Spin-Orbit Contributions to the H^3-He^3 Magnetic Moments

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The contributions to the magnetic moments of the triton and He^3 from the Signell-Marshak and Gammel-Thaler phenomenological spin-orbit potentials have been calculated using the Pease and Feshbach wave functions. The results indicate that the isotopic spin dependence of the spin-orbit potential should be of the form $(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$ and also that the spin-orbit potential contributions are too small by an order of magnitude to account for the approximately 0.2 nm anomalies in the H^3-He^3 magnetic moments.

I. INTRODUCTION

THE success of the Signell-Marshak[1] and Gammel-Thaler[2] phenomenological potentials in producing theoretical agreement with the nucleon-nucleon scattering data, has led to application of these potentials to various other nuclear problems. The potentials are still, however, somewhat ambiguous, in particular, with respect to the isotopic spin dependence of their spin-orbit parts. Feshbach[3] has shown that both the Signell-Marshak and Gammel-Thaler spin-orbit (S.O.) potential terms in their original forms give rise to too large a contribution to the magnetic moment of the deuteron, and Sessler and Foley[4] have pointed out a similar difficulty in the hfs of deuterium. However, de Swart, Marshak, and Signell[5] have shown that the scattering-data agreement is relatively insensitive to the choice of S.O. interaction in the singlet isotopic spin state and therefore adjustment of this potential can be made so as to yield no magnetic moment contribution in the deuteron. They have proposed that the tail of their S.O. potential be reversed in the singlet isotopic spin state at a radial distance such that the integrated contribution to the deuteron magnetic moment is zero. Another procedure which maintains the agreement with the scattering data[6] is to set the S.O. potential equal to zero in this state.

A calculation of the S.O. potential contribution to the magnetic moments of H^3 and He^3 can be expected to help resolve the question of the nature of the isotopic spin dependence, if, as in the deuteron, the contribution from a non-isotopic-spin-dependent potential in the singlet-isotopic-spin-state part of the ground state is again too large. Then the difference between the radial dependences of the deuteron and H^3-He^3 wave functions would make it unreasonable to expect that the same choice made by de Swart, Marshak, and Signell[5] for the deuteron would cause the magnetic moment contribution to vanish in all three nuclei. Thus, one might conclude that the isotopic spin dependence is a multiplicative term in the potential.

In addition, there will be a magnetic moment contribution from the triplet-isotopic-spin-state parts of the H^3 and He^3 wave functions which must remain. It would be of considerable interest to determine whether these contributions have the character and magnitude to account for the deviations of the H^3 and He^3 magnetic moments from the Schmidt limits (or odd-nucleon magnetic moment value). These deviations are approximately 0.2 nuclear magneton in H^3 and -0.2 nm in He^3. Actually, Sachs and Schwinger[7] have shown that if the ground state has about $4\% D$ state then the magnetic moment deviations to be accounted for are increased to approximately 0.25 nm in H^3 and -0.25 nm in He^3.

The usual explanation for these magnetic moment deviations has been that they are due to magnetic moment contributions arising from meson exchange currents.[8] The possible forms and magnitudes of these exchange moments has been derived phenomenologically[9] and it has been shown that they can give rise to fairly large primarily isotropic contributions to the deuteron photodisintegration cross section.[10] Such a contribution would not be consistent with the calculations of deuteron photodisintegration by de Swart and Marshak[11] based on the Signell-Marshak potential. Their calculation accounts for the large experimentally observed isotropy in the differential cross section without including any exchange moment effects. Thus a calculation of the H^3 and He^3 moments indirectly tests the consistency of these phenomenological potentials.

