting other results for completeness, for a $2S$ level,

$$\begin{aligned}\Delta E_{2,\,\mathrm{dispersive}} &= 0.20e^4|\phi(0)|^2 \text{ fermi/Mev}\\ &= 2.9\times 10^{-11}\epsilon_0 = 0.02 \text{ Mc/sec},\\ \Delta E_{2,\,\mathrm{static}} &= 0.001 \text{ Mc/sec},\\ \Delta E_1 &= (2\pi/3)e^2R_0^2|\phi(0)|^2,\\ \Delta E &= E-\epsilon_0 = 0.88 \text{ Mc/sec}.\end{aligned} \tag{17}$$

Comparing with the earlier, more approximate procedure, we note that

$$\Delta E_2(\text{approximate}) = (1-0.1)\Delta E_2(\text{accurate}).$$

The deuteron calculation also yields the information that

$$g(E_0)\sim R_0^2/9,\quad g(1/4R_0)\sim 0.8(R_0^2/3),\quad g(1/R_0)\sim R_0^2/3,$$

as an indication of the reliability of approximation (16). The logarithmic dependence on the limits is clearly a saving feature of the approximation.

Assuming that the correction required to bring the approximate and accurate calculations into agreement is a linear addition to $\ln(1/R_0E_0)$ in (16), and is independent of Z, we may use the deuteron calculation to obtain a slightly improved value for $\Delta E_{2,\,\mathrm{disp}}$ for helium for a $2S$ level.

$$\begin{aligned}\Delta E_{2,\,\mathrm{disp}}(\text{helium}) &= 0.16 \text{ Mc/sec},\\ \Delta E_1 &= 7.08 \text{ Mc/sec},\\ \Delta E = \Delta E_1+\Delta E_2 &= 7.2 \text{ Mc/sec}\\ &\quad \pm 0.7 \text{ Mc/sec}.\end{aligned}$$

The error comes from the present experimental error in determining the nuclear radius R_0.

IV. DISCUSSION OF RESULTS

Table I and Table II indicate that the contribution to the cross section due to dispersion is generally negative, and between a few percent and ten percent for the light elements. The tabulated results for the heavier elements indicate, of course, the breakdown of the Born approximation, and the simultaneous decrease in the importance of dispersion. The contribution from the so-called remainder is generally larger than the contribution from the crude approximation (2). The unusual largeness of the dispersive contribution for the deuteron, compared with the static terms in the same order, which was observed in the calculation of energy shift for bound states, is reflected also in the scattering calculations. The ratio of static to dispersive contributions to second-order perturbation theory is much smaller for scattering from the deuteron than from heavier elements, even when the extra factor of Z occurring in the static part is divided out.

The ideal way to use the cross sections derived here would be to subtract them from the experimental values, and fit the difference with an appropriate static model, using the Stanford partial-wave techniques. Some summary of the cross sections may be made by indicating the change in R_0 deduced from experiment when dispersive effects are included. We may write $F_0(q)$ as the form factor deduced from experiment using a first Born approach to scattering, and $F_1(q)$ as the form factor deduced using first and second Born approximation. Then, for the deuteron, with the Hulthén wave function $N(e^{-\alpha r}-e^{-\mu r})/r$, we have

$$\begin{aligned}\sigma_{\mathrm{experiment}} &\equiv F_0^2(q)\sigma_{\mathrm{point\ charge}} \equiv F_1^2(q)\sigma_{\mathrm{point}}+\sigma_2,\\ \Delta F^2 &\equiv -\sigma_2/\sigma_{\mathrm{point}},\\ \Delta F^2 &= [7.4F-63(q^2+0.86)^{-1}(q^2+10)^{-1}]\Delta\alpha\\ &\quad +[0.44-122(q^2+30)^{-1}(q^2+9)^{-1}]\Delta\mu,\\ \Delta R_0^2 &= -20.0\Delta\alpha-1.4\Delta\mu.\end{aligned}$$

A similar approach was used for the shell model on He^4 and C^{12} with the results

	ΔR_0 (in fermis)	R_0 (in fermis)
deuteron	−0.010	2.10±0.04
helium	−0.08	1.53
carbon	−0.05	2.32

The experimental errors will soon be reduced.

It is clear here that the scattering calculation, giving

[11] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

a correction to R_0, has an influence on the first-order change in the bound-state energy, ΔE_1, which was seen to depend on R_0^2.

It should be pointed out, in conclusion, that the approximations used here in solving the second Born approximation are based on the fact that in the region of electron energy considered, the nucleons remain nonrelativistic. For incoming electrons with energies of the order of the nucleon rest mass energy, or more, this method will fail.

