Electron and Ion Runaway in a Fully Ionized Gas. I*

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Hydrodynamic equations are used to describe the flow of the electrons and ions of a fully ionized gas under the action of an electric field, **E**, of arbitrary magnitude. The dynamical friction force exerted by the electrons and ions upon each other through the agency of two-body Coulomb encounters is evaluated. In this connection the electrons and ions have been assigned Maxwellian velocity distributions which are displaced from each other by their relative drift velocity. This treatment yields a dynamical friction force which maximizes when the relative drift velocity is equal to the sum of the most probable random electron and ion speeds. For relative drift velocities in excess of this value the friction force decreases rapidly. As a consequence, it is found that a fully ionized gas cannot exhibit the steady-state behavior characterized by time independent drift velocities which has previously been accredited to it by other authors. Instead, it is shown that the electron and ion currents flowing parallel to the existing magnetic fields increase steadily in time (i.e., runaway) as long as a component of the electric field persists along the magnetic field. Drift velocities which greatly exceed the random speeds of the plasma particles can be created in this manner.

The theory yields a critical electric field parameter, E_c, which is proportional to the plasma density and inversely proportional to its temperature. It is a measure of the electric field which is required if the drift velocities are to increase and exceed the most probable random speeds in the gas in one mean free collision time. For electric fields in excess of E_c runaway proceeds even faster. In smaller fields runaway occurs when Joule heating has depressed E_c sufficiently. Several interpretations of E_c are given in terms of the collisional phenomenon involved.

Within the framework of the hydrodynamic equations it is shown that the well-known (temperature)$^{\frac{3}{2}}$ electrical conductivity law can be recovered, provided $E \ll E_c$ and the electron temperature is held constant.

Numerical solutions giving electron temperature and drift velocity as a function of time are presented for a range of the ratio E/E_c. The assumption of the displaced Maxwellian distribution is justified on the basis of a comparison between the rate of Joule heating and the rate of equipartition of random speeds. Moreover, it is found that the use of an anisotropic velocity distribution does not affect the runaway phenomenon in any important way.

The possibility of runaway induced across magnetic fields by steep pressure gradients and its relation to diffusion across magnetic fields is examined and discussed in detail.

I. INTRODUCTION

IN evaluating the electrical conductivity of a fully ionized gas the conventional treatment follows the methods of Lorentz[1] or Chapman and Enskog.[2] The starting point of these calculations is the time-independent Boltzmann equation, and the method of solution follows a perturbation scheme in which the electron velocity distribution is expressed as a sum of spherical harmonics, or in powers of some small expansion parameter. Generally, it is assumed that encounters between particles occur with sufficient frequency to bring about a Maxwellian distribution in the absence of the electric field. The introduction of a weak electric field is then found to give rise to an electrical current which is linearly related to the perturbing field. This method of solution is assumed to yield correct answers provided the average random electron speed is very much larger than the electron drift velocity. In this limit the electrical conductivity is the well-known (temperature)$^{\frac{3}{2}}$ law, and the numerical results obtained by several authors[3] are in good agreement. The assumption that a steady-state velocity distribution is attained several mean free collision times after the electrical field is turned on is basic to these time independent treatments.

In the present paper we wish to consider the problem of electrons moving through a gas of positive ions under the influence of a static uniform electric field of arbitrary magnitude. From the start we avoid the usual steady-state assumption and search for the time-dependent behavior. Our main results indicate that the conventional concept of an electrical conductivity along magnetic field lines, which is based upon a Stokes law dynamical friction force, is of limited applicability. The rapid variation of the Rutherford cross section with energy is responsible for strong deviations from a Stokes' law and rules out the existence of a collision controlled steady-state drift velocity for the electrons and ions. Instead, we find that the drift velocities of these particles steadily increase with time as long as a component of the electric field persists along the direction of the total magnetic field, and relativistic effects can be ignored.

This is the so-called "runaway" effect which has become important in connection with controlled fusion research.

The results presented in this paper[4] are based upon

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] H. A. Lorentz, *The Theory of Electrons* (Teubner, Leipzig, 1909, and G. E. Stechert and Company, New York, 1923).

[2] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1939).

[3] Cohen, Spitzer, and Routley, Phys. Rev. **80**, 230 (1950); L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953); R. Landshoff, Phys. Rev. **76**, 904 (1949); R. Landshoff, Phys. Rev. **82**, 442 (1951); T. G. Cowling, Proc. Roy. Soc. (London) **A183**, 453 (1945).

[4] For earlier results, see H. Dreicer, Ph.D. thesis, Physics Department, Massachusetts Institute of Technology, 1955 (unpublished). Results of this work are also reported in W. P. Allis', *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 21, p. 436. The results of the present paper were first reported at a classified joint British-American Conference on Controlled Thermonuclear Research held at Princeton University, April, 1957 (unpublished). See also H. Dreicer, Proceedings of the Third

the two-fluid (electron-ion) hydrodynamic equations. These equations yield most of the important runaway effects. However, when the problem is examined in more detail, additional effects come to light which we have treated by means of the Boltzmann equation. These results are to be presented in a subsequent paper, where we also discuss the experimental implications of the runaway effect.

II. RUNAWAY IN THE TWO-FLUID HYDRODYNAMIC APPROXIMATION

A. Derivation of Equations

In this section we shall consider a completely ionized gas of infinite extent subjected to a magnetic field, $\mathbf{B}$, and placed under the influence of an electric field, $\mathbf{E}$, at some initial instant of time. The pressure of the gas is, in some parts of the calculation, assumed to vary from point to point in space. For simplicity we assume that in addition to the electrons only protons or deuterons are present. Their respective masses will be denoted by m and M. The two Boltzmann equations which we must consider are

$$\frac{\partial F_e}{\partial t}+\mathbf{c}\cdot\nabla F_e+\left(\frac{-e}{m}\right)(\mathbf{E}+\mathbf{c}\times\mathbf{B})\cdot\nabla_c F_e=\left(\frac{\partial F_e}{\partial t}\right)_c,\quad (1)$$

$$\frac{\partial F_i}{\partial t}+\mathbf{c}\cdot\nabla F_i+\left(\frac{e}{M}\right)(\mathbf{E}+\mathbf{c}\times\mathbf{B})\cdot\nabla_c F_i=\left(\frac{\partial F_i}{\partial t}\right)_c.\quad (2)$$

Here $F_e(\mathbf{r},\mathbf{c},t)$ and $F_i(\mathbf{r},\mathbf{c},t)$ are, respectively, the electron and ion velocity distribution functions, $\mathbf{r}$ and $\mathbf{c}$ are their space and velocity coordinates, and ∇_c is the gradient operator in velocity space. The collision terms $(\partial F_e/\partial t)_c$ and $(\partial F_i/\partial t)_c$ are in this paper represented by the Fokker-Planck equation

$$\left(\frac{\partial F_\alpha}{\partial t}\right)_c=\sum_{\beta=e,i}\left\{-\frac{\partial}{\partial c_k}[F_\alpha\langle\Delta c_k\rangle_{\alpha\beta}]+\frac{1}{2}\frac{\partial^2}{\partial c_k\partial c_j}[F_\alpha\langle\Delta c_k\Delta c_j\rangle_{\alpha\beta}]\right\},\quad (3)$$

where repeated Latin indices are summed over, and the summation of β over e and i indicates that each type of particle encounters both electrons and ions. The average velocity increments are related to the Rosenbluth H and G potentials[5] through

$$\langle\Delta c_k\rangle_{\alpha\beta}=\partial H_{\alpha\beta}/\partial c_k,\quad (4)$$

$$\langle\Delta c_k\Delta c_j\rangle_{\alpha\beta}=\partial^2 G_{\alpha\beta}/\partial c_k\partial c_j,\quad (5)$$

where

$$H_{\alpha\beta}(\mathbf{r},\mathbf{c},t)=\frac{m_\alpha+m_\beta}{m_\beta}\Gamma_\alpha\int\frac{F_\beta(\mathbf{r},\mathbf{c}',t)}{w}d^3c',\quad (6)$$

$$G_{\alpha\beta}(\mathbf{r},\mathbf{c},t)=\Gamma_\alpha\int wF_\beta(\mathbf{r},\mathbf{c}',t)d^3c',\quad (7)$$

and

$$w=|\mathbf{c}-\mathbf{c}'|,\quad (8)$$

$$\Gamma_\alpha=4\pi\left(\frac{e^2}{4\pi\epsilon_0 m_\alpha}\right)^2\log_e\left(\frac{\lambda}{p_0}\right),\quad (9)$$

$\epsilon_0=1/36\pi\times10^{-9}$ coulomb volt^{-1} meter^{-1}, m_α= mass of the α constituent, λ= Debye radius[6], and p_0= average impact parameter for a 90° Coulomb deflection. Only short-range two-body Coulomb encounters between a given particle and its neighbors inside of a Debye sphere have been counted in the evaluation of H and G.

