processes, but it is not clear that these interpretations can account for all the results from inelastic scattering.[29]

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[29] T. Tamura and D. C. Choudhury, Phys. Rev. **113**, 552 (1959).

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Nuclear States in the RaE β-Decay*

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It is shown that the β-transforming neutron of Bi^{210} most probably has an $i_{11/2}$ character, despite a $g_{9/2}$ character of the ground-state neutron in Pb^{209}. This makes a critical difference to the RaE spectrum parameter, $\xi=i\langle\mathbf{r}\rangle/\langle\boldsymbol{\sigma}\times\mathbf{r}\rangle$, yielding $\xi\approx+1$ rather than $\xi\approx-1/10$. The effect of configuration mixing is also investigated but does not affect ξ appreciably.

To arrive at the above conclusions, it is necessary to show that the neutron-proton attraction in the $(h_{9/2}i_{11/2})_1$ state is the large amount, 840 kev, greater than in the $(h_{9/2}g_{9/2})_0$ state. The resulting shell-model problem has interest independent of the β-theory application which was the original objective of this work. True and Ford had found that two neutrons extra to the doubly-magic core, Pb^{208}, as against nucleons deep in the core matter, interact with about the same strength and range of force as do two free nucleons. The problem here checks the extension of that important finding to neutron-proton and proton-proton pairs. The force strength is consequently *not* used as an adjustable parameter, as it has been in previous approaches to such problems.

The True-Ford problem involved only singlet, central forces between like nucleons in an essential way. The RaE daughter, Po^{210}, investigated here, has only Coulomb repulsion superposed. The resultant comparison with experiments is about as good as that obtained by True and Ford.

The extension to the neutron proton pair of Bi^{210} is far more complex, since triplet forces, an exchange character, and noncentral forces, may now come into play. We find that the finite-range, central forces alone cannot give substantially more attraction in the $(h_{9/2}i_{11/2})_1$ state than in $(h_{9/2}g_{9/2})_0$. Even using the strength as a parameter cannot help significantly. However, the tensor forces produce attraction in the former and repulsion in the latter state. Hence, the two-body forces must be imitated even in this detail in order to yield an explanation of the RaE level scheme. This is unfortunate for quantitative results, because the strength of the two-body tensor force seems never to have been determined unambiguously for potentials without cores.

Our final conclusion is that the two-body neutron-proton force may be well represented by zero-range forces of the same volume energy as found experimentally. Tensor effects vanish identically in this limit and so an unambiguous representation of the strength can be obtained. The results for the relative positions of the $J=0$ and 1 states, used as the test above, now turn out in almost perfect agreement with the observations. Configuration mixing plays a role in this result, and, in consequence, the work includes a generalization of de-Shalit's formulas, for the interaction energies with zero-range forces, to nondiagonal matrix elements.

INTRODUCTION

THE distinctive characteristics of the RaE decay (nonstatistical spectrum, anomalous electron polarization, prolonged lifetime) have made it an important test case.[1] The radiation seems to be generated through at least the β-moments,[2] $\langle\mathbf{r}\rangle$, $\langle\boldsymbol{\sigma}\times\mathbf{r}\rangle$, and $\langle\boldsymbol{\alpha}\rangle$. These have been treated as completely arbitrary parameters, with attendant uncertainties of interpre-

* Supported by a grant from the National Science Foundation.

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[1] A. G. Petschek and R. E. Marshak, Phys. Rev. **85**, 608 (1952); M. Yamada, Progr. Theoret. Phys. (Kyoto) **10**, 252 (1953); E. A. Plassman and L. M. Langer, Phys. Rev. **96**, 1593 (1954); R. Nataf, J. phys. radium **17**, 480 (1956); G. E. Lee-Whiting, Phys. Rev. **97**, 463 (1955).

[2] Takebe, Nakamura, and Taketani, Progr. Theoret. Phys. (Kyoto) **14**, 317 (1955); Fujita, Yamada, Matumoto, and Nakamura, Progr. Theoret. Phys. (Kyoto) **20**, 287 (1958). The investigation reported in the last paper seems to indicate that the experimental RaE spectrum may be consistent with $\xi\lesssim-1$, or with $\xi\gtrsim+6$, but not with $\xi\approx+1$, which is the value we find best consistent with the RaE level scheme. We consider our evidence for $\xi\approx+1$ the stronger for the following reasons. To find the above ranges of ξ, Fujita *et al.*, effectively treat the matrix element $\langle\boldsymbol{\alpha}-3\hat{r}(\boldsymbol{\alpha}\cdot\hat{r})\rangle$ as having roughly the same magnitude as $\langle\boldsymbol{\alpha}\rangle$ rather than using it as an independent parameter, as they do $\langle\boldsymbol{\alpha}\rangle$, $\langle\mathbf{r}\rangle$ and $\langle\boldsymbol{\sigma}\times\mathbf{r}\rangle$. Actually, all four matrix elements are independent "spherical tensors," subject to different selection rules, e.g., on orbital momentum, hence should be treated on the same footing. The consequent intrusion of a fourth parameter, into a spectrum-fitting scarcely able to determine the original three, has the result that no value of ξ is provably inconsistent with the observed spectrum. This was one reason we sought independent evidence for ξ, directly from the nuclear states it is supposed to characterize. Actually, we find that our result, $\xi\approx+1$, can fit the spectrum only with very large values of the ratio $\langle\boldsymbol{\alpha}-3\hat{r}(\boldsymbol{\alpha}\cdot\hat{r})\rangle/\langle\boldsymbol{\alpha}\rangle$. These seem implausible, but we could adduce no decisive argument against them. We believe that these developments reduce the conventional theory of the RaE spectrum to an unsatisfactory state and may indicate that R. Feynman and M. Gell-Mann [Phys. Rev. **109**, 193 (1958)] are correct in attributing "Fermi charge" to the pion clouds surrounding nucleons. Their theory will require extensive revision of the expectations for forbidden spectra.

tation, because of ignorance concerning the initial and final nuclear states.

Of particular importance for the interpretations has been the ratio

$$\xi = i\langle \mathbf{r} \rangle / \langle \boldsymbol{\sigma} \times \mathbf{r} \rangle. \quad (1)$$

Unlike the β-moment $\langle \boldsymbol{\alpha} \rangle$, this ratio should not require knowledge of relativistic corrections to nuclear states in an essential way. Current treatments of nuclear states based on the shell model may have some validity for the determination of ξ.

From the shell-model viewpoint, the states involved in the RaE decay have some gratifyingly simple characteristics. The parent nucleus, Bi^{210}, has just one neutron and one proton outside the doubly-magic Pb^{208} core. The Po^{210} product has just two protons. This is clearly a case for which the shell-model should be particularly applicable, and we set out to find as reliable a characterization of these nuclear states as seemed feasible on that basis.

The values of ξ for various, assumed, pure shell-model configurations have been given long ago.[3] The final state is, quite clearly, predominatly $(h_{9/2}{}^2)_0$, but at least two configurations, $(h_{9/2}g_{9/2})_1$ and $(h_{9/2}i_{11/2})_1$, must be considered seriously as the possible dominant configuration of the initial state. The first candidate gives $\xi = -1/10$, while the second yields $\xi = +1$, hence the critical nature of the choice is evident. Lee-Whiting[3] presented arguments for the second choice, but they depended heavily on the easier consistency of that choice with the value of ξ as "measured" by the RaE spectrum. It has become questionable[2] that the spectrum can provide any such "measurement"; the situation now seems rather to require independent information about ξ to check the consistency of the whole interpretation of the spectrum. We try to obtain such independent information by finding which configurations can give the observed energy-level structure of RaE. Moreover, we consider possible configuration mixing.

The problem has interest independent of our initial objective, i.e., the application to β-decay theory. True and Ford[4] have shown that if two *neutrons* (holes) extra to the Pb^{208} core interact through forces of about the same strength and range as do two free neutrons, then a quantitative explanation of a large amount of data about Pb^{206} is obtained. The implication that extra-core nucleons, as against those embedded in the nuclear core medium, interact through "vacuum forces" has an obvious importance. Our problem will involve a test of this finding for two protons, and for a neutron-proton pair.