Therefore, calculations of the phenomenological S.O. potentials contributions to the magnetic moments of H^3 and He^3 nuclei have been made using the Signell-Marshak potentials and the Gammel-Thaler potential

[1] P. S. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958).
[2] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).
[3] H. Feshbach, Phys. Rev. **107**, 1626 (1957).
[4] A. M. Sessler and H. M. Foley, Phys. Rev. **110**, 995 (1958).
[5] de Swart, Marshak, and Signell, Nuovo cimento **6**, 1189 (1957).
[6] R. E. Marshak (private communication).
[7] R. G. Sachs and J. Schwinger, Phys. Rev. **70**, 41 (1946).
[8] See J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 252, for references.
[9] J. M. Berger and L. L. Foldy, Technical Report No. 18 of the Nuclear Physics Laboratory, Case Institute of Technology, Cleveland, 1952 (unpublished); also R. K. Osborn and L. L. Foldy, Phys. Rev. **79**, 795 (1950); also R. G. Sachs, Phys. Rev. **74**, 433 (1948).
[10] J. M. Berger, Phys. Rev. **94**, 1698 (1954).
[11] J. J. de Swart and R. E. Marshak, Phys. Rev. **111**, 272 (1958).

with the Pease and Feshbach[12] H^3-He^3 wave functions. The calculations were performed for both a non-isotopic-spin-dependent S.O. potential and for the isotopic spin dependence of $\frac{1}{4}(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$ which vanishes in the singlet state. These two forms exhaust the choices consistent with charge independence since any other suitable form may be obtained by a linear combination of these two. Various other calculations were performed to examine the effects of altering the potentials by either inverting the tail or changing the range, and also of altering the wave function. All of these additional calculations were done only with the Signell-Marshak potential.

In the following section a description of the calculations is given and in the final section the results are presented and discussed. In the Appendix are given the detailed integrals which were numerically evaluated.

II. CALCULATION

The spin-orbit potentials, V_{so}, used in the calculations are of the form

$$V_{so}=V(x)\tfrac{1}{2}(\mathbf{L}\cdot\mathbf{S}T+T\mathbf{L}\cdot\mathbf{S}),\quad x\geqslant x_0 \qquad (1)$$
$$=0,\quad x<x_0$$

where

$$\mathbf{L}\cdot\mathbf{S}=\tfrac{1}{2}(\mathbf{r}_i-\mathbf{r}_j)\times(\mathbf{p}_i-\mathbf{p}_j)\cdot(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j),$$

and $\mathbf{r}_i$, $\mathbf{p}_i$, and $\boldsymbol{\sigma}_i$ are the position vector, momentum vector, and ordinary spin operator, respectively, referring to the ith nucleon, and T is the isotopic spin operator[13] which takes either of the two forms consistent with charge independence:

$$T^1{}_{ij}=\tfrac{1}{4}(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j),$$

or

$$T_{ij}{}^2=1.$$

The resultant magnetic moment operator for the three-nucleon system is

$$M^k=\sum_{i>j=1}^{3}\frac{e}{4c}V(x)\tfrac{1}{2}\{[(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)\cdot(\tau_i{}^p\mathbf{r}_i-\tau_j{}^p\mathbf{r}_j)(z_i-z_j)$$
$$-(\mathbf{r}_i-\mathbf{r}_j)\cdot(\tau_i{}^p\mathbf{r}_i-\tau_j{}^p\mathbf{r}_j)(\sigma_{iz}-\sigma_{jz})]T_{ij}{}^k$$
$$+\text{Hermitian conjugate}\},\quad k=1,2,\qquad (2)$$

where the static magnetic field is in the z direction, e is the charge of the proton, c is the speed of light, and τ^p is the proton projection operator.

The primary calculations were done for $V(x)$ given by the following:

(a) the latest[6] Signell-Marshak potential, where

$$V(x)=\frac{V_0}{x}\frac{d}{dx}\left(\frac{\exp(-ax)}{x}\right),\qquad (3a)$$

and $x=r_{ij}/r_c$, $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$, $a=2$, r_c is the π-meson Compton wavelength, $V_0=21$ Mev, and $x_0=0.37$.

(b) the Gammel-Thaler potential, where

$$V(x)=V_0e^{-x}/x,\qquad (3b)$$

and $x=\gamma r_{ij}$, $\gamma=3.7\times10^{+13}\ \text{cm}^{-1}$, $x_0=1.48$, and $V_0=7300$ Mev.