For problems involving electric fields of arbitrary magnitude there is at present little hope for obtaining an analytic solution to the Boltzmann equation in a closed form. Nevertheless, with the help of the conservation laws of momentum and energy we can make some general statements which are helpful in understanding the basic features of the problem. In this way we are also led to a useful approximate treatment. We first consider a plasma which is distributed uniformly in space and is under the influence of uniform and static electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. In strong magnetic fields the charged particles circle magnetic lines many times between collisions, and the entire electrical current is produced by the component of $\mathbf{E}$ along $\mathbf{B}$. The component of $\mathbf{E}$ perpendicular to $\mathbf{B}$ leads merely to a translation of the plasma as a whole with the drift velocity $(\mathbf{E}\times\mathbf{B})/B^2$. We shall ignore this motion by choosing $\mathbf{E}$ parallel to $\mathbf{B}$.

The electrical current density is given by

$$\mathbf{j}=ne(\mathbf{v}_i-\mathbf{v}_e),$$

where n is the electron (or ion) particle density, and the drift velocities are defined by

$$\mathbf{v}_\alpha=\frac{1}{n}\int F_\alpha\mathbf{c}d^3c.\quad (10)$$

The first-moment equations are obtained by multiplying the Boltzmann Eqs. (1) and (2) by $m\mathbf{c}$ and integrating over all velocity space. This results in

$$m\frac{\partial\mathbf{v}_e}{\partial t}+e\mathbf{E}=\frac{m}{n}\int F_e(\mathbf{c},t)\nabla_c H_{ei}d^3c,\quad (11)$$

$$M\frac{\partial\mathbf{v}_i}{\partial t}-e\mathbf{E}=\frac{M}{n}\int F_i(\mathbf{c},t)\nabla_c H_{ie}d^3c.\quad (12)$$

International Conference on Ionization Phenomena in Gases, Societa Italian di Fisica, Venice, 1957, p. 249; and H. Dreicer, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, September, 1958* (United Nations, Geneva, 1958).

[5] Rosenbluth, MacDonald, and Judd, Phys. Rev. **107**, 1 (1957).

[6] P. Debye and E. Hückel, Z. Physik **24**, 190 (1923).

These equations state that the time rate of change of momentum along **B** is for each constituent gas a balance between the electric force and the dynamical friction force arising from electron-ion encounters. Encounters between like particles do not alter the total momentum of the parent gas and therefore do not contribute to the dynamical friction. The total momentum of a neutral plasma subjected to a steady electric field must be conserved. To prove that this is so we need merely show that the dynamical friction force obeys Newton's third law. With the help of Eqs. (6), (8), and (9), we find

$$m\int F_e\,\nabla_c H_{ei}d^3c=(m+M)\frac{m}{M}\Gamma_e\int\int F_e(\mathbf{c},t)F_i(\mathbf{c}',t)\,\nabla_c\times\left(\frac{1}{w}\right)d^3c d^3c'$$

$$=-M\int F_i(\mathbf{c}',t)\,\nabla_{c'}H_{ie}d^3c',$$

and addition of Eqs. (11) and (12) yields

$$\frac{\partial \mathbf{v}_e}{\partial t}=-\frac{M}{m}\frac{\partial \mathbf{v}_i}{\partial t}. \tag{13}$$

This result shows that the electrons carry nearly the entire current generated by the electric field.

With the help of Eq. (11) we find that the power fed into the electron gas from the electric field can be divided into the following two terms:

$$-ne\mathbf{E}\cdot\mathbf{v}_e=n\frac{d}{dt}\left(\frac{mv_e^2}{2}\right)-m\mathbf{v}_e\cdot\int F_e\,\nabla_c H_{ei}d^3c. \tag{14}$$

The first term on the right-hand side gives the increase of electron drift energy with time. The second term describes the rate with which electron-ion encounters convert electron drift energy into random energy. Most of this random energy is stored in the electron gas, however, because of the finite ion mass, some of it is transferred to the ions. The total rate of Joule heating, Q, is given by

$$Q=m(\mathbf{v}_e-\mathbf{v}_i)\cdot\int F_e\,\nabla_c H_{ei}d^3c. \tag{15}$$

Joule heating involves the conversion of drift to random energy. Accordingly, since encounters between like particles cannot alter the drift energy of the gas, Joule heating occurs as a result of electron-ion collisions only. In collisions between like particles random energy is, however, exchanged very efficiently, and a Maxwellian velocity distribution, displaced by the instantaneous drift velocity, tends to be established. We see, therefore, that the rate of Joule heating and the dynamical friction force which gives rise to it can be indirectly influenced by like-particle encounters, inasmuch as the precise form of the velocity distributions plays a role in the *rate* of electron-ion encounters.

In the limit of weak and strong electric fields this problem exhibits certain simplifying features. In strong fields we may consider the effect of electron-ion encounters to be a small perturbation on the motion which the electrons and ions execute in the applied electric field. To a good approximation, then, the electrons and ions are accelerated independently and at a constant rate. Moreover, if we remember that the velocity distributions are, under these conditions, altered largely by collisions between like particles, then it becomes apparent that these distributions will tend asymptotically in time to Maxwellian distributions which are centered about the drift velocities.

These notions have led us to the consideration of the *displaced* Maxwellian distribution

$$F_\alpha(\mathbf{r},\mathbf{c},\mathbf{v}_\alpha(t))=n(\mathbf{r})[\beta_\alpha(\mathbf{r})/\pi]^{\frac{3}{2}}\exp[-\beta_\alpha|\mathbf{c}-\mathbf{v}_\alpha(t)|^2], \tag{16}$$

where

$$\beta_\alpha=m_\alpha/2kT_\alpha(\mathbf{r}),$$

as an approximate solution which satisfies the Boltzmann equation on the average. Subsequent analysis shows that this distribution leads to many correct results even in the limit of weak electric fields. The H_{ei} function required for the solution of the first moment Eqs. (11) and (12) may now be evaluated by substituting Eq. (16) into Eq. (6). Straightforward integration results in

$$H_{ei}=n\Gamma_e\frac{m+M}{M}\frac{\mathcal{E}_2(\beta_i^{\frac{1}{2}}q)}{q}, \tag{17a}$$

where

$$\mathcal{E}_2(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp(-t^2)dt,$$

$$q=|\mathbf{c}-\mathbf{v}_i|, \qquad \beta_i=M/2kT_i. \tag{17b}$$

In many problems the average random electron speed greatly exceeds all random ion speeds, and in this limit

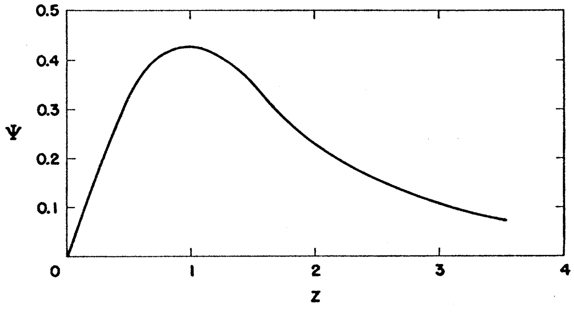

FIG. 1. The variation of the dynamical friction function Ψ with the relative electron-ion drift velocity (expressed in units of $\beta_e^{-\frac{1}{2}}$).