The extension to the neutron-proton pair of Bi^{210} should be particularly revealing. The two *like* nucleons treated by True and Ford interact almost exclusively through a central, singlet force. Unlike nucleons should also interact through triplet forces. Moreover, the exchange and noncentral characteristics of the forces which are observed for two unlike free nucleons, now have a chance to come into play.

Even the two-proton problem of Po^{210} entails a nontrivial extension of the True-Ford findings. Again, the specifically nuclear forces are expected to be almost purely singlet central, but Coulomb repulsion is superposed. It is of interest to see whether this makes a distinguishable difference from the two-neutron case.

The first treatment of the Bi^{210} problem was given by Pryce.[5] He used zero-range forces and restricted himself to pure jj-coupled states. He thus obtained level schemes arising from each configuration separately, but did not relate these level schemes with each other. That will be one of our principal problems.

TABLE I. "Zero-order" levels in Bi^{210} and Po^{210}.[a-d]

Configuration	Bi^{210} (kev)	Configuration	Po^{210} (kev)
$h_{9/2}g_{9/2}$	0	$h_{9/2}{}^2$	0
$h_{9/2}i_{11/2}$	790	$h_{9/2}f_{7/2}$	900
$f_{7/2}g_{9/2}$	900	$h_{9/2}i_{13/2}$	1600
$h_{9/2}$ ($d_{5/2}$ or $g_{7/2}$)	1560	$f_{7/2}{}^2$	1800
$i_{13/2}g_{9/2}$	1600	$f_{7/2}i_{13/2}$	2500
$f_{7/2}i_{11/2}$	1690	$i_{13/2}{}^2$	3200
$h_{9/2}d_{3/2}$	2010		
$h_{9/2}p_{1/2}$	2130		

[a] In the symbols for the configurations the proton character is put first.
[b] R. M. Kiehn and C. Goodman, Phys. Rev. **95**, 989 (1954).
[c] Strominger, Stephens, and Rasmussen, Phys. Rev. **103**, 748 (1956).
[d] See reference 7.

"ZERO-ORDER" LEVEL SCHEMES

In "zeroth approximation," the two nucleons extra to the Pb^{208} core will have just the sum of their energies when each is alone with that core, as in Bi^{209} and Pb^{209} nuclei. One uses the observed values of these energies, to avoid reliance on any theory of the interaction of a nucleon with a core. The lowest of the resulting "zero-order" energy levels are listed in Table I. The assignments for the 83rd proton of Bi^{209} have been established for some time.[6] The evidence that the lowest state of the 127th nucleon in Pb^{209} is $g_{9/2}$, rather than $i_{11/2}$, has been produced at Indiana University.[7]

The *observed* level schemes of Bi^{210} and Po^{210} are shown in Fig. 1. The states of primary interest to us are the 5-day, β-emitting, RaE state, which has a measured spin 1, and the 0^+ ground state of Po^{210} into which it decays. The RaE state has a 2.6×10^6 year, α-emitting, isomer which may be the lowest state of Bi^{210}, as indicated. This state is reported[8] to have a β-decay branch of 0.4%, and on that basis has $ft \approx 10^{19}$ sec. This indicates a spin of 4 or 5 units.

[3] See Lee-Whiting, reference 1, for the most explicit presentation.
[4] W. True and K. Ford, Phys. Rev. **109**, 1675 (1958).
[5] M. H. L. Pryce, Proc. Phys. Soc. (London) **A65**, 773 (1952).
[6] J. E. Mack, Revs. Modern Phys. **22**, 64 (1950).
[7] McEllistrem, Martin, Miller, and Sampson, Phys. Rev. **111**, 1636 (1958).
[8] H. B. Levy and I. Perlman, Phys. Rev. **94**, 152 (1954).

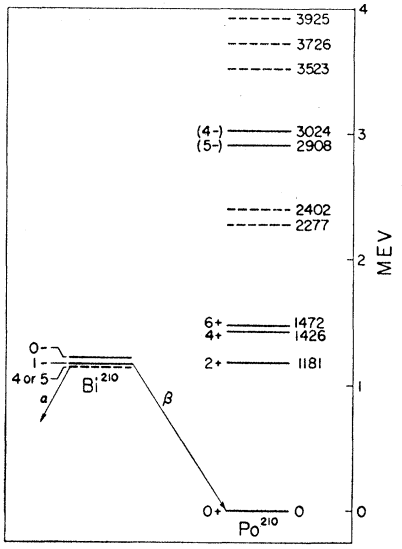

FIG. 1. Experimentally known low-lying energy levels in Bi^{210} and Po^{210}.

The $h_{9/2}g_{9/2}$ configuration of Table I is expected to have a total spin $J=0$ in its lowest energy state; i.e., after one takes into account the attraction of the proton for the neutron. The calculations presented below confirm this expectation. Similarly, the lowest state of the $h_{9/2}i_{11/2}$ configuration is expected to have $J=1$. It is probable, therefore, that this is the dominant configuration of our β-decaying state. In consequence, our principal problem will be to show that the $(h_{9/2}i_{11/2})_1$ state will fall some 47 kev below the $(h_{9/2}g_{9/2})_0$ state when the neutron-proton pair interaction in the two states is taken into account. The attraction in the $(h_{9/2}i_{11/2})_1$ state will have to exceed that in the $(h_{9/2}g_{9/2})_0$ state by (790+47) kev.

The difference of attraction in the two states, which will be the test here, is a severe one in that this difference turns out to be very insensitive to large modifications of finite-range central forces. Mere adjustment of parameters will not suffice to compensate for basically incorrect assumptions.

ASSUMPTIONS ABOUT THE PAIR-INTERACTIONS

Because of the True-Ford[4] finding that the two neutron holes of Pb^{206} interact with about the same potential strength and range as two free neutrons, we shall in general try to make our assumed interactions imitate those which are consistent with nucleon-nucleon scattering, and the deuteron data. We thus avoid using the interaction strength as a parameter. We follow the practice[4] of incorporating the data mentioned in potentials of the Gaussian form,

$$U(r)=-D\exp(-\beta r^2). \tag{2}$$

This is adequate in view of the uncertainties as to the correct forms for the radial wave functions (see below). For the same reason, there is no point in retaining the troublesome distinction between the singlet and triplet ranges; we used

$$\beta^{-\frac{1}{2}}=\beta_s^{-\frac{1}{2}}=\beta_t^{-\frac{1}{2}}=1.6 \text{ fermi}, \tag{3}$$

where 1 fermi (f) $=10^{-13}$ cm. We then adopted potential depths which preserve the correct individual singlet and triplet volume energies,[9] i.e., maintained $D\beta^{-\frac{3}{2}}$. That yields

$$D=D_t=60 \text{ Mev}, \quad pD\equiv D_s=46 \text{ Mev}, \tag{4}$$

$p=0.765$ will represent the singlet-to-triplet strength ratio.

True and Ford used purely singlet forces, in conformity with the fact that the triplet forces between like nucleons are relatively negligible. This implies a Serber exchange mixture for a charge-independent internucleon force and so we adopted the central interaction potential

$$V(1,2)=U(r)[\pi_t+p\pi_s]\times\tfrac{1}{2}(1+P_M), \tag{5}$$

where $\pi_{t,s}$ are the triplet and singlet projection operators, while P_M stands for the Majorana exchange operator. Actually, we made extensive investigations of variations in the exchange mixture, as well as in the singlet-to-triplet ratio, p, but found it no significant help.

It turned out to be necessary to discuss the effect of a tensor force admixture,

$$V_T=U_T(r)S_{12}\equiv U_T(3\boldsymbol{\sigma}_1\cdot\hat{r}\boldsymbol{\sigma}_2\cdot\hat{r}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2), \tag{6}$$

as will be seen later.