PHYSICAL REVIEW VOLUME 115, NUMBER 2 JULY 15, 1959

Anticommutator for a Nonlinear Field Theory

F. L. SCARF*
CERN, Geneva, Switzerland
(Received February 26, 1959)

The anticommutator for the Thirring model is computed by ordering the operator $\psi(x)\psi^*(x')$ and evaluating its renormalized vacuum expectation value. The infrared divergence is defined by introducing an *ad hoc* cutoff. The final expression does not agree with the approximations obtained by using perturbation theory or by using expansion over intermediate states (with the same cutoff). It is also found that Heisenberg's procedures cannot be applied to this two-dimensional problem.

I

IN recent years, there has been much discussion concerning the form of the anticommutator for bare-particle spinor operators which satisfy nonlinear equations of motion such as[1]

$$i\gamma^\mu\partial_\mu\psi+2g(\bar{\psi}\psi)\psi=0. \quad (1)$$

It has been suggested[2] that

$$S_{\alpha\beta}'(x_1,x_2)=i\langle 0|\{\psi_\alpha(x_1),\bar{\psi}_\beta(x_2)\}|0\rangle$$

resembles a classical solution of Eq. (1) modified by the addition of a mass term near the light cone, and that (in four dimensions) it is "effectively" more regular in this region than $S_{\alpha\beta}(x_1,x_2)$, the corresponding free-field function. The soluble two-dimensional Thirring model[3] allows one to check on the first of these speculations for a nonlinear theory, but since g is dimensionless in Thirring's case and $[g]=L^{-2}$ in Heisenberg's, the actual forms of $S'(x)$ in the two problems cannot be directly compared.

At first sight it would seem that the calculation is trivial for the two-dimensional example. Since no lengths are present, it has been pointed out[4] that the most general form for Lehmann's spectral function[5] is $\rho(m)=a(g)\delta(m)+b(g)|m|^{-1}$, giving $S'(x)=S(x)[a(g)+b(g)I]$, $I=\mathrm{P}\int_0^\infty dx/x(1-x)$. Perturbation theory and expansion over intermediate states also yield this expression,[3,6] which does not resemble a c-number solution of Eq. (1). However, the infrared divergence in I leads to an ill-defined space-time dependence for S'. If one tries to specify the divergent term precisely by introducing a cutoff, $k_{\min}=K$, the dimensional argument fails and singular contributions to $\rho(m)$ such as $m^{-1}\times\sin(K/m)$, etc., cannot be excluded, even as $K\to 0$.

In order to resolve any ambiguity, it is desirable to write the operator $\psi_\tau(x)\psi_{\tau'}{}^*(x')$ as an ordered functional of the free-field operators ϕ_τ and $\phi_{\tau'}{}^*$, as in Glaser's treatment[7] for ψ_τ. This is done in Secs. II, III. In Sec. IV, the renormalized vacuum expectation value is computed and a "covariant" infrared cutoff is introduced. The result is

$$S_{12}'(x_1,t_1;x_2,t_2)=S_{12}(v_1-v_2)\exp\{[gg'/(2\pi)^2]\ln(L/|u_1-u_2|)\} \quad u_1,u_2\ll L, \quad (2)$$

where $v=x-t$, $u=x+t$, $g'=g+2\pi n$, so that $|g'/2\pi|<1$, and L is a constant (with dimension of length) which transforms as $L'=\gamma L(1-\beta)$ under a Lorentz transformation.

The functional dependence of Eq. (2) does not agree with the predictions of perturbation theory (there is, in fact, an essential singularity at $u/L=0$) or of the intermediate state expansion. Furthermore, it is shown in the last section that Heisenberg's techniques cannot be applied to this two-dimensional example.

II

In Thirring's two-component representation ($\gamma^{(1)}=i\sigma_1$, $\gamma^{(2)}=\beta=\sigma_2$), Eq. (1) becomes

$$\partial\psi_1/\partial u=ig\psi_2{}^*\psi_2\psi_1, \quad \partial\psi_2/\partial v=-ig\psi_1{}^*\psi_1\psi_2, \quad (3)$$

* U. S. National Science Foundation Post-doctoral Fellow on leave from the University of Washington, Seattle, Washington.

[1] $\hbar=c=1$.
[2] W. Heisenberg, Revs. Modern Phys. **29**, 269 (1957).
[3] W. E. Thirring, Ann. Phys. **9**, 91 (1958).
[4] F. L. Scarf, Nuclear Phys. **11**, 475 (1959).
[5] H. Lehmann, Nuovo cimento **11**, 342 (1954).
[6] W. E. Thirring, Nuovo cimento **9**, 1007 (1958).
[7] V. Glaser, Nuovo cimento **9**, 990 (1958).