The wave function for the three-nucleon system that was used in the calculations was that obtained by Pease and Feshbach using a variational method for a noncentral Yukawa potential. This wave function gave essentially exact agreement with the triton binding energy, but a discrepancy of 25% in the Coulomb energy in He^3. The calculations were actually performed with the fully antisymmetrized wave function, which for the triton may be written as

$$\Psi_{H^3}=\psi_{12}\eta_1\eta_2\xi_3+\psi_{23}\eta_2\eta_3\xi_1+\psi_{31}\eta_3\eta_1\xi_2,\qquad (4)$$

where ψ_{12}, following the notation of Pease and Feshbach, is given by

$$\psi_{12}=A_{S1}\psi_{S1}+A_{S2}\psi_{S2}+[A_D+\rho A_{D^+}+(r_1+r_2)A_{D^0}]\psi_D$$
$$+[A_{D'}+\rho A_{D'^+}+(r_1+r_2)A_{D'^0}]\psi_{D'}+A_{D''}\psi_{D''},\qquad (5)$$

where

$$\psi_{Si}=\chi_S\exp[-\tfrac{1}{2}\lambda_i(r_1+r_2+\rho)],\quad i=1,2$$
$$\psi_D=\chi_D\exp[-\tfrac{1}{2}\mu(r_1+r_2+\rho)],$$
$$\psi_{D'}=\chi_{D'}(r_1-r_2)\exp[-\tfrac{1}{2}\nu(r_1+r_2+\rho)],$$
$$\psi_{D''}=\chi_{D''}(r_1-r_2)\exp[-\tfrac{1}{2}\omega(r_1+r_2+\rho)],$$

and

$$\chi_S=6^{-\frac{1}{2}}[\alpha(1)\beta(2)-\beta(1)\alpha(2)]\alpha(3),$$
$$\chi_D=[r_1{}^2S_{13}+r_2{}^2S_{23}]\chi_S,$$
$$\chi_{D'}=[r_1{}^2S_{13}-r_2{}^2S_{23}]\chi_S,$$
$$\chi_{D''}=[3(\boldsymbol{\sigma}_1\cdot\mathbf{r}_1\times\mathbf{r}_2)(\boldsymbol{\sigma}_3\cdot\mathbf{r}_1\times\mathbf{r}_2)-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)(\mathbf{r}_1\times\mathbf{r}_2)\cdot(\mathbf{r}_1\times\mathbf{r}_2)]\chi_S,$$

and

$$S_{ij}=r_i{}^{-2}[3(\boldsymbol{\sigma}_i\cdot\mathbf{r}_i)(\boldsymbol{\sigma}_j\cdot\mathbf{r}_i)]-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j,$$

where η_i is the neutron state isotopic spin wave function for the ith particle, ξ is the proton state, $\alpha(i)$ and $\beta(i)$ are the spin wave functions of particle i, and $\mathbf{r}_1$ is the vector distance from particle 3 to particle 1, $\mathbf{r}_2$ is the vector distance from particle 3 to particle 2, $\boldsymbol{\rho}$ is the vector distance from particle 1 to particle 2, and the unit of distance is $r_c=1.184\times10^{-13}$ cm. The magnetic moment operator must be rewritten in this coordinate system, but it was more convenient to keep the two coordinate systems until after the isotopic spin and spin operations had been performed.

The values of the parameters used in the calculations are:

$\lambda_1=0.9,$	$A_{S1}=1.08,$	$A_{D^+}=0.45,$
$\lambda_2=1.8,$	$A_{S2}=3.01,$	$A_{D'^+}=-0.136,$
$\mu=2.2,$	$A_D=1.0,$	$A_{D^0}=-0.27,$
$\nu=2.5,$	$A_{D'}=-1.29,$	$A_{D'^0}=0.20.$
$\omega=2.8,$	$A_{D''}=0.91,$	

[12] R. L. Pease and H. Feshbach, Phys. Rev. 88, 945 (1952).
[13] Blanchard, Avery, and Sachs, Phys. Rev. 78, 292 (1950).

In calculating the expectation value of M^k with the totally antisymmetrized ground state wave function, it is necessary to consider only one of the three terms in M corresponding to any single pair of the particles 1, 2, and 3. The performance of the isotopic spin and spin operations yields the following interesting results. The expectation value of M^1 with the isotopic spin dependence $T_{ij}{}^1=\frac{1}{4}(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$, has the following properties:

$$\langle\psi_S|M^1|\psi_S\rangle_{\mathrm{H}^3}=\langle\psi_S|M^1|\psi_S\rangle_{\mathrm{He}^3}=0, \tag{6a}$$

$$\langle\psi_S|M^1|\psi_D\rangle_{\mathrm{He}^3}=-\langle\psi_S|M^1|\psi_D\rangle_{\mathrm{H}^3}, \tag{6b}$$

where both (6a) and (6b) hold for any of the S-state parts of the total wave function, and (6b) is also true with ψ_D replaced by $\psi_{D'}$ or $\psi_{D''}$.