In what was originally intended as preliminary work only, we investigated the zero-range limit ($\beta\rightarrow\infty$):

$$U_0=-D(\pi/\beta)^{\frac{3}{2}}\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{7}$$

So long as $D\beta^{-\frac{3}{2}}$ has a value determined by (3) and (4) the correct volume energy will be maintained. The work with (7) is vastly simplified because, in this limit,

$$\tfrac{1}{2}(1+P_M)\rightarrow 1 \quad \text{and} \quad S_{12}\rightarrow 0, \tag{8}$$

effectively. Several investigators[10] have achieved considerable success using zero-range forces to represent the pair-interactions. However, they have usually used the strength as an adjustable parameter, whereas we adhere to the same volume energies as are found for the interaction of free nucleon pairs.

ASSUMPTIONS ABOUT RADIAL WAVE FUNCTIONS

We follow the widespread practice[11] of using harmonic-oscillator radial wave functions. These at least

[9] H. A. Bethe and P. Morrison, *Elementary Nuclear Theory* (John Wiley & Sons, Inc., New York, 1956), second edition.

[10] D. E. Alburger and M. H. L. Pryce, Phys. Rev. **95**, 1482 (1954); R. W. Hoff and J. M. Hollander, Phys. Rev. **109**, 447 (1958); A. de-Shalit, Phys. Rev. **91**, 1479 (1953); also see reference 5.

[11] D. Kurath, Phys. Rev. **80**, 98 (1950); **87**, 218 (1952); **91**, 1430 (1953). E. H. Kronheimer, Phys. Rev. **90**, 1003 (1953); L. W. Longdon, Phys. Rev. **90**, 1125 (1953); A. de-Shalit and M. Goldhaber, Phys. Rev. **92**, 1211 (1953); M. G. Redlich, Phys. Rev. **99**, 1421 (1955); G. E. Tauber and T. Y. Wu, Phys. Rev. **94**, 1307 (1954); **105**, 1772 (1957). B. C. Carlson and I. Talmi,

allow recognition of centrifugal and boundary effects. The results are not expected to be sensitive to details of the radial functions since only weighted integrals over the radial functions enter the calculations.

The nodeless functions are

$$R(0l)=N_l r^l \exp(-\nu r^2), \quad N_l^2=\frac{2^{l+2}(2\nu)^{l+\frac{3}{2}}}{\pi^{\frac{1}{2}}(2l+1)!!}, \tag{9}$$

where $(2l+1)!!\equiv 1\times 3\times 5\times\cdots\times(2l+1)$. We shall also need the 1-node function

$$R(1l)=(l+\tfrac{3}{2})^{\frac{1}{2}}R(0l)-(l+\tfrac{5}{2})^{\frac{1}{2}}R(0l+2). \tag{10}$$

The parameter ν measures the spread of the radial distribution and is related to the classical "turning-point." If the latter is at radius R then $\nu=(l+\frac{3}{2})/R^2$ in the nodeless state, and $\nu=(l+\frac{7}{2})/R^2$ in the 1-node state. For R we take the charge radius $R=1.2A^{\frac{1}{3}}$ fermis, thus using the proton spread to determine a common value for ν. This value of ν is clearly the same for the 1-node $f_{7/2}$ proton state as for the nodeless $h_{9/2}$ state and is used for all the extra-core single particle states. The value of ν so obtained is such that $(2\nu)^{-\frac{1}{2}}=2.0$ fermis.

The radial functions enter the calculations through the "Slater integrals,"

$$F_k=\int_0^\infty dr_1\, r_1^2 R_1(r_1)R_1'(r_1)\int_0^\infty dr_2\, r_2^2 R_2(r_2)R_2'(r_2) \times\int_{-1}^{1}(d\mu/2)P_k(\mu)U(|\mathbf{r}_1-\mathbf{r}_2|), \tag{11}$$

which measure the intensity of "orbital momentum transfers," k.

For the zero-range potential (7)

$$F_k\equiv F^0=-(D/4\pi)(\pi/\beta)^{\frac{3}{2}}\int_0^\infty dr r^2 R_1R_1'R_2R_2'. \tag{12}$$

Thus it is clear that two potentials of the Gaussian form (2) will have the same zero-range limit if $D\beta^{-\frac{3}{2}}$ is the same for the two potentials. This provides the justification for preserving $D\beta^{-\frac{3}{2}}$ when comparing different potentials.

Pryce and Alburger[10] avoid the special assumptions involved in a choice of radial wave-functions by estimating F^0—a different F^0 for each configuration $(l_{1,2}'=l_{1,2})$. In the present case, the experimental information is too fragmentary for such a procedure to be profitable. Moreover configuration mixing $(l_{1,2}'\neq l_{1,2})$ is of particular concern here since our objective is the evaluation of β-decay matrix elements which depend on the purity of the states.

Phys. Rev. **96**, 436 (1954); C. Levinson and K. W. Ford, Phys. Rev. **99**, 792 (1955); **100**, 13 (1955). H. Horie and A. Arima, Phys. Rev. **99**, 778 (1955); W. True, see reference 4; J. B. French and B. J. Raz, Phys. Rev. **104**, 1411 (1956).

TREATMENT OF THE PAIR-INTERACTIONS

The discussion here will be suited to n-p pairs. Obvious modifications are needed for like-nucleon pairs (such as the two protons of Po^{210}). The discussion of Po^{210} will be left to a later section.

We symbolize with $|\alpha\rangle,|\alpha'\rangle,\cdots$ the jj-coupled states which are used as a basis:

$$|\alpha\rangle\equiv R_1(r_1)R_2(r_2)\,|\,(l_1s_1)j_1,(l_2s_2)j_2,JM\rangle. \tag{13}$$

These states are eigenfunctions of the energy, $\epsilon(l_1j_1l_2j_2)$, in the field of the core, before the pair interaction is "turned on." The energies are obtained from the experimental information on Pb^{209} and Bi^{209}. Relative to $\epsilon(h_{9/2}g_{9/2})=0$, we have: $\epsilon(h_{9/2}i_{11/2})=0.79$ Mev, $\epsilon(f_{7/2}g_{9/2})=0.90$ Mev, and $\epsilon(f_{7/2}i_{11/2})=1.69$ Mev, as listed in Table I. The further levels shown in Table I will not be of interest for our problem of determining the $J=0$, 1 states, since they do not contribute to those angular momenta.

The pair-interaction removes the degeneracy of the states with various J arising from a given configuration. The resultant energies, E_J, and the corresponding state amplitudes, $\langle\alpha|EJM\rangle$, are to be obtained from the secular equation

$$\sum_{\alpha'}[\langle\alpha|V|\alpha'\rangle-(E-\epsilon_{\alpha'})\delta_{\alpha\alpha'}]\langle\alpha'|EJM\rangle=0. \tag{14}$$

Because the information on the ϵ's is limited, we approximate by restricting the summation to the four configurations $(h_{9/2}g_{9/2})$, $(h_{9/2}i_{11/2})$, $(f_{7/2}g_{9/2})$ and $(f_{7/2}i_{11/2})$. We therefore improve on the first-order perturbation energies,

$$E\approx\epsilon_\alpha+\langle\alpha|V|\alpha\rangle, \tag{15}$$

only to that extent. Pryce and Alburger, and Hoff and Hollander, restrict themselves to the first order, (15), ignoring configuration mixing completely.

Because $\pi_t=1-\pi_s$ in (5), we need only evaluate singlet and spin-independent interaction energies. Moreover

$$\langle\alpha|UP_M|\alpha'\rangle = (-)^{j_1'+j_2'+J}Q_{12}'[\langle\alpha|U|\alpha'\rangle-2\langle\alpha|U\pi_s|\alpha'\rangle],$$

and

$$\langle\alpha|U\pi_sP_M|\alpha'\rangle=-(-)^{j_1'+j_2'+J}Q_{12}'\langle\alpha|U\pi_s|\alpha'\rangle, \tag{16}$$

where Q_{12}' interchanges $l_1'\leftrightarrow l_2'$ and $j_1'\leftrightarrow j_2'$ in the "primed" or "initial" state. Thus the matrix elements of (5) are

$$\langle\alpha|V|\alpha'\rangle=\tfrac{1}{2}[1+(-)^{j_1'+j_2'+J}Q_{12}']\langle\alpha|U|\alpha'\rangle -\tfrac{1}{2}[(1-p)+(1+p)(-)^{j_1'+j_2'+J}Q_{12}']\langle\alpha|U\pi_s|\alpha'\rangle. \tag{17}$$

This reduces to

$$\langle\alpha|V_0|\alpha'\rangle=\langle\alpha|U_0|\alpha'\rangle-(1-p)\langle\alpha|U_0\pi_s|\alpha'\rangle \tag{18}$$

for zero-range forces, when $P_M\equiv 1$.