we can simplify H_{ei} appreciably by dealing with an ion gas at zero temperature. We adopt this procedure and also neglect all terms of order m/M compared to unity. This leads to

$$H_{ei}=n\Gamma_e/q, \tag{18}$$

and after another integration Eq. (11) takes the form

$$m\left(\frac{\partial v_e}{\partial t}\right)+eE=-eE_c\Psi(z), \tag{19}$$

where

$$\Psi(z)=\frac{\mathcal{E}_2(z)-zd\mathcal{E}_2/dz}{z^2},$$

$$z=\beta_e^{\frac{1}{2}}|\mathbf{v}_e-\mathbf{v}_i|,$$

$$\beta_e=m/2kT_e, \tag{20}$$

$$eE_c=nm\Gamma_e\beta_e.$$

The Ψ function presented in Fig. 1 accounts for the variation with relative drift velocity of the dynamical friction force exerted by the ions on the electron gas. The total magnitude of this force depends also upon the coefficient E_c which, as we shall demonstrate shortly, plays the role of a critical electric field parameter.

The velocity dependence of $\Psi(z)$ can be understood with the help of potential theory, since the potential $H_{\alpha\beta}$ and the velocity distribution F_β bear the same relation to each other in velocity space as do electrostatic potential ϕ and electric charge density distribution ρ in real space. From Eq. (6) we find the exact analogy to be

$$\phi(\mathbf{R})\leftrightarrow H_{\alpha\beta}(\mathbf{c}),$$
$$\rho(\mathbf{R})\leftrightarrow\frac{m_\alpha+m_\beta}{m_\beta}\Gamma_\alpha F_\beta(\mathbf{c}), \tag{21}$$

where

$$\phi(\mathbf{R})=\int\frac{\rho(\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|}d^3R'.$$

In particular, when F_e is spherically symmetric about $\mathbf{v}_e$, we can obtain the dynamical friction force, $M\nabla_cH_{ie}$, which acts on an ion moving with the velocity $\mathbf{v}_i$ by making use of Gauss' theorem. Moreover, we can immediately state that the contributions arise only from the electrons whose velocity $\mathbf{c}$ is interior to the sphere defined by

$$c^2=|\mathbf{v}_e-\mathbf{v}_i|^2=z^2/\beta_e.$$

A further simplification results from the circumstance that as far as the calculation of the force is concerned, all of the electrons in the sphere act as if they were moving with the velocity $\mathbf{v}_e$. Gauss' theorem taken together with Eqs. (6), (9), and (21) then yields

$$[M\,\nabla_cH_{ie}(\mathbf{c})]_{\mathbf{c}=\mathbf{v}_i}$$
$$=\frac{M^2}{m}\Gamma_i n\beta_e\frac{4}{(\pi)^{\frac{1}{2}}z^2}\int_0^z t^2\exp(-t^2)dt$$
$$=mn\Gamma_e\beta_e\frac{4}{(\pi)^{\frac{1}{2}}z^2}\int_0^z t^2\exp(-t^2)dt, \tag{22a}$$

and this is just $eE_c\Psi(z)$. In the limit of small z, the number of electrons in the sphere is proportional to z^3, and this force takes the form of a Stokes' law:

$$\Psi(z)\rightarrow\frac{4}{3\sqrt{\pi}}z. \tag{22b}$$

For $z\gg1$ essentially all electrons are inside of the sphere

$$\int_0^z t^2\exp(-t^2)dt\rightarrow\int_0^\infty t^2\exp(-t^2)dt,$$

and

$$\Psi(z)\rightarrow1/z^2. \tag{22c}$$

The maximum of $\Psi(z)$ occurs at the most probable speed $z=1$. Above this speed the inverse square law falls off faster with z than the number of electrons which contribute to the force can increase with z.

B. Uniform Plasma in Parallel Electric and Magnetic Fields

1. Critical Runaway Field at Constant Temperature

We can solve Eqs. (13) and (19) by a method of successive approximations. To zero order in m/M, v_e is the solution of

$$\frac{\partial v_e}{\partial t}+\frac{e}{m}E_c\Psi(x)=-\frac{e}{m}E, \tag{23}$$

where

$$x=\beta_e^{\frac{1}{2}}v_e,$$

and to first order, conservation of momentum yields

$$v_i=-(m/M)v_e.$$

With the electron temperature held fixed Eq. (23) yields two different kinds of solutions. The first is characteristic of an applied to critical field ratio which obeys the inequality

$$E/E_c<\Psi(1)=0.43. \tag{24a}$$

In this case, v_e, starting from zero initial value tends monotonically to a terminal value $v_{et}(<\beta_e^{-\frac{1}{2}})$ which is the solution of the transcendental equation

$$E=E_c\Psi(\beta_e^{\frac{1}{2}}v_{et}). \tag{25}$$

The second type of solution is associated with the opposite inequality

$$E/E_c>\Psi(1). \tag{24b}$$

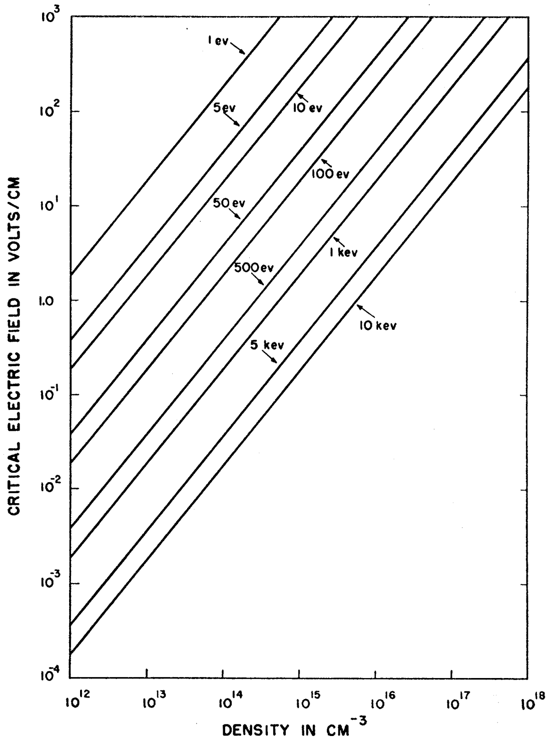

FIG. 2. Critical electric field, E_c, as a function of the particle density with the average electron energy, $\frac{3}{2}kT_e$, treated as a parameter. The cutoff factor, $\ln(\lambda/p_0)$, was assumed to be 10 for all conditions.

Starting from any initial condition, we find in this case that the particle acceleration is always positive. Consequently, as time proceeds v_e and v_i increase without limit, i.e., the electrons and ions "run away." We find it useful to speak of electric fields which are appreciably larger than, or smaller than, E_c as "strong" and "weak" fields, respectively.

It is useful to ascribe to the critical field a physical interpretation in terms of the collisional phenomena involved. Let us consider electron-ion encounters. For electrons moving with the most probable random speed, $\beta_e^{-\frac{1}{2}}$, these occur approximately with the frequency

$$\nu = n\Gamma_e\beta_e^{\frac{3}{2}} = (e/m)E_c\beta_e^{\frac{1}{2}}. \tag{26}$$

Thus, in order to double the speed of an average electron in the mean free time between collisions, the acceleration required is

$$\beta_e^{-\frac{1}{2}}\nu = (e/m)E_c,$$

and the required applied field must therefore equal E_c.