Another somewhat surprising result was that the terms in M^1 corresponding to $T\mathbf{L}\cdot\mathbf{S}$ in Eq. (1) do not contribute at all to the matrix elements indicated in (6a) and (6b). The symmetrized form was used in order to have a Hermitian magnetic moment operator in isotopic spin space. That this is required is readily seen from Eq. (2) where the τ^p, which arises in $\mathbf{L}$ through the replacement of $\mathbf{p}$ by $\mathbf{p}-\tau^p(e\mathbf{A}/c)$, does not commute with T^1. The matrix elements of M^1 between the strictly D-state parts of the total wave function were not calculated because of the small percentage of D state. Their contributions would be expected to be small compared to those arising from the S- and D-state parts.

In the case of the expectation value of M^2, the purely S-state terms do not vanish, and these contributions are identical for H^3 and He^3. That is

$$\langle\psi_S|M^2|\psi_S\rangle_{\mathrm{H}^3}=\langle\psi_S|M^2|\psi_S\rangle_{\mathrm{He}^3}\neq 0 \tag{7}$$

for any combination of ψ_{S1} an ψ_{S2}. In this case no further terms were calculated since the S-state terms will clearly dominate.

The six integrations over the center-of-mass coordinates and the three angular variables may be done quite simply, yielding the seven triple integrals for M^1 and the two triple integrals for M^2 that are given in Appendix I. These integrals still exhibit the explicit exponentials appearing in the wave functions. Also given in Appendix I are the complete normalization integrals after reduction to triple integrals. The triple integrals can, in principle, be evaluated analytically; however, the algebra involved is prohibitive. Therefore, all of the integrals were evaluated numerically on the IBM 704. Since high accuracy was not desired, the integrals were evaluated to within one percent using the trapezoidal rule over 50 points in each dimension, with an interval $\Delta=0.3$. The accuracy was checked by comparison to the analytic evaluation of the S-state normalization integrals and by repeating parts of the calculation using a smaller Δ, and more points. The results of the computations, expressed in nuclear magnetons, are: (a) for the Signell-Marshak potential given in (3a),

$$\begin{aligned} M^1{}_{\mathrm{H}^3}&=-M^1{}_{\mathrm{He}^3}=0.008\text{ nm}\pm10\%,\\ M^2{}_{\mathrm{H}^3}&=M^2{}_{\mathrm{He}^3}=1.11\text{ nm}\pm10\%; \end{aligned} \tag{8a}$$

and (b) for the Gammel-Thaler potential given in (3b),

$$\begin{aligned} M^1{}_{\mathrm{H}^3}&=-M^1{}_{\mathrm{He}^3}=0.004\text{ nm}\pm10\%,\\ M^2{}_{\mathrm{He}^3}&=M^2{}_{\mathrm{H}^3}=1.35\text{ nm}\pm10\%. \end{aligned} \tag{8b}$$

III. DISCUSSION AND RESULTS

The results of the calculations described in Sec. II together with the results of supplementary calculations are summarized in Table I. The supplementary calculations were undertaken in order to determine the degree of reliability of the conclusions drawn from cases 1 and 2, and are discussed below.

1. Cases 1 and 2.—These calculations of the S.M. and G.T. spin-orbit potential magnetic moment contributions are those described in detail above. From the results for M^2 we conclude that the isotopic spin dependence T of the spin-orbit potential must be essentially $T=\frac{1}{4}(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$ since the M^2 results are two orders of magnitude too large. The alternative to this conclusion is that the radial dependence of the potential is different from that given above for the isotopic spin singlet states of the nuclei, such that there is a null contribution to the magnetic moment. This could be accomplished by reversing the sign of a part of the potential[5] to give cancellation in the integrals. However, the same behavior is required in the deuteron and it would seem to be fortuitous if the integrals of the same function vanished when weighted in one case by the wave function of the deuteron and in the other case by the triton-He^3 wave function. It is likely that hfs measurements[4] can distinguish between these two possibilities. It is worth noting that either interpretation appears consistent with the present fit of the scattering data.[6] Cases 3 and 4 summarize two calculations bearing on this question.