TABLE II. $|\langle j_1 j_2 \tfrac{1}{2} -\tfrac{1}{2}|J0\rangle|^2$.[a]

J \ (j_1,j_2)	(9/2,9/2)	(9/2,11/2)	(7/2,11/2)
0	1/10	0	0
1	1/3×10×11	3/22	0
2	(−) 4/3×11	5/6×11×13	(−) 25/11×12
3	(−) 12/5×11×13	(−) 49/3×11×13	(−) 25/11×12×13
4	81/5×11×13	(−) 3/11×13	3×25/4×11×13
5	3/5×13	4/3×13	7/3×4×13
6	(−) 16/3×5×11	4×7/3×11×17	(−) 7/66
7	(−) 16×49/3×11×13×17	(−) 5×6×7/11×13×17	(−) 3×7×25/2×11×13×17
8	49/5×11×13	(−) 6×49/11×13×19	21/2×11×13
9	49×81/5×11×13×17	3×49/11×13×17	7×21/2×13×17
10	0	3×9×49/13×17×19	0

[a] When the unsquared coefficient is negative this is indicated in parenthesis before the squared value.

0-RANGE PAIR INTERACTIONS

For several large nuclei with a Pb^{208} core, Pryce, Alburger[10] and Hoff and Hollander[10] have had considerable success in approximating finite range nuclear forces by a zero-range potential. We therefore investigated this relatively simple case first.

de-Shalit[10] has derived convenient expressions for the diagonal energies, $\langle\alpha|V_0|\alpha\rangle$, for the zero-range potential (7). We present a generalization of his results which includes the nondiagonal elements needed for configuration mixing. They can be derived in the same way as de-Shalit's formulas or by going to the zero-range limit of our results for the finite-range forces presented later.

We have for the singlet part,

$$\langle\alpha|U_0\pi_s|\alpha'\rangle = aF^0(-)^{j_1+j_1'+l_1+l_1'+1}, \qquad (19)$$

if *both* $J+l_1+l_2$ and $J+l_1'+l_2'$ are even; the singlet interaction vanishes otherwise. F^0 is given by (12) and

$$a = ww'/2(2J+1), \qquad (20)$$

with

$$w = [(2j_1+1)(2j_2+1)]^{\frac{1}{2}}\langle j_1 j_2 \tfrac{1}{2} -\tfrac{1}{2}|J0\rangle. \qquad (21)$$

The last factor is the conventional vector addition coefficient.

The spin independent part also vanishes unless $l_1+l_2+l_1'+l_2'$ is even (parity conservation). When it does not thus vanish, it is

$$\langle\alpha|U_0|\alpha'\rangle = aF^0[(-)^{j_1+j_1'+l_1+l_1'+1}+uu'/4J(J+1)], \qquad (22)$$

with

$$u = (2j_1+1)+(2j_2+1)(-)^{j_1+j_2+J}. \qquad (23)$$

Both (19) and (22) agree with de-Shalit's formulas for his case of $l_{1,2}'=l_{1,2}$ and $j_{1,2}'=j_{1,2}$. They have the proper behavior in interchanges of primed and unprimed quantities and of the indices 1,2.

The required coefficients $\langle j_1 j_2 \tfrac{1}{2} -\tfrac{1}{2}|J0\rangle$ were partially tabulated by de-Shalit. For our purposes we had to extend his tables to $j_2=11/2$. Moreover, we found his values for $j_1=j_2=9/2$, $J=3$, 4 in error. We therefore supplement his table with our Table II. The parentheses at the head of each column give the values of $(j_1 j_2)$. When the unsquared coefficient is negative this is indicated in parenthesis before the squared value. The sum of the numbers in each column is unity as it should be (this is the test which revealed de-Shalit's errors).

Table III gives the values of $F^0(l_1 l_1' l_2 l_2')$ computed from (12). The nodeless functions (9) were used for $h_{9/2}$ and $i_{11/2}$, the one-node functions (10) for $f_{7/2}$ and $g_{9/2}$. The numbers given are in units of the diagonal F^0 for $h_{9/2}i_{11/2}$, which contains no radial nodes:

$$F^0(5566) = -0.11D(\nu/\beta)^{\frac{3}{2}} = -1.20 \text{ Mev}, \qquad (24)$$

with the values $D=60$ Mev, $\beta^{-\frac{1}{2}}=1.6$ f and $(2\nu)^{-\frac{1}{2}}=2.0$ f adopted above. Of course the ratios in Table III are independent of these parameters.

The diagonal energies (15) of the configurations have been given by Pryce.[5] As expected, the lowest level of $(h_{9/2}g_{9/2})$ has $J=0$, and the lowest of $(h_{9/2}i_{11/2})$ has $J=1$. Pryce did not attempt to predict the relative position of the levels from different configurations as we do through the use of Table III.

With the interaction strength $F^0=-1.20$ Mev of (24) we find that the diagonal n-p attraction in the $(h_{9/2}g_{9/2})_0$ state is $\langle\alpha|V|\alpha\rangle=-3.47$ Mev, while it is $\langle\alpha'|V|\alpha'\rangle=-4.14$ Mev, in the $(h_{9/2}i_{11/2})_1$ state. The lesser attraction in the $J=0$ state must be attributed to the misfit of the nodeless $h_{9/2}$-proton radial distribution with the one-node $g_{9/2}$-neutron distribution. The effect is in the right direction to explain why the $J=1$ state is observed to fall 0.047 Mev lower than the $J=0$ state in Bi^{210}. However, it is not great enough to overcome the initial 0.79 Mev greater binding of the $J=0$ state, since the excess of pair-attraction is only $4.14-3.47=0.67$ Mev. Yet, in comparison with the parametrization of Pryce and Alburger, we are already

TABLE III. Zero-range Slater integrals, in units of the diagonal F^0 for $h_{9/2}i_{11/2}$.

	$h_{9/2}g_{9/2}$	$h_{9/2}i_{11/2}$	$f_{7/2}g_{9/2}$	$f_{7/2}i_{11/2}$
$h_{9/2}g_{9/2}$	0.578	−0.111	0.0482	0.491
$h_{9/2}i_{11/2}$	−0.111	1	0.483	−0.509
$f_{7/2}g_{9/2}$	0.0482	0.483	0.973	−0.174
$f_{7/2}i_{11/2}$	0.491	−0.509	−0.174	0.590

using a larger energy scale than they when we put $F^0=-1.20$ Mev.

There is still another effect which should push down the $J=1$ state relative to $J=0$. The latter is unaffected by configuration mixing since only one configuration, $(h_{9/2}g_{9/2})$, can yield $J=0$. On the other hand, three configurations give rise to $J=1$ states: not only $(h_{9/2}i_{11/2})$ but also $(h_{9/2}g_{9/2})$ and $(f_{7/2}g_{9/2})$. The configuration mixing of these $J=1$ levels will produce the effect. We computed it, still using the pre-chosen parameters as given above. Our result was that the lowest $J=1$ level is pushed down 0.04 Mev below the $J=0$ level, almost precisely as observed! Of course the accuracy of this agreement is fortuitous in view of the arbitrariness involved in representing the forces in vacuo with 0-range potentials. The configuration mixing push is $0.79+0.04-0.67=0.16$ Mev.