A somewhat different interpretation can also be put forward. In terms of the Debye radius

$$\lambda = \left(\frac{\epsilon_0 kT_e}{ne^2}\right)^{\frac{1}{2}},$$

we can write E_c in the form

$$E_c = \frac{e}{4\pi\epsilon_0\lambda^2}\ln\left(\frac{\lambda}{p_0}\right).$$

Since the value of the logarithm ranges roughly from 5 to 20, we see that E_c is equal to the electric field at a distance of about $\lambda/2$ to $\lambda/5$ from a positive ion. In weak fields, therefore, most electrons are scattered in ionic fields whose magnitude exceeds that of the applied field. The opposite holds true for strong fields. These statements can be expected to hold even though the Debye sphere about an ion becomes distorted and somewhat displaced in a strong field.[7] A graph giving E_c as a function of density and electron temperature is shown in Fig. 2.

2. *The (Temperature)$^{\frac{3}{2}}$ Law*

In the weak-field limit we can solve Eq. (23) with the help of Eq. (22b). At constant temperature, the solution which satisfies the initial condition $v_e(0)=0$ is given by

$$v_e(t) = \frac{3(\pi)^{\frac{1}{2}}}{4}\frac{E}{E_c}\beta_e^{-\frac{1}{2}}[1-\exp(-4\nu t/3\sqrt{\pi})]. \tag{27}$$

After a few mean free collision times, the corresponding electrical conductivity takes the form of the well-known (temperature)$^{\frac{3}{2}}$ law

$$\sigma = nev_e = \frac{3(\pi)^{\frac{1}{2}}e^2}{4\Gamma_e m}\left(\frac{2kT_e}{m}\right)^{\frac{3}{2}}. \tag{28}$$

Comparison of σ with the conductivity, σ_0, calculated on the basis of conventional perturbation theory yields

$$\sigma_0/\sigma = 1.977,$$

for an electron-proton mixture.[8] The origin of this discrepancy probably originates in our assumption that the distribution can be described by a displaced Maxwellian distribution, whereas in fact electron-electron encounters probably do not occur frequently enough to make the distribution completely Maxwellian.

For arbitrary applied electric fields in the range

$$E \leq E_c\Psi(1),$$

a more accurate electrical conductivity than σ_0 is obtained by solving the transcendental Eq. (25). We then find that the conductivity is a function not only of temperature, but of applied electric field and density as well.

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[7] B. B. Kadomtsev, J. Exptl. Theoret. Phys. U.S.S.R. **33**, 151 (1957) translation: Soviet Phys. JETP **6**, 117 (1958).
[8] L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).

3. *Joule Heating*

Dynamical friction heats the electron gas at the rate

$$P_J(x)=\frac{d}{dt}(\tfrac{3}{2}kT_e)$$
$$=eE_c\beta_e^{-\frac{1}{2}}x\Psi(x). \tag{29}$$

Our choice of the displaced Maxwellian distribution in this problem implies that electron-electron encounters partition this random energy equally and instantly into all degrees of freedom. Actually, we must expect the relative growth rates of the random energies in each degree of freedom to be a balance between the Joule heating rate and the rate of electron-electron encounter. If the Joule heating rate exceeds the latter, then the velocity distribution will appear distorted from spherical symmetry when viewed in the coordinate frame moving with the drift velocity. A rough measure of the accuracy of our approximation can thus be obtained from a comparison of these rates. Figure 3 shows that $x\Psi(x)$ maximizes with an amplitude of 0.525 at $x=1.5$. The quantity

$$P_j(1.5)=0.525eE_c\beta_e^{-\frac{1}{2}},$$

therefore represents the maximum Joule heating rate. The rate of mutual encounter between electrons of average random speed in a Maxwellian distribution of electrons is given by the frequency ν, and the amount of random energy exchanged by these in unit time is approximately

$$P_e\simeq 2kT_e\nu=eE_c\beta_e^{-\frac{1}{2}}.$$

A comparison of P_e and $P_j(x)$ shows that P_e is roughly equal to $P_j(1.5)$, and it therefore exceeds P_j appreciably for almost all values of x. This result indicates that the use of the displaced Maxwellian distribution should be qualitatively correct as far as the gross features of this problem are concerned.

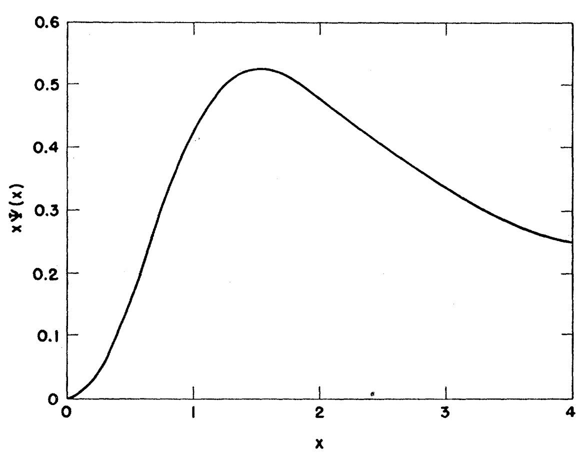

FIG. 3. Dependence of the Joule heating rate expressed in units of $eE_c\beta_e^{-\frac{1}{2}}$ upon the relative electron-ion drift velocity. The latter is expressed in units of the most probable random electron speed, $\beta_e^{-\frac{1}{2}}$.

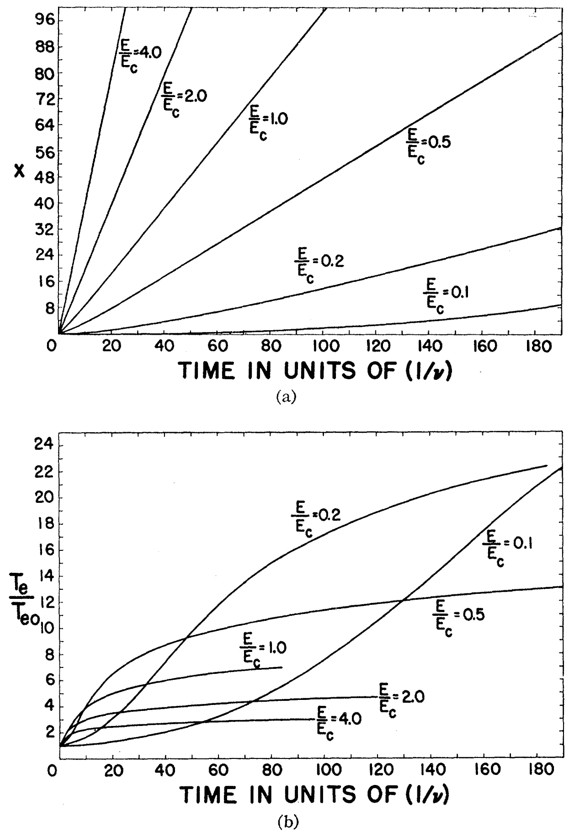

FIG. 4(a). The variation of electron drift velocity (expressed in units of $\beta_e^{-\frac{1}{2}}$) with time for a range of the parameter E/E_c. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$. (b). The variation of electron temperature (expressed in terms of the initial temperature T_{e0}) with time for a range of the parameter E/E_c. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$.

An important consequence of Joule heating is that runaway can occur for any nonzero applied electric field. Indeed, if initially the gas is subjected to a weak field, i.e., $E<E_c\Psi(1)$, then the monotonic increase of temperature with time guarantees the eventual reversal of this inequality, and in general what we consider to be a weak field will in the course of time evolve into a strong field even though E remains constant.