The results for M^1 are interesting in that they have the proper behavior to account for the magnetic moment discrepancies but are unfortunately at least an order of magnitude too small. Numerically, part of the smallness of these terms is due to cancellations from among the various D-state parts. Thus, it might be hoped that a different wave function would yield considerably different results. This question is examined in cases 6–12. It appears unlikely that the results can be appreciably changed. If that is the case it raises a further question about the S.M. potential. The argument proceeds as follows. If the spin-orbit potential contribution to the magnetic moment of H^3 and He^3 is insufficient to account for the discrepancies in the moments, then one must still resort to exchange moments for this effect. Phenomenologically one can construct exchange moments that will account for the observed moments of the light nuclei; however, these

TABLE I. Summary of the calculated results. "Wave function modification" describes the changes made in the parameters of the Pease and Feshbach wave function, and P_D gives the percentage D state for the wave function used in each case. The magnetic moment results are expressed in nuclear magnetons. The accuracy of all calculated results is $\pm 10\%$.

Case	S.O. potential	Wave function modification	$M^1{}_{H^3}=-M^1{}_{He^3}$ (nm)	$M^2{}_{H^3}=M^2{}_{He^3}$ (nm)	P_D
1	S.M.	None	0.008	1.1	5%
2	G.T.	None	0.004	1.4	5%
3	S.M. with tail reversed at 1.4 fermi	None	0.008	0.39	5%
4	Old S.M. with tail reversed at 1.4 fermi	None	0.007	0.78	5%
5	Old S.M. ($a=1$, $V_0=30$ Mev, $x_0=0.15$)	None	0.007	1.2	5%
6	S.M.	$\lambda_1=1.4$, $A_{S1}=3.23$	0.010	2.2	5%
7	S.M.	$A_D=2.0$	0.029	1.0	17%
8	S.M.	$A_D=2.0$, $A_{D^0}=-0.135$, $A_{D'}=-0.645$	0.041	0.9	25%
9	S.M.	$A_{D^+}=1.35$	0.030	0.9	26%
10	S.M.	$A_{D^0}=0$	0.024	1.0	15%
11	S.M.	$A_{D'}=0$	0.027	1.1	7%
12	S.M.	$A_{D^0}=0$, $A_{D'}=0$	0.039	1.0	18%

exchange moments will, in general, also contribute in the photodisintegration of the deuteron and primarily to the isotropic part. This isotropic term contribution to the total cross section has been estimated[10] to be about 30 microbarns for photon energies above 20 Mev. These contributions if added to the terms calculated by de Swart and Marshak[11] from the S.M. potentials would lead to a theoretical estimate of the isotropy of the differential cross section of deuteron photodisintegration that would be considerably larger than that observed.

2. Cases 3 and 4.—M^2 as calculated has contributions only from the singlet isotopic spin states. Therefore, to test the degree to which cancellation is obtained in the He^3-H^3 case, these terms were calculated with the tail of the S.M. spin-orbit potential reversed in sign for distances greater than 1.4 fermis (1 fermi$\equiv 10^{-13}$ cm). This was the procedure adopted by de Swart, Marshak, and Signell[5] to overcome the deuteron magnetic moment difficulty using their original spin-orbit potential. In case 3 we have used the newer S.M.S.O. potential and in case 4 we have used the original S.M.S.O. potential. The results in both cases are still too large, although the possibility of this method, or some variation, for obtaining satisfactory cancellation of the deuteron, H^3, and He^3 moments is not necessarily precluded.

3. Case 5.—In this case the calculations were made using the older S.M.S.O. potential in which the range is about twice and the core cutoff distance is about one-half those of the current potential. The insensitivity of the results to these changes together with the agreement between cases 1 and 2 leads to the speculation that any of the S.O. potentials yielding agreement with the scattering data will yield essentially these same magnetic moment results.