The structure found for the lowest $J=1$ state was

$$\psi=0.936|h_{9/2}i_{11/2},J=1\rangle +0.134|h_{9/2}g_{9/2},J=1\rangle+0.327|f_{7/2}g_{9/2},J=1\rangle. \quad (25)$$

Its success in giving the right energy when the forces are represented by zero-range potentials should perhaps give it some significance, at least as measuring the maximum[12] impurity of the β-decaying RaE state. The large magnetic moment[3] of the $(f_{7/2}g_{9/2})_1$ state confirms that this is a maximum. One gets $\mu=-0.75$ for the mixture (25) whereas $\mu=-0.36$ for a pure $(h_{9/2}i_{11/2})_1$ state. The latter value is perhaps in better conformity with the failure to detect a magnetic moment in Bi^{210}.[13]

The remaining $J=1$ states arising from the two mixtures orthogonal to (25) were found to lie, respectively, at 1.25 Mev and 1.77 Mev above the lowest $J=1$ state.

The next higher states on this scale but computed without configuration mixing (diagonal energies only) are

$$E(h_{9/2}i_{11/2},\ J=10)=2.07 \text{ Mev},$$
$$E(h_{9/2}g_{9/2},\ J=9)\ =2.43 \text{ Mev},$$
$$E(f_{7/2}g_{9/2},\ J=8)\ =2.44 \text{ Mev},$$
$$E(h_{9/2}i_{11/2},\ J=2)\ =2.56 \text{ Mev}.$$

The fact that the highest-spin levels are next above the $J=0$ and $J=1$ states was already pointed out by Pryce. He conjectured (see also Brink[14]) that they might account for the high-spin, alpha-decaying state of Bi^{210} which is observed to lie in the neighborhood of the $J=1$ β-decaying state (perhaps 25 kev below it). Unfortunately, we find that the configuration mixing cannot bring the high-spin states down appreciably from the positions listed above. Moreover, the observations indicate that the "high" spin of the α-decaying state is $J=4$ or 5 rather than $J=8$, 9 or 10. Our lowest $J=4$ or 5 state is a $J=5$ state at 2.92 Mev and several lower-spin states lie below it. We therefore can throw no further light on the origin of the observed α-decaying state.

[12] It is a general experience that zero-range central forces lead to a maximum of configuration mixing. Forces of infinite range (constant potentials) make nondiagonal matrix elements vanish because of the orthogonality of the wave functions. For forces of finite range, one obtains intermediate results.

[13] Fred, Tomkins, and Barnes, Phys. Rev. **92**, 1324 (1953).

[14] D. M. Brink, Proc. Phys. Soc. (London) **A67**, 757 (1954).

FINITE-RANGE CENTRAL FORCES

True and Ford achieved their success with the two-neutron problem (Pb^{206}) by using a pair-interaction which had not only about the same volume-energy as the forces between individual like nucleons ("in vacuo"), but also about the same finite range. We therefore also tried the effect of extending the range from the zero extent used in the last section. We first employed the central interaction form specified by (2) and (5).

To evaluate the matrix elements $\langle\alpha|V|\alpha'\rangle$, we first analyzed the potential into spherical tensors, in the usual way. We could then apply the general formula[15]

$$\langle J\|(R_r\odot S_s)_k\|J'\rangle=\langle j_1\|R_r\|j_1'\rangle\langle j_2\|S_s\|j'_2\rangle \times[(2k+1)(2J+1)(2J'+1)]^{\frac{1}{2}}\begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ r & s & k \end{Bmatrix}, \quad (26)$$

and its various specializations. Here J may be any angular momentum decomposable into $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2$ where $\mathbf{j}_{1,2}$ refer to separate sets of degrees of freedom. R_r is any spherical tensor operator which operates in the domain of the first set of freedoms, while S_s operates in the other. We use the symbol of Jahn *et al.*,[16]

$$(R_r\odot S_s)_{k\kappa}\equiv\sum_\rho\langle rs\rho\ \kappa-\rho|k\kappa\rangle R_{r\rho}S_{s\kappa-\rho}, \quad (27)$$

to denote the composite spherical tensor operator. The 9-j symbol is defined in many places. The "reduced matrix elements" are here defined by

$$\langle j\mu|R_{r\rho}|j'\mu'\rangle \equiv\langle j\|R_r\|j'\rangle(-)^{j'-\mu'}\langle jj'\mu-\mu'|r\rho\rangle(2r+1)^{-\frac{1}{2}}. \quad (28)$$

The formula (26) is the result of a line of generalization initiated by Racah. We shall need it in full generality only for treating the tensor forces in the next section. In most steps, specializations, in which the 9-j symbol reduces to a Racah coefficient (6-j symbol), are adequate.

Our result for the spin-independent part of the interaction is

$$\langle\alpha|U|\alpha'\rangle =(-)^{j_2+j_2'+J}[(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)]^{\frac{1}{2}} \times\sum_k F_k\langle j_1j_1'\tfrac{1}{2}-\tfrac{1}{2}|k0\rangle\langle j_2j_2'\tfrac{1}{2}-\tfrac{1}{2}|k0\rangle \times W(j_1j_1'j_2j_2';kJ), \quad (29)$$

in which the summation over k must be restricted to

[15] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

[16] H. A. Jahn and J. Hope, Phys. Rev. **93**, 318 (1954).

TABLE IV. Diagonal interaction energies, in Mev.

	$\langle\alpha\|U\|\alpha\rangle$	$\langle\alpha\|UP_M\|\alpha\rangle$	$\langle\alpha\|U\pi_s\|\alpha\rangle$	$\langle\alpha\|U\pi_sP_M\|\alpha\rangle$	$\langle\alpha\|V\|\alpha\rangle$[a]	$\langle\alpha\|V_0\|\alpha\rangle$[b]
$h_{9/2}g_{9/2}$, $J=0$	−2.30	−0.29	0	0	−1.30	−3.47
$h_{9/2}g_{9/2}$, $J=1$	−1.75	−0.052	−0.02	−0.003	−0.90	−1.78
$h_{9/2}i_{11/2}$, $J=1$	−2.73	−0.19	−1.43	−0.28	−1.26	−4.14
$f_{7/2}g_{9/2}$, $J=1$	−2.04	−0.44	−1.25	−0.33	−1.05	−3.28

[a] With Serber exchange mixture, singlet to triplet ratio $p=0.765$, range $\beta^{-1}=1.60$ f, depth $D=60$ Mev, "radial spread" $(2\nu)^{-\frac{1}{2}}=2.0$ f.
[b] Zero-range results.

positive integers for which $k+l_1+l_1'$ and $k+l_2+l_2'$ are both even. Thus the interaction vanishes unless $l_1+l_1'+l_2+l_2'$ is even. The same restrictions apply to the result for the singlet interaction

$$\langle\alpha|U\pi_s|\alpha'\rangle=\tfrac{1}{2}(-)^JAA'\sum_k F_k\langle l_1l_1'00|k0\rangle \times\langle l_2l_2'00|k0\rangle W(l_1l_1'l_2l_2';kJ), \quad (30)$$

with

$$A=(-)^{j_1-\frac{1}{2}}[(2l_1+1)(2l_2+1)(2j_1+1)(2j_2+1)]^{\frac{1}{2}} \times W(l_1j_1l_2j_2;\tfrac{1}{2}J). \quad (30a)$$

The summations over k, in both (29) and (30) can be carried out analytically for zero-range forces when all $F_k=F^0$. The results are the formulas (22) and (19).

To continue the evaluation of (29) and (30) for finite-range forces, one must resort to numerical work.[17] The greatest part of the labor is then the evaluation of the Slater integrals, F_k. We were able to reduce this part of the labor by a large factor when we managed to carry through the analytic integration of the Slater integrals.[18]

Some of the results obtained with the finite-range central forces are shown in Table IV. The most noteworthy fact is that the success we had with the zero-range approximation in explaining the lowering of the $J=1$ RaE state below the $J=0$ state is now lost. The main factor in this loss is the reduction of the general scale of energies through the spreading of the interaction. There is an initial gap of 0.79 Mev between the $J=0$ and $J=1$ states to overcome; the effectiveness of a larger pair-interaction in the $J=1$ over the $J=0$ state depends on the scale of energies involved.