Equations (23) and (29) have been solved on a 704 IBM digital computer for the variation of electron temperature and drift velocity with time. Typical results are presented in Figs. 4(a) to 9(b). The quantities x, E_c, and ν are defined by Eqs. (20), (23), and (26), in terms of the electron temperature T_{e0} at time $t=0$, i.e.,

$$x(t)=(m/2kT_{e0})^{\frac{1}{2}}v_e(t)=\beta_{e0}^{\frac{1}{2}}v_e(t),$$
$$E_c=n(m/e)\Gamma_e\beta_{e0},$$
$$\nu=n\Gamma_e\beta_{e0}^{\frac{3}{2}}.$$

The initial conditions are described in the caption of each figure. We find that the time required for the electron drift velocity to increase from zero to $\beta_e^{-\frac{1}{2}}$ is

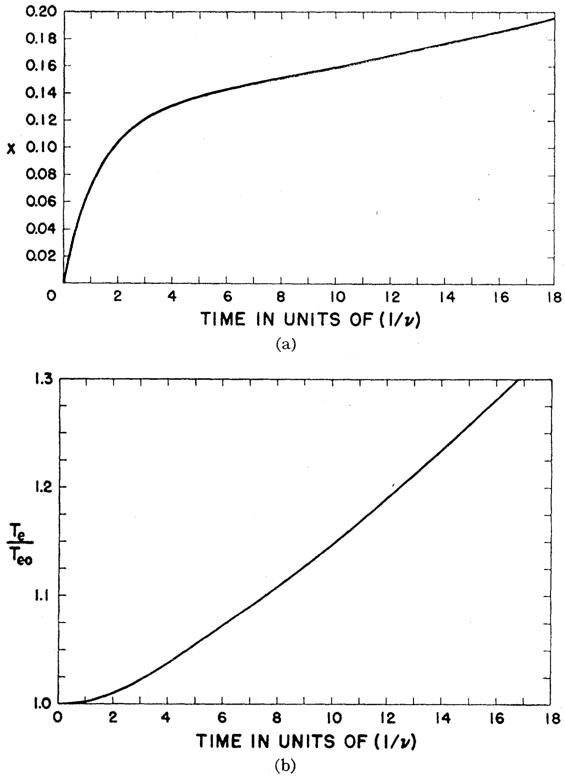

FIG. 5(a). The variation of electron drift velocity with time for $E/E_c=0.1$. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$. (b). The variation of electron temperature with time for $E/E_c=0.1$. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$.

roughly one mean free collision time provided that the applied electric field is approximately equal to E_c. This result is in agreement with the interpretation we have assigned to E_c. The Joule heating processes presented in Fig. 4(b) show a steady initial increase in T_e/T_{e0} followed by a leveling off to an asymptotic value. This behavior can be understood by comparing the ratio of P_j to the power P_d going into drift motion. With the help of Eq. (23) we find

$$\frac{P_j}{P_d}=\frac{\Psi(x)}{(E/E_c)-\Psi(x)}.$$

This relation shows that the energy gained from the applied field can be very largely stored in the form of drift or directed energy since in the limit of runaway we have

$$\frac{P_j}{P_d}\rightarrow\frac{E_c}{E}\frac{2kT_e}{mv_e^2},$$

and this expression approaches zero with increasing v_e. For $E/E_c\ll 1$, x quickly reaches the terminal value it would assume in the absence of Joule heating. Its subsequent increase follows the rise in temperature and thus takes place more slowly. This behavior is illustrated in Figs. 5(a) and 5(b) and to a lesser extent in Figs. 6 and 7.

The energy stored in drift motion may eventually be dissipated and converted to random energy by the removal of the electric field or the growth of magnetic fields perpendicular to $\mathbf{E}$. Examples of several such cases are presented in Fig. 8. These curves show that the rate of Joule heating decreases with increasing temperature.

Figures 9(a) and 9(b) illustrate the behavior of x and T_e/T_{e0} when the electrical field is initially exceeded by the dynamical friction force. At first the drift velocity decreases, however, in time Joule heating decreases the dynamical friction, and the situation is reversed.

Runaway may be avoided by use of alternating electric fields. Ultimately, however, its frequency must increase with time in a manner dictated by the amount of Joule heating which takes place.

4. *Displaced Anisotropic Velocity Distribution*

In order to test the effect of incomplete equipartition of the random energy, we have repeated some of the calculations carried out so far, making the new assumption that the electron velocity distribution has the cylindrical form

$$F_e(\mathbf{c},v_e(t);a,b)=\frac{n}{2\pi a^2 b}\qquad (v_e-b)\leq c_z\leq(v_e+b)\quad c_r\leq a$$

$$=0\quad\text{for all other velocities.}$$

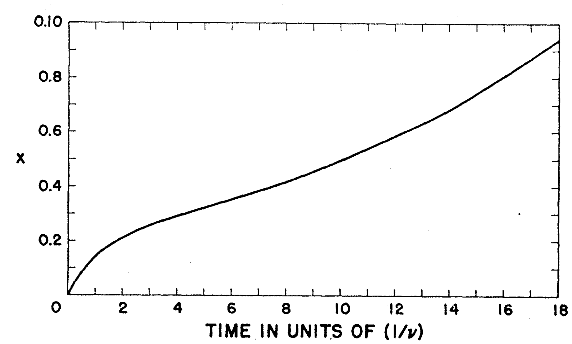

FIG. 6. The variation of electron drift velocity with time for $E/E_c=0.2$. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$.

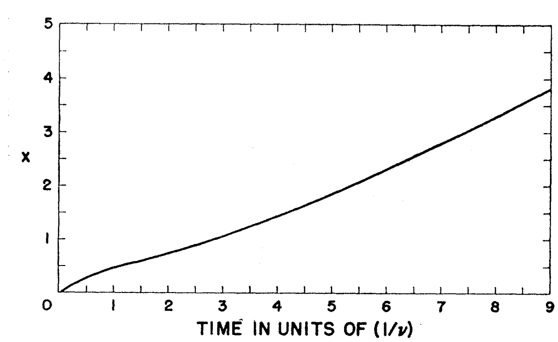

FIG. 7. The variation of electron drift velocity with time for $E/E_c=0.6$. The initial conditions are $x(0)=0$ and $T_e(0)=T_{e0}$.

The variable c_r is the radial velocity component, and a, b defines the thermal spread of the distribution in the two orthogonal directions. Again the ions are assumed to be stationary, and with the help of Eq. (18) we have

$$H_{ei}=n\Gamma_e/(c_r^2+c_z^2)^{\frac{1}{2}}.$$

The dynamical friction force defined by the right-hand side of Eq. (11) is now easily evaluated with the result

$$mdv_e/dt+eE=-eE_c^*\Phi(\xi,\zeta),$$

where

$$eE_c^*=nm\Gamma_e/a^2,$$
$$\Phi(\xi,\zeta)=[(\xi+1)^2+\zeta^2]^{\frac{1}{2}}-[(\xi-1)^2+\zeta^2]^{\frac{1}{2}}-2,$$
$$\xi=v_e/b,$$
$$\zeta=a/b.$$

In the limit of large ξ, we find as before

$$\Phi(\xi,\zeta)\to\frac{1}{\xi^2}.$$

Weak and strong electric fields are defined as in Eqs. (24a) and (24b) by their relation to the electric field defined by

$$E=E_c^*\Phi(1,\zeta).$$

With the help of $\Phi(\xi,\zeta)$ presented in Fig. 10 we note that $E_c^*\Phi(1,\zeta)$ is nearly proportional to ζ. This observation permits us to conclude that anisotropic distributions will not alter our previous conclusions in any important way.

5. *Effect of Random Ion Motion*

So far in our discussion we have restricted the ion temperature to be zero. Clearly, we shall obtain very similar results provided

$$\beta_e^{\frac{1}{2}}\ll\beta_i^{\frac{1}{2}}.$$

In studying the effect of random ion motion the relevant parameter is therefore

$$\gamma=(\beta_e/\beta_i)^{\frac{1}{2}}. \tag{30}$$

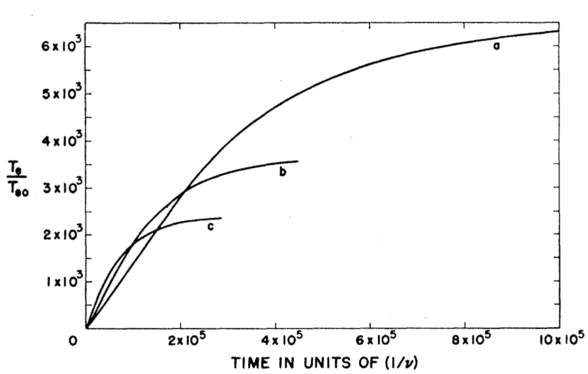

FIG. 8. The variation of electron temperature with time for $E/E_c=0$. The initial and final conditions shown are: (a) $x(0)=100$, $T_e(0)=T_{e0}$, $x(10^6)=24$; (b) $x(0)=75$, $T_e(0)=T_{e0}$, $x(4.6\times10^5)=15$; (c) $x(0)=60$, $T_e(0)=T_{e0}$, $x(2.8\times10^5)=8$.