4. Case 6.—In this case, a part of the S state of the P.F. (Pease-Feshbach) wave function has been modified by increasing the range by an amount approximately equal to that of the hard-core radius, while keeping the percentage D state fixed. This was done to compensate for the fact that the P.F. wave function is based on the Yukawa potential and therefore there is a greater concentration of S state near the origin than would be expected in a wave function derived from a hard-core potential. Thus, one would expect a greater $S-D$ state overlap in the case of the hard-core potential, and therefore a larger magnetic moment contribution. The results reflect this effect but the smallness of the change leads to the conclusion that even with a proper hard-core wave function, this effect will not be sufficient to obtain agreement with the magnetic moment data.

5. Cases 7–12.—As already noted, the smallness of the results for M^1 occurs partly through cancellations among the contributions from the different D-state terms in the wave function. It might therefore be argued that a different wave function with a different distribution of the D-state terms might give a significantly larger result. In order to obtain some insight into this possibility, cases 7 through 12 were calculated. In these calculations the D-state parts of the P.F. wave function were arbitrarily altered in such a way as to remove the cancellation between terms. In Table I the column headed "Wave function modification" indicates which of the wave function parameters were changed and what new values were used. The column "P_D" shows the percentage D state corresponding to the modified wave function. It is seen from Table I that no alteration chosen yielded a result nearly large enough to account for the magnetic moment discrepancy and with the exception of case 11, all corresponded to much too large a D-state probability. Such calculations as these cannot, of course, be conclusive and it would be desirable to recalculate these results with a wave function derived from the new phenomenological potentials. However, these results make it appear unlikely that such a calculation will yield qualitatively different results.

APPENDIX

The expectation value of $M^1{}_{H^3}$ is given by

$$M^1{}_{\mathrm{H}^3}=\frac{-e\pi^2}{6\hbar c}\frac{r_s^2}{\langle\Psi|\Psi\rangle}\sum_{i=1}^{2}A_{Si}(A_DI_{1i}+A_{D^+}I_{2i}+A_{D^0}I_{3i}+A_{D'}I_{4i}+A_{D'^+}I_{5i}+A_{D'^0}I_{6i}+A_{D''}I_{7i}),\quad\text{(A.1)}$$

where

$$I_{ji}=\int_0^\infty r_1dr_1\int_0^\infty r_2dr_2\int_{|r_1-r_2|}^{(r_1+r_2)}f_j(r_1,r_2,\rho)V_{12}(\rho)\times\exp[-\alpha_{ji}(r_1+r_2+\rho)]d\rho,\quad\text{(A.2)}$$

and $V_{12}(\rho)=V(x)$ with $r_{ij}=\rho$, $r_s=1.184\times10^{-13}$ cm, and

$\alpha_{1i}=\frac{1}{2}(\lambda_i+\mu)\quad f_1=[-(1/6)r_1^4-(1/6)r_2^4+(1/3)r_1^2r_2^2+(1/2)\rho^4-r_1^2\rho^2-r_2^2\rho^2]$,

$\alpha_{2i}=\alpha_{1i},\quad f_2=\{r_2g_1+r_1g_2\}$,

$g_1=[(1/4)r_2^4-(5/12)r_1^4+(1/6)r_1^2r_2^2+(1/4)\rho^4+(1/6)r_1^2\rho^2-(7/6)r_2^2\rho^2]$,

$g_2=[(1/4)r_1^4-(5/12)r_2^4+(1/6)r_1^2r_2^2+(1/4)\rho^4+(1/6)r_2^2\rho^2-(7/6)r_1^2\rho^2]$,

$\alpha_{3i}=\alpha_{1i},\quad f_3=\{(\rho+r_1)g_1+(\rho+r_2)g_2\}$,

$\alpha_{4i}=\frac{1}{2}(\lambda_i+\nu),\quad f_4=\{(r_1-\rho)g_3+(\rho-r_2)g_4\}$,

$g_3=[(1/4)r_2^4-(5/12)r_1^4+(1/6)r_1^2r_2^2+(1/4)\rho^4-(7/6)r_1^2\rho^2+(1/6)r_2^2\rho^2]$,