Actually the change from zero-range to finite range central forces has more effect than simply the reduction of energy scale. The "surplus" of attraction in the $J=1$ state over the $J=0$ state, which we found in the zero-range approximation, is here substantially reduced. This comes from the "misfit" of the nodeless and one-node radial functions in the $J=0$ state and the misfit is less serious to a "nonlocal", finite-range attraction. Further, only a small part of the initial gap was overcome with the help of configuration mixing in the zero-range case (0.16 Mev of the total of 0.79+0.05 Mev). Now we must expect the configuration mixing to have still less effect for reasons already mentioned (see reference 12).

These results bring one to either of two conclusions. It may be that the interaction of "extra-core" nucleons should not be expected to imitate their interaction when alone ("vacuum forces"). The fact that we found that an attraction of the same volume-energy as the vacuum forces worked in the zero-range approximation may be entirely accidental; the further imitation in range as well as strength should perhaps not be expected. Alternatively, it may be that we do not yet imitate the vacuum forces sufficiently well. The latter have another well-known property, namely at least a partial tensor character. Tensor forces are usually ignored in treating the internucleonic interactions in complex nuclei, partly because of the labor of evaluating them, and partly because it is hoped that in states centered on the nucleus, rather than on the relative motion of the particles, their effect can be considered to be smeared out sufficiently so that they can be replaced by central forces. It may be that RaE is a case in which this is a bad assumption.

The success of the zero-range approach in RaE may be attributable to the fact that tensor force effects disappear in that limit; they may then be replaceable by a δ-function potential of total strength sufficient to represent the full volume-energy of the total interaction. On the other hand extending the range of central forces weakens their effect, on our findings, and the reverse is true of tensor forces. We therefore made an exploration of tensor effects.

TENSOR FORCE EFFECTS

The formal work of evaluating the tensor potential operator (6) into spherical tensors has been done by Talmi[19] and by Hope[20] and Longdon.[21] Applying (26), one then gets

$$\langle\alpha|U_TS_{12}|\alpha'\rangle =\sum_{LL'}({}^3L_J)({}^3L_J')\langle\beta LS\|U_TS_{12}\|\beta'L'S'\rangle, \quad (31)$$

[17] We originally developed a method by which the summations over k in (29) and (30) could be performed analytically also for finite range forces. This amounted to a development of the interaction in powers of the squared ratio of force range to nuclear radius. In view of the success of the zero-range approximation this should have worked out very well. Actually we could not get convergence of the development for realistic values of the range. This also seems to have been the experience of Brink[14] with a similar development; he completes his evaluation only for much smaller force ranges than he himself initially suggests as most reasonable. There seems to be little advantage in taking an artificially small range over working with the zero-range limit.

[18] K. W. Ford and E. J. Konopinski, Nuclear Phys. **9**, 218 (1959).

[19] I. Talmi, Phys. Rev. **89**, 1065 (1953).

[20] J. Hope, Phys. Rev. **89**, 884 (1953).

[21] L. W. Longdon, Phys. Rev. **90**, 1125 (1953).

where

$$ {}^3L_J \equiv [3(2L+1)(2j_1+1)(2j_2+1)]^{\frac{1}{2}} \begin{Bmatrix} l_1 & l_2 & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j_1 & j_2 & J \end{Bmatrix}, \quad (32) $$

represents the triplet-L fraction of the state, and

$$ \langle \beta LS \| U_T S_{12} \| \beta' L' S' \rangle \equiv 2(5)^{\frac{1}{2}} (-)^{J+L+1} W(LL'11;2J) \times \sum_{kij} \langle \beta | u_k r_i r_j | \beta' \rangle \langle L \| \mathcal{L}_2(ij) \| L' \rangle. \quad (33) $$

Here $i,j=1,2$ refer to the interacting particles and

$$ \langle \beta | u_k r_i r_j | \beta' \rangle = (2k+1) \int_0^\infty dr_1 r_1^2 R_1 R_1' \times \int_0^\infty dr_2 r_2^2 R_2 R_2' r_i r_j \int_{-1}^{1} (d\mu/2) P_k U_T / r^2 \quad (34) $$

stands for the radial integral. The expressions for $\langle L \| \mathcal{L}_2(ij) \| L' \rangle$ are, respectively,

$$ \langle L \| \mathcal{L}_2(11) \| L' \rangle = [\tfrac{2}{3}(2L+1)(2L'+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}} \times \sum_x (-1)^x (2x+1)^{\frac{1}{2}} \langle k200 | x0 \rangle \langle xl_1'00 | l_10 \rangle \langle kl_2'00 | l_20 \rangle \begin{Bmatrix} l_1' & l_2' & L' \\ x & k & 2 \\ l_1 & l_2 & L \end{Bmatrix}, \quad (35a) $$

$$ \langle L \| \mathcal{L}_2(22) \| L' \rangle = [\tfrac{2}{3}(2L+1)(2L'+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}} \times \sum_x (-)^x (2x+1)^{\frac{1}{2}} \langle k200 | x0 \rangle \langle kl_1'00 | l_10 \rangle \langle xl_2'00 | l_20 \rangle \begin{Bmatrix} l_1' & l_2' & L' \\ k & x & 2 \\ l_1 & l_2 & L \end{Bmatrix}, \quad (35b) $$

$$ \langle L \| \mathcal{L}_2(12) \| L' \rangle = [5(2L+1)(2L'+1)(2l_1'+1)(2l_2'+1)]^{\frac{1}{2}} \times \sum_{xy} (-)^y [(2x+1)(2y+1)]^{\frac{1}{2}} \langle k100 | x0 \rangle \langle k100 | y0 \rangle W(11xy;2k) \times \langle xl_1'00 | l_10 \rangle \langle yl_2'00 | l_20 \rangle \begin{Bmatrix} l_1' & l_2' & L' \\ x & y & 2 \\ l_1 & l_2 & L \end{Bmatrix}. \quad (35c) $$

As expected, the entire expression (31) vanishes for a zero-range potential, U_T, when the radial integral (34) becomes proportional to $(2k+1)$ as its sole k-dependence.

We undertook some evaluation of these formidable expressions only for the two states of primary interest to us: $(h_{9/2}g_{9/2})_0$ and $(h_{9/2}i_{11/2})_1$. The $J=0$ state is pure ${}^3P_0$ while

$$ |h_{9/2}i_{11/2}; J=1\rangle = (5/11)^{\frac{1}{2}}({}^1P_1) + (5/22)^{\frac{1}{2}}({}^3P_1) + (7/22)^{\frac{1}{2}}({}^3D_1). \quad (36) $$

It is the numerical coefficients in this expression which are given by (32).[22]

Initially, we further simplified the work by neglecting all terms except those with $k=0$. This is the more valid the longer the potential range. At the opposite extreme of zero-range (all k-values given equal weight in terms of Slater integrals), the tensor effects vanish, hence we expect to at least get the right sign of the interaction by our procedure. Exploration of a similar procedure for the simpler central forces shows that the errors introduced are well within our uncertainty as to the proper strength and range to adopt for the tensor potential, U_T. It is known that the range of the tensor force in the deuteron must be substantially longer than the central force range.

The result for the $(h_{9/2}g_{9/2})_0 \equiv {}^3P_0$ state is

$$ \langle \alpha | U_T S_{12} | \alpha \rangle = -(4/9)\langle \beta | u_0 r_1^2 | \beta \rangle - (8/33)\langle \beta | u_0 r_2^2 | \beta \rangle, \quad (37) $$

where $\langle \beta | u_0 r_i^2 | \beta \rangle$ is defined by (34). This is a positive-definite repulsion when the tensor force has the same sign as in the deuteron ($U_T<0$). On the other hand, we found attraction in the $(h_{9/2}i_{11/2})_1$ state:

$$ \langle \alpha' | U_T S_{12} | \alpha' \rangle = +(4/11)\langle \beta' | u_0 r_1^2 | \beta' \rangle + (42/121)\langle \beta' | u_0 r_2^2 | \beta' \rangle. \quad (38) $$

The contributing components of the state, $({}^3P_1)^2$, $({}^3D_1)^2$ and $({}^3P_1)$, $({}^2D_1)$, each give attraction separately, so there is no question of a balancing which could change with the range adopted. Thus, without making the detailed assumptions required for a complete evaluation we already find that the tensor force helps to push the $J=1$ state of Bi^{210} below the $J=0$ state.