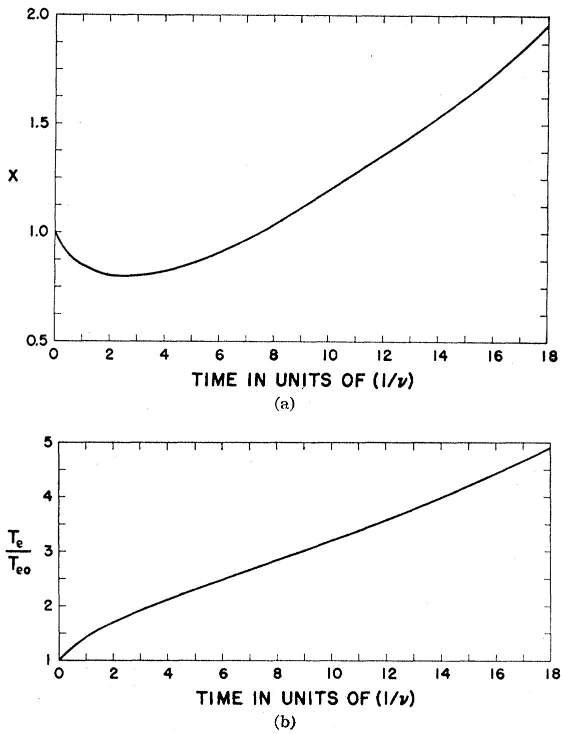

FIG. 9(a). The variation of electron drift velocity with time for $E/E_c=0.2$. The initial conditions are $x(0)=1.0$ and $T_e(0)=T_{e0}$. (b). The variation of electron temperature with time for $E/E_c=0.2$. The initial conditions are $x(0)=1.0$ and $T_e(0)=T_{e0}$.

The extreme limit $\gamma\gg1$ is easily obtained by distributing the electrons into a delta function at $\mathbf{v}_e$ and distributing the ions according to a Maxwellian distribution characteristic of the temperature T_i and displaced by the drift velocity $\mathbf{v}_i$. The dynamical friction force is then obtained from Gauss' law in the form

$$f_{ei}=\gamma^{-2}eE_c\Psi(\gamma^{-1}z),$$

where E_c is again defined by Eq. (20). This expression has a maximum at the most probable random ion speed, and the critical field $\gamma^{-2}E_c$ is inversely proportional to kT_i/M.

The extension of the theory to a plasma whose electrons and ions possess random speeds of the same magnitude, i.e., $\gamma\approx1$, is readily carried out. Details of this calculation are presented in Appendix I. [See Eq. (42) and Fig. 11]. As one might expect, the sum of the most probable random electron and ion speeds enters into these results. The maximum of the dynamical friction occurs approximately when the relative drift velocity between the electrons and ions is equal to $[(2kT_i/M)+(2kT_e/m)]^{\frac{1}{2}}$, and the critical electric field is proportional to the inverse square of this quantity. We may conclude, therefore, that the

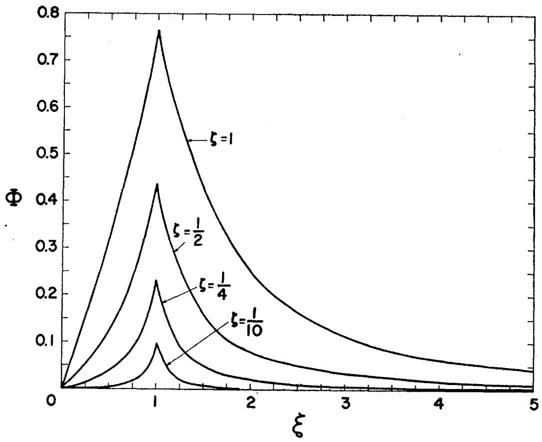

FIG. 10. The variation of the dynamical friction function $\Phi(\xi,\zeta)$ with the electron drift velocity for a range of values of the anisotropy parameter ζ.

random motion of the ions does not alter the phenomenon of runaway in any essential way.

6. Comparison with Other Work

The main results obtained in this paper are based upon the strong deviation of the dynamical friction force from a Stokes law, especially for large drift velocities. This behavior, embodied in the Ψ function, has led to fundamentally new results. In this connection it is useful to compare this work with the earlier work of Giovanelli[9] which is based upon a Stokes law over the entire velocity range and also results in a critical electric field. Giovanelli considers the balance between the rate at which electrons gain energy from the electric field and the rate at which they lose energy as a result of elastic collision with ions. He concludes that for electric fields in excess of a certain critical field E_c^{**} the electron energy grows in time without limit. E_c^{**} depends upon the ion mass and ion temperature and is given by

$$E_c^{**}=\frac{2\sqrt{3}}{9}\frac{\pi\alpha^2 m^{\frac{3}{2}} n}{u_0^2 e M^{\frac{1}{2}}},$$

where u_0 is the velocity of electrons whose kinetic energy equals the mean kinetic energy of the ions. The quantity α is related to the mean free path, l, for electron-ion encounter through the expressions

$$l=1/(\pi n\Omega^2),$$
$$\Omega=\alpha/u_1^2,$$

where the electron speed is represented by u_1. Giovanelli takes α to be equal to 1.57×10^9 cm^3-sec^{-2}.

This result may be compared with the critical field derived in this paper only if we let $\pi\Omega^2$ stand for the momentum transfer cross section. This yields

$$\frac{\pi\alpha^2}{u_1^4}=2\pi\frac{e^4}{4m^2u_1^4}\int_{\theta_{\text{minimum}}}^{\pi/2}(1-\cos\theta)\csc^4(\theta/2)\sin\theta d\theta,$$

or

$$\pi\alpha^2=\frac{4\pi e^4}{m^2}\ln\left(\frac{\lambda}{p_0}\right).$$

With this result Giovanelli's critical field becomes

$$E_c^{**}=\frac{8\pi\sqrt{3}}{9}\frac{e^3\ln(\lambda/p_0)}{u_0^2(mM)^{\frac{1}{2}}}.$$

There is a functional difference between this expression and our result [Eq. (20)] which arises in the following way. Giovanelli retains the Stokes law for all velocities and restricts his treatment to weak electric fields so that the energy gained by electrons between collisions is small compared to the average random speed. Under these circumstances the increase in the magnitude of the drift velocities is caused by Joule heating alone. He then finds that a runaway in electron energy develops when the electrons gain energy from the field faster than they can transfer it to the positive ions by elastic encounter. This effect depends strongly upon the mass ratio m/M and the ion temperature,

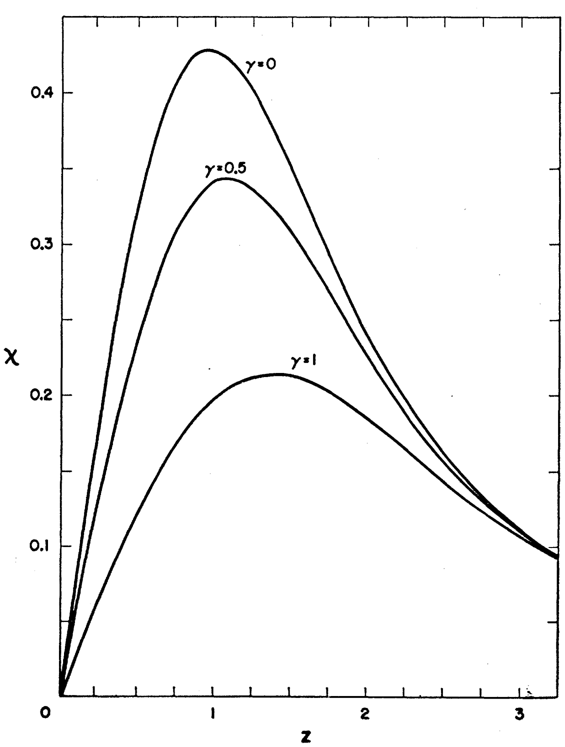

FIG. 11. The variation of the generalized dynamical friction function $\chi(\gamma;z)$ with relative electron-ion drift velocity for several values of the parameter γ.