$g_4=[(5/12)r_2^4-(1/4)r_1^4-(1/4)\rho^4-(1/6)r_1^2r_2^2+(7/6)r_2^2\rho^2-(1/6)r_1^2\rho^2]$,

$\alpha_{5i}=\alpha_{4i},\quad f_5=\{r_2(r_1-\rho)g_3+r_1(\rho-r_2)g_4\}$,

$\alpha_{6i}=\alpha_{4i},\quad f_6=\{(r_1^2-\rho^2)g_3+(\rho^2-r_2^2)g_4\}$,

$\alpha_{7i}=\frac{1}{2}(\lambda_i+\omega),\quad f_7=(r_1-r_2)[-(1/12)r_1^6+(1/12)r_2^6+(1/4)r_1^4r_2^2-(1/4)r_1^2r_2^4-(1/12)r_1^2\rho^4+(1/12)r_2^2\rho^4+(1/6)r_1^4\rho^2-(1/6)r_2^4\rho^2]$.

Also

$$M^2{}_{\mathrm{H}^3}=\frac{4\pi^2e}{3\hbar c}\frac{r_s^2}{\langle\Psi|\Psi\rangle}\sum_{i=1}^{2}\sum_{j=1}^{2}(A_{Si}A_{Sj}K_{ij}),\quad\text{(A.3)}$$

where

$$K_{ij}=\int_0^\infty r_1dr_1\int_0^\infty r_2dr_2\int_{|r_1-r_2|}^{(r_1+r_2)}\rho^3V_{12}(\rho)e^{-\alpha_{ij}(r_1+r_2+\rho)}d\rho,$$

and

$$\alpha_{ij}=\tfrac{1}{2}(\lambda_i+\lambda_j).$$

The normalization integral is given by

$$\langle\Psi|\Psi\rangle=8\pi^2\sum_{N=1}^{9}J_N,\quad\text{(A.4)}$$

where

$$J_N=\int_0^\infty r_1dr_1\int_0^\infty r_2dr_2\int_{|r_1-r_2|}^{(r_1+r_2)}\rho F_N(r_1,r_2,\rho)e^{-\beta_N(r_1+r_2+\rho)}d\rho,$$

with

$\beta_1=\lambda_1,\quad F_1=A_{S1}^2$,

$\beta_2=\frac{1}{2}(\lambda_1+\lambda_2),\quad F_2=2A_{S1}A_{S2}$,

$\beta_3=\lambda_2,\quad F_3=A_{S2}^2$,

$\beta_4=\mu,\quad F_4=6[A_D+\rho A_{D^+}+(r_1+r_2)A_{D^0}]^2\times[r_1^4+r_2^4+r_1^2r_2^2(1-3\cos^2\theta)]$,

$\beta_5=\frac{1}{2}(\mu+\nu),\quad F_5=\{6[A_D+\rho A_{D^+}+(r_1+r_2)A_{D^0}]\times[A_{D'}+\rho A_{D'^+}+(r_1+r_2)A_{D'^0}]\times(r_1-r_2)(r_1^4-r_2^4)\}$,

$\beta_6=\frac{1}{2}(\mu+\omega),\quad F_6=-3[A_D+\rho A_{D^+}+(r_1+r_2)A_{D^0}]\times A_{D''}(r_1-r_2)(r_1^2-r_2^2)r_1^2r_2^2\times(1-\cos^2\theta)$,

$\beta_7=\nu,\quad F_7=6[A_{D'}+\rho A_{D'^+}+(r_1+r_2)A_{D'^0}]^2\times(r_1-r_2)^2[r_1^4+r_2^4-r_1^2r_2^2\times(1-3\cos^2\theta)]$,

$\beta_8=\frac{1}{2}(\nu+\omega),\quad F_8=-3\{[A_{D'}+\rho A_{D'^+}+(r_1+r_2)A_{D'^0}]\times A_{D''}(r_1-r_2)^2(r_1^2+r_2^2)r_1^2r_2^2\times(1-\cos^2\theta)\}$,

$\beta_9=\omega,\quad F_9=6A_{D''}^2(r_1-r_2)^2[r_1^2r_2^2(1-\cos^2\theta)]^2$,

$$\cos^2\theta=(r_1^2+r_2^2-\rho^2)^2/(4r_1^2r_2^2).$$