The important result here is the finding of opposite signs for the tensor interaction in the $h_{9/2}g_{9/2}$ and $h_{9/2}i_{11/2}$ configurations. That was confirmed by an extension of the investigations from $k=0$, only, to all values of the orbital momentum transfer, k. Almost all the 9-j symbol values necessary for this were kindly supplied us by Kenneth Smith. The effects of remaining ones were estimated by a process of extrapolation. It was also necessary to develop new procedures for

[22] We found a simpler way to evaluate the 9-j symbols for the triplet fractions than those known to us from the literature (e.g., Edmonds 6.4.17). We reduced the evaluation to as few vector-addition coefficients as there are Racah coefficients in the usual methods. We could show that

$$ \langle l_1 l_2 00 | L0 \rangle \begin{Bmatrix} l_1 & l_2 & L \\ \frac{1}{2} & \frac{1}{2} & S \\ j_1 & j_2 & J \end{Bmatrix} = \frac{(-)^{S+l_1+\frac{1}{2}-i_1}}{[(2J+1)(2S+1)]^{\frac{1}{2}}} \cdot \frac{\langle j_1 j_2 \frac{1}{2} -\frac{1}{2} | J0 \rangle}{[2(2l_1+1)(2l_2+1)]^{\frac{1}{2}}} \times \{ \langle JS00 | L0 \rangle + \langle JS-1+1 | L0 \rangle u (-)^{J+l_2+\frac{1}{2}-i_1} / [2J(J+1)]^{\frac{1}{2}} \}, $$

where u is given by (23).

TABLE V. Coulomb interaction.

J	$\langle h_{9/2}h_{9/2}\|e^2/r_{12}\|h_{9/2}h_{9/2}\rangle$ (Mev)
0	+0.341
2	+0.293
4	+0.263
6	+0.249
8	+0.252

evaluating Slater integrals, because the occurrence of singularities made our method for the central forces inapplicable. Space will not be taken to present the large volume of these results in further detail. This is partially because approximations we tried for the Slater integrals, though they appeared adequate for determining the signs of the interactions, have a large range of uncertainty in absolute magnitude. The decisive reason, however, is that we could find no satisfactory way to represent the specific relative strengths of the tensor and central forces as required by the free-nucleon data. There has been a remarkable avoidance of this question in almost all treatments[23] of the tensor force in the literature. This is presumably because the low-energy data, though requiring existence of the tensor forces, nevertheless is inadequate for determining a specific choice out of a large range of values.

Our conclusion from these investigations is the following. One can represent the full volume-energy of the interactions of free nucleons either by a finite-range or a zero-range central force. As True and Ford found, there is a *small* improvement in extending the range from zero to a finite value, for singlet forces. There is no tensor interaction to consider for singlet states. Similarly, no tensor force operates at zero range. However, when extending the interaction range in triplet states, it makes a critical difference whether a central force character is retained or whether part of the volume energy is put into a now operative tensor potential. The latter can operate as an attraction or a repulsion, depending on the particular state. We found the signs to be in the right direction to confirm our results in the zero range approximation, although the formidable difficulties of the problem prevent quantitative use of the finite range central plus tensor force picture. We can only rely on True and Ford's finding that the finite-range forces yielded only minor changes from the zero-range representations in singlet states, to assume that our zero-range results for triplet states have a comparable validity.

RESULTS FOR Po^{210}

Much of our discussion for Bi^{210} carries over to the case of Po^{210}. Now we are dealing with the Pb^{208} core plus two protons. The experimental energy level scheme as reported by Hoff and Hollander is shown on the right in Fig. 1.

Antisymmetric wave functions were constructed from the zero-order states given in Table I. The internucleon potential is given by (5) and (2) with $\beta^{-1}=1.6$ f as previously. Configuration mixing was computed for all levels arising from the $h_{9/2}^2$ configuration. The levels having $J=2$, 4, 6, were computed by diagonalizing the corresponding 4×4 matrices involving the $h_{9/2}^2$, $h_{9/2}f_{7/2}$, $f_{7/2}^2$ and $i_{13/2}^2$ configurations. The $J=0$ level involved diagonalizing in the space spanned by the $h_{9/2}^2$, $f_{7/2}^2$ and $i_{13/2}^2$ configurations.

The diagonal matrix elements of the Coulomb interaction were calculated exactly for the $h_{9/2}^2$ states. The results are given in Table V.

As expected the variation with J is small. The results of calculation are compared with experiment in Fig. 2. For purposes of clarity Fig. 2 contains only the Hoff-Hollander levels which were given spin and parity assignments by them. The right-hand column contains the Hoff-Hollander experimental level scheme. Directly to the left of this is our best fit including configuration mixing and Coulomb interaction (diagonal only in $(h_{9/2}^2)_J$ states). The Coulomb interaction was not computed for the 4− and 5− single-particle states arising from the $h_{9/2}i_{13/2}$ configuration. The Coulomb interaction for these states was set at +250 kev, on the basis of exact calculations for states of similar spin arising from the $h_{9/2}^2$ configuration (see Table V).

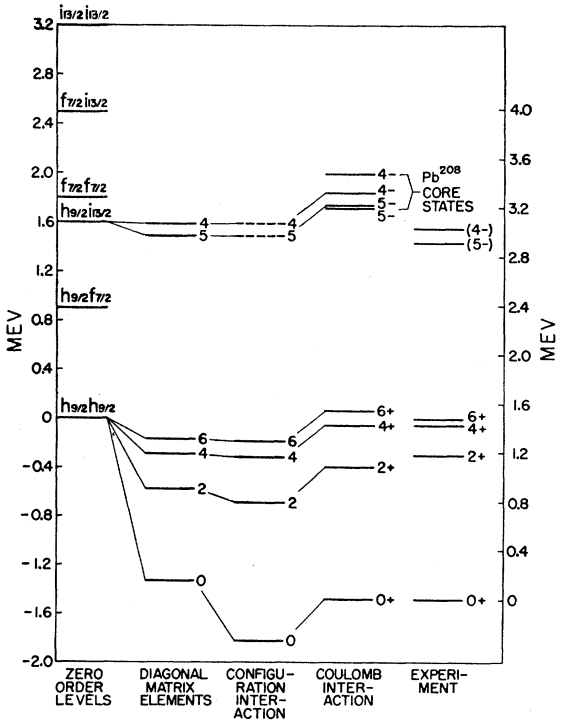

FIG. 2. Comparison of theory and experiment in Po^{210}. The coulomb interaction was calculated exactly only for the states arising from the $h_{9/2}^2$ configuration. See also Tables V, VI, and VII.

[23] The use of repulsive cores in their potentials, by Gammel and Thaler, makes a translation into the customary representations without cores quite ambiguous on this point.

The calculated levels given in Fig. 2 are those which we believe correspond to the experimental levels. To keep Fig. 2 uncluttered we list *all* our calculated levels in Table VI. Table VII gives the eigenfunctions for those states in which mixing was taken into account. Our best fit as shown in Fig. 2 was obtained for a well-depth, $D=-57$ Mev. The use of a Serber mixture with totally antisymmetric wave functions implies interaction in singlet states only. Thus the singlet well-depth is

$$-D_s=-pD=-0.765\times 57=-43.6 \text{ Mev}. \quad (39)$$

This "best fit" well-depth along with the interaction range used ($\beta^{-\frac{1}{2}}=1.6$ f) combine to give an interaction with the same volume energy as the "correct" gaussian singlet interaction ((2) with $D=33.4$ Mev and $\beta^{-\frac{1}{2}}=1.75$ f). This is interesting in view of the fact that the fit obtained is quite sensitive to well-depth—a change of 1 or 2 Mev either way producing a noticeably poorer fit.

The calculated ground-state wave function for Po^{210} is

$$\psi=0.943\,|\,h_{9/2}h_{9/2},\ J=0\rangle+0.101\,|\,f_{7/2}f_{7/2},\ J=0\rangle -0.317\,|\,i_{13/2}i_{13/2},\ J=0\rangle. \quad (40)$$

From mass spectroscopic data the pairing energy of the two extra-core protons is 1503 ± 120 kev (corrected for Coulomb interaction with the core). The calculated value of 1485 kev seems to be in good agreement with this. The Coulomb repulsion plays an essential role in this agreement.