[9] R. G. Giovanelli, Phil. Mag. 40, 206 (1949).

and accounts for the appearance of M and u_0 in E_c^{**}. In fact, for infinitely massive ions (i.e., $m/M=0$) E_c^{**} vanishes in a manner which is independent of the electron temperature and the particle density. The considerations which led to the critical field, E_c, derived in this paper are based upon the transfer of momentum between electrons and ions, and therefore our results are independent of the ion mass to the extent that m/M can be neglected compared to unity. In particular E_c does not vanish for the case of infinitely massive ions unless $n/T_e=0$. It is important to note that our critical field appears in the theory only because the dynamical friction force has a maximum when considered as a function of the drift velocity. This deviation from a Stokes law can in general result in the runaway of the drift velocities, even though little or no Joule heating takes place. Furthermore, as we have shown, this kind of runaway can develop in periods which are very short compared to the characteristic electron-ion energy exchange times which enter into the mechanism considered by Giovanelli.

C. Currents Across Magnetic Fields

In a uniform plasma subjected to a strong magnetic field, charged particles circle magnetic lines many times between collisions, and the application of a time independent electric field, perpendicular to $\mathbf{B}$ gives rise to the same drift velocity, $(\mathbf{E}\times\mathbf{B})/B^2$, for all charged particles. Under these circumstances, the dynamical friction force between the electron and ion gases is zero, and the question of runaway does not arise.

However, in the presence of a pressure gradient, ∇P, at right angles to the magnetic field the situation is somewhat more involved. If we ignore collisions there is a magnetization current

$$\mathbf{j}(\mathbf{r})=\nabla\times\mathbf{M}(\mathbf{r}), \tag{31}$$

where

$$\mathbf{M}=-\left[\frac{nk(T_e+T_i)}{B}\right]\frac{\mathbf{B}}{B},$$

which is at right angles both to B and ∇P at $\mathbf{r}$. The quantity $\mathbf{M}$ is the magnetic moment per unit volume which arises from the circular motion of electrons and ions about magnetic field lines. The current $\mathbf{j}$ is due to the incomplete cancellation of elementary particle currents at $\mathbf{r}$ and arises when a pressure gradient exists. Although the charges do not actually drift perpendicularly to $\mathbf{B}$ and ∇P, we can still assign drift velocities $\mathbf{v}_e$ and $\mathbf{v}_i$ to the electrons and ions which are a measure of this incomplete cancellation. The current $\mathbf{j}$ is then given by

$$\mathbf{j}=ne(\mathbf{v}_i-\mathbf{v}_e). \tag{32}$$

If we perturb this situation by introducing collisions, we give rise to a dynamical friction force, f_{ei}, which acts to oppose the relative drift between electrons and ions. First order orbit theory then gives rise to a diffusion drift[10]

$$\mathbf{v}_d=\frac{\mathbf{f}_{ei}\times\mathbf{B}}{eB^2},$$

which is in the same direction for both electrons and ions, namely along $(-\nabla P)$.

With the particles distributed according to displaced Maxwellian distributions, f_{ei} is given by Eq. (42) in Appendix I. However, within the accuracy of our treatment we may follow the comments in Sec. 5 and use the approximate formula given in Eq. (43), i.e.,

$$f_{ei}=\frac{eE_c}{\gamma^2+1}\Psi\left[\frac{z}{(\gamma^2+1)^{\frac{1}{2}}}\right], \tag{33}$$

where z is defined by Eq. (19). By combining Eqs. (31), (32), and (33) we find that the magnitude of Ψ depends upon the steepness of the pressure gradient.

Let us estimate the pressure gradient which results in the maximum value of Ψ. We proceed from the first-moment equations which have the form

$$m\frac{\partial\mathbf{v}_e}{\partial t}+m(\mathbf{v}_e\cdot\nabla)\mathbf{v}_e+\frac{1}{n}\nabla(nkT_e)+\frac{(\mathbf{v}_e-\mathbf{v}_i)}{|\mathbf{v}_e-\mathbf{v}_i|}f_{ei}=-e(\mathbf{E}+\mathbf{v}_e\times\mathbf{B}),$$

$$M\frac{\partial\mathbf{v}_i}{\partial t}+M(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i+\frac{1}{n}\nabla(nkT_i)+\frac{(\mathbf{v}_i-\mathbf{v}_e)}{|\mathbf{v}_e-\mathbf{v}_i|}f_{ei}=e(\mathbf{E}+\mathbf{v}_i\times\mathbf{B}).$$

For simplicity, B is now assumed to be along the z-axis and $\nabla(nkT_{e,i})$ along the x-axis of a Cartesian coordinate system. In the absence of collisions we find for the static drifts

$$(v_{e,i})_y=-\frac{1}{neB}\frac{\partial}{\partial x}(nkT_{e,i})-\frac{E_x}{B}, \tag{34a}$$

$$(v_{e,i})_x=E_y/B. \tag{34b}$$

In the presence of collisions these become modified to the form

$$v_{ey}=-\frac{1}{neB}\frac{\partial}{\partial x}(nkT_e+nmv_{ex}^2)-\frac{E_x}{B}, \tag{35a}$$

$$v_{iy}=-\frac{1}{neB}\frac{\partial}{\partial x}(nkT_i+nMv_{ix}^2)-\frac{E_x}{B}, \tag{35b}$$

$$v_{ex}=\frac{E_y}{B}+\frac{f_{ei}}{eB}+\frac{1}{neB}\frac{\partial}{\partial x}(nmv_{ex}v_{ey}), \tag{36a}$$

$$v_{ix}=\frac{E_y}{B}+\frac{f_{ei}}{eB}+\frac{1}{neB}\frac{\partial}{\partial x}(nMv_{ix}v_{iy}). \tag{36b}$$

[10] L. Spitzer, Astrophys. J. **116**, 299 (1952).

In the laboratory phenomena generally encountered,[11] f_{ei}/eB is smaller than the relevant random speeds. Moreover, the drift E_y/B due to induced electric fields must by definition be small in a static situation. The diffusion drifts v_{ex} and v_{ix} are therefore also smaller than the relevant random speeds, and we can neglect the drift energy compared to the random energy in Eqs. (35a) and (35b). The relative drift velocity is now given by

$$|v_{ey}-v_{iy}|=\frac{1}{neB}\left|\frac{\partial}{\partial x}[nk(T_e+T_i)]\right|. \quad (37)$$

The maximum of Ψ occurs when its argument becomes unity, or

$$\left(\frac{\beta_i\beta_e}{\beta_e+\beta_i}\right)^{\frac{1}{2}}|v_{ey}-v_{iy}|=1. \quad (38)$$

By combining Eqs. (37) and (38) we find that this condition requires appreciable variation in the particle pressures in a distance of one Larmor radius. For even steeper gradients the dynamical friction decreases, and the diffusion velocity v_d induced by electron-ion encounters falls off as the inverse square of the pressure gradient. In this event the diffusion caused by like-particle encounters remains.[12] Equations (33), (36a), and (36b) indicate the deviations which we may expect from the usual linear diffusion law. These deviations turn out to be relatively unimportant as long as the pressure variations occur over many Larmor radii.

ACKNOWLEDGMENTS

The author finds it a pleasure to express his appreciation to J. L. Tuck for his constant support and encouragement during the course of this work.

Alan Feldstein carried out the solution of Eqs. (23) and (29) on the 704 IBM at Los Alamos. His contribution to this problem is gratefully acknowledged.

APPENDIX I. EVALUATION OF THE DYNAMICAL FRICTION FORCE

In this section we calculate the dynamical friction force when both the ions and electrons are distributed according to displaced Maxwellian distributions. Following the analogy prescribed in Eq. (21) we easily obtain the potential H_{ei} given in Eq. (17). The dynamical friction force exerted upon an electron moving with the velocity $\mathbf{c}$ by all of the ions is given by

$$m\,\nabla_c H_{ei}=-eE_c\beta_e^{-1}\frac{\mathbf{q}}{q}\frac{d}{dq}\left[\frac{\mathcal{E}_2(\beta_i^{\frac{1}{2}}q)}{q}\right], \quad (39)$$

with the relative speed q defined as in Eq. (17b). The force exerted upon the entire electron gas acts in the direction of the relative drift velocity $(\mathbf{v}_i-\mathbf{v}_e)$, and we need only sum over the component of 39 in this direction. In the subsequent integration over all electron velocities we introduce a spherical coordinate system whose origin is at $\mathbf{v}_i$, and whose polar axis coincides with the unit vector $\mathbf{k}$ defined by

$$\mathbf{k}=(\mathbf{v}_e-\mathbf{v}_i)/|\mathbf{v}_e-\mathbf{v}_i|.$$

The cosine of the polar angle is then given by

$$\mu=(\mathbf{q}/q)\cdot\mathbf{k}.$$

The total dynamical friction force averaged over all electrons is given by

$$-\left(\frac{\beta_e}{\pi}\right)^{\frac{3}{2}}m\int\mathbf{k}\cdot\nabla_c H_{ei}\exp[-\beta_e|\mathbf{c}-\mathbf{v}_e|^2]d^3c$$
$$=-eE_c(2/\sqrt{\pi})\exp(-z^2)D(\gamma;z), \quad (40)$$

where

$$D(\gamma;z)=\int_0^\infty\int_{-1}^{+1}\mu\frac{\partial}{\partial x}\left[\frac{\mathcal{E}_2(x/\gamma)}{x}\right]\times\exp(-x^2+2zx\mu)d\mu x^2dx$$

$$z=\beta_e^{\frac{1}{2}}|\mathbf{v}_e-\mathbf{v}_i.$$

Integration over μ results in

$$D(\gamma;z)=\int_{-\infty}^{+\infty}\exp(-x^2+2zx)\frac{\phi(x)}{2zx}dx-\int_{-\infty}^{+\infty}\exp(-x^2+2zx)\frac{\phi(x)}{4z^2x^2}dx,$$

where we have used the definition

$$\phi(x)=x^2\frac{d}{dx}\left[\frac{\mathcal{E}_2(x/\gamma)}{x}\right],$$

and the oddness property $\phi(-x)=-\phi(x)$. Integration by parts results in

$$D(\gamma;z)=\frac{1}{2z^2}\int_{-\infty}^{+\infty}\phi(x)\exp(-x^2+2zx)dx-\frac{1}{4z^2}\int_{-\infty}^{+\infty}\frac{1}{x}\frac{d\phi}{dx}\exp(-x^2+2zx)dx,$$

and with the use of the definition

$$J(z;\gamma)=\int_{-\infty}^{+\infty}\mathcal{E}_2(x/\gamma)\exp(-x^2+2zx)dx,$$

[11] See, for example, J. Honsaker *et al.*, Nature **181**, 231 (1958); L. C. Burkhardt *et al.*, Nature **181**, 224 (1958); P. C. Thoneman *et al.*, Nature **181**, 217 (1958); T. Coors *et al.*, Physics of Fluids **1**, 411 (1958).

[12] C. L. Longmire and M. N. Rosenbluth, Phys. Rev. **103**, 507 (1956).

we can rewrite D in the form

$$D(\gamma;z)=\frac{\gamma^2+1}{(\pi)^{\frac{1}{2}}\gamma^3z^2}\int_{-\infty}^{+\infty}x\exp\left[-\left(\frac{\gamma^2+1}{\gamma^2}\right)x^2+2zx\right]dx -\frac{1}{2z^2}J(\gamma;z).$$

The first integral on the right-hand side of this equation can be evaluated analytically, and D is finally given by

$$D(\gamma;z)=\frac{1}{(\gamma^2+1)^{\frac{1}{2}}z}\exp\left[\left(\frac{\gamma^2}{\gamma^2+1}\right)z^2\right]-\frac{J(\gamma;z)}{2z^2}. \quad (41)$$

The total dynamical friction force is obtained by combining Eqs. (40) and (41):

Dynamical friction force

$$=-eE_c\left[\frac{2}{\sqrt{\pi}}\exp(-z^2)D(\gamma;z)\right]$$
$$=-eE_c\chi(\gamma;z). \quad (42)$$

The bracketed term which is represented by $\chi(\gamma;z)$ plays the role of the Ψ function introduced in Eq. (19). Indeed for $\gamma=0$ it is precisely the same function.

We have evaluated $J(z;\gamma)$ numerically, and $\chi(\gamma;z)$ is presented in Fig. 11 as a function of z with γ as a parameter. Within the accuracy of our treatment these curves may be represented by the single formula

$$\chi(\gamma;z)=\frac{1}{\gamma^2+1}\Psi\left[\frac{z}{(\gamma^2+1)^{\frac{1}{2}}}\right]. \quad (43)$$

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Third Law of Thermodynamics

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A new formulation of the third law is proposed stating a universal connection between the lower limits of the energy and the entropy of any physical system. As consequences of the new theorem are derived the Nernst heat theorem, a theorem concerning the lowest energy state of mixtures, and the nondegeneracy of the energetic ground state of physical systems.

1. INTRODUCTION

A WELL-KNOWN formulation of Nernst's heat theorem (NHT) is the following[1]: At absolute zero the entropy S of a chemically pure substance assumes the value zero. The term "chemically pure" requires a few remarks. The theorems of statistical thermodynamics state that a mixture of chemically different substances, a mixed crystal for instance, has an entropy different from zreo at any temperature and hence also at $T=0$. The same applies to a crystal consisting of two isotopes of the same element as long as the spatial arrangement of the isotopes shows the characteristics of a statistical mixture. If, finally, isotopes are excluded and only one sort of nuclei is allowed, one can assume internal degrees of freedom—nuclear spin or general state variables of the nuclei—and again these degrees of freedom can be the source of a statistical disorder which prevents the entropy from vanishing as T tends to zero. These considerations seem to prove that the NHT is not very valuable from a merely practical point of view because an unambiguous application to a given physical system requires all its degrees of freedom to be known. On the other hand, the theorem has a very successful history of applications even to systems of unknown internal degrees of freedom. This shows that the weakness of the theorem does not lie in its content but merely in its formulation.

Simon[2] has defined the term "chemically pure" as "being in internal equilibrium." It seems certain that this definition covers all cases where the NHT is valid.

To call the NHT the third law of thermodynamics however does not seem appropriate because of the explicit use of the concept of temperature. The actual meaning of the third law is a universal connection between the energy and the entropy of *any* physical system. Hence the explicit use of a property which cannot be defined for all physical systems is certainly not suitable to formulate a law of such generality.

The first and the second law of thermodynamics can be considered as theorems concerning the existence of the two quantities energy and entropy. These laws are

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[1] M. Planck, *Vorlesungen über Thermodynamik* (W. de Gruyter, Berlin and Leipzig, 1930), nineth edition, Chap. IV, p. 6.

[2] F. Simon, Z. Physik **41**, 806 (1927); Ergeb. exakt. Naturw. **9**, 222 (1930).