TABLE VI. Calculated levels in Po^{210}.[a]

Configuration	J	Energy (kev)
$h_{9/2}^2$	0	0[b]
	2	1085[c]
	4	1430[c]
	6	1545[c]
	8	1636
$h_{9/2}f_{7/2}$	1	2702
	2	2634[c]
	3	2633
	4	2557[c]
	5	2597
	6	2450[c]
	7	2573
	8	2030
$h_{9/2}i_{13/2}$	2	3367
	3	3308
	4	3337
	5	3229
	6	3326
	7	3145
	8	3329
	9	2988
	10	3329
	11	Not calculated
$f_{7/2}^2$	0	1806[b]
	2	2983[c]
	4	3305[c]
	6	3390[c]
$f_{7/2}i_{13/2}$	3–10	Not calculated
$i_{13/2}^2$	0	4478[b]
	2	4481[c]
	4	4646[c]
	6	4731[c]
	8, 10, 12	Not calculated

[a] See Table V for Coulomb effects for $h_{9/2}^2$. Coulomb repulsions for other states were set equal to the repulsion in the $h_{9/2}^2$ state having the same J.
[b] Mixing of $h_{9/2}^2$, $f_{7/2}^2$, and $i_{13/2}^2$.
[c] Mixing of $h_{9/2}^2$, $h_{9/2}f_{7/2}$, $f_{7/2}^2$, and $i_{13/2}^2$.

TABLE VII. Calculated eigenfunctions in Po^{210}.

Energy (kev)	Eigenfunctions $h_{9/2}^2$	$f_{7/2}^2$	$i_{13/2}^2$	$h_{9/2}f_{7/2}$
$J=0$				
0	0.9432	0.1008	0.3167	
1806	0.0390	0.9127	0.4068	
4478	0.3301	−0.3960	0.8569	
$J=2$				
1085	0.9821	0.0417	0.0542	0.1755
2634	−0.0478	0.9958	0.0773	0.0067
2983	−0.0068	−0.0737	0.9672	0.2431
4481	0.1820	0.0334	0.2360	0.9540
$J=4$				
1430	0.9927	0.0690	0.0373	0.0919
2557	−0.0738	0.9960	0.0391	0.0330
3305	−0.0233	−0.0366	0.9914	0.1231
4646	0.0928	0.0442	0.1188	0.9876
$J=6$				
1545	0.9935	0.0945	0.0250	0.0590
2450	−0.0979	0.9936	0.0390	0.0410
3390	−0.0174	−0.0380	0.9968	0.0676
4731	0.0560	0.0491	0.0646	0.9951

In the present work no account has been taken of the 4− and 5− states arising from the Pb^{208} core (see Fig. 2). Mixing of these states with the corresponding states from the $(h_{9/2}i_{13/2})$ and $(f_{7/2}i_{13/2})$ configurations will tend to bring the lowest lying 4− and 5− states into closer agreement with the experimentally observed 4− and 5− states.

EVALUATION OF ξ

The best characterizations of the initial and final nuclear states which we have been able to attain are given by (25) and (40), respectively. The point of primary significance about the results is the evidence they provide that it is the $(h_{9/2}i_{11/2})$ configuration which predominates in the initial state. That is enough to show that the spectrum parameter ξ of (1), is of order $+1$ rather than $-1/10$ of the alternatives anticipated in the introduction.

We can also see how much the configuration mixing may change ξ from $+1$. If we express the initial and final nuclear states as

$$\psi_i(J=1)=a\,|\,h_{9/2}i_{11/2}\rangle+b\,|\,h_{9/2}g_{9/2}\rangle+c\,|\,f_{7/2}g_{9/2}\rangle, \quad (41a)$$

$$\psi_f(J=0)=a'\,|\,h_{9/2}h_{9/2}\rangle+b'\,|\,f_{7/2}f_{7/2}\rangle+c'\,|\,i_{13/2}i_{13/2}\rangle, \quad (41b)$$

we may write

$$\xi_{\text{mixed}}=\frac{1+(b\langle\mathbf{r}\rangle'/a\langle\mathbf{r}\rangle)+(cb'\langle\mathbf{r}\rangle''/aa'\langle\mathbf{r}\rangle)}{\xi^{-1}+(b\langle\mathbf{r}\rangle'/\xi'a\langle\mathbf{r}\rangle)+(cb'\langle\mathbf{r}\rangle''/\xi''aa'\langle\mathbf{r}\rangle)}, \quad (42)$$

where we have written

$$\langle\mathbf{r}\rangle=\langle h_{9/2}h_{9/2},\ J=0\,|\,\mathbf{r}\,|\,h_{9/2}i_{11/2},\ J=1\rangle, \quad (43a)$$

$$\langle\mathbf{r}\rangle'=\langle h_{9/2}h_{9/2},\ J=0\,|\,\mathbf{r}\,|\,h_{9/2}g_{9/2},\ J=1\rangle, \quad (43b)$$

$$\langle\mathbf{r}\rangle''=\langle f_{7/2}f_{7/2},\ J=0\,|\,\mathbf{r}\,|\,f_{7/2}g_{9/2},\ J=1\rangle. \quad (43c)$$

ξ is defined in (1) and ξ' and ξ'' have obvious meanings. Calculation yields

$$\xi_{\text{mixed}} = \frac{1+(b/a)[-(39)^{\frac{1}{2}}/117]+(cb'/aa')[11(390)^{\frac{1}{2}}/234]}{1+(b/a)[10(39)^{\frac{1}{2}}/117]+(cb'/aa')[-11(390)^{\frac{1}{2}}/234]}. \quad (44)$$

By using the amplitudes given by (25) and (40), we find

$$\xi_{\text{mixed}} = +0.99. \quad (45)$$

If one takes account of mixing in the initial state only ($b'=c'=0$) one finds $\xi=+0.92$, whereas taking account of both initial and final state mixing yields $\xi=+0.99$. Thus, in spite of the fairly substantial effect of the configuration mixing on the position of the energy levels of initial and final states, the value of ξ remains virtually unchanged.

GENERAL CONCLUSIONS

The initial objective of the work was to use the evidence about the $J=0$ and $J=1$ energy level positions in Bi^{210} to decide whether the transforming nucleon is better characterized as $g_{9/2}$ or $i_{11/2}$. We conclude that it is $i_{11/2}$ despite the fact that the $g_{9/2}$ level of Pb^{209} is the lowest. This was the critical point in determining the spectrum parameter ξ, as of order $+1$, rather than the very disparate order $-1/10$.

The above conclusions required an explanation of why the attraction in the $(h_{9/2}i_{11/2})_1$ state should be a large amount greater than in the $(h_{9/2}g_{9/2})_0$ state. This problem turns out to serve as a severe test of the True-Ford finding that extra-core nucleons interact about as they do in vacuo.

In Po^{210}, the True-Ford finding was confirmed for proton pairs, which interact essentially only through singlet central forces. The extension to neutron-proton pairs in Bi^{210} brought triplet, exchange and tensor effects into play. We found that the finite-range central forces alone could not account for the large surplus of attraction in the $J=1$ over the $J=0$ state. No appreciable help was afforded by adjustments of strength, range, triplet-to-singlet ratio or exchange mixture within all plausible ranges. It was necessary to imitate the vacuum forces even in the detail that they have a partial noncentral tensor character. The outcome made this fact unsurprising, since it was found that the tensor forces actually changed sign in going from the $J=0$ to $J=1$ state. We could show that the sign change was in the right direction. A quantitative check was prevented not so much by the complexity of the application, as because there seems never to have been any unambiguous determination of the tensor force strength for nucleon pairs in vacuo using potentials without cores.

Finally, we have found that a promising method of representing the vacuum forces is through the use of a zero-range approximation in which volume energies are maintained. Tensor and exchange effects then disappear. The method yields relative positions for the $J=0$ and $J=1$ states of Bi^{210} which are in almost precise agreement with those observed, if configuration mixing is taken into account.

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