Structure Theorem for the Photon Propagator

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A simple theorem relating the structure of the bare and dressed photon commutators and propagators is derived, and its implication, with respect to the choice of photon gauge, is discussed.

1. INTRODUCTION

IN a recent paper, Bogoljubov and Sirkov[1] have criticized the form of the photon propagator assumed by Gell-Mann and Low,[2] and have emphasized the need of using a gauge-invariant form in order to obtain a consistent renormalization procedure. Such a gauge-invariant propagator has also been utilized by Umezawa *et al.*,[3,4] in their discussion of the renormalization program. Because of this evident need to consider a gauge-invariant quantity (thereby avoiding the danger of mishandling gauge-variant terms), it seems worthwhile to state a simple but pertinent theorem relating the structure of the bare photon propagator to that of the dressed (unrenormalized) propagator. The physical content of this theorem may be expressed by the statement that the gauge-variant portion of the bare propagator is unchanged by the fermion-photon interaction; and that although one usually expects to renormalize only the gauge-invariant part of the dressed propagator, nevertheless its gauge-variant portion must also be renormalized if the dressed photon operators are to be proportional, but not equal, to the renormalized photon operators. This observation is made plausible by an examination of the propagator constructed in reference 3; but since the theorem also provides a similar relation connecting the bare and dressed commutators, it perhaps merits a formal statement and proof.

2. DERIVATION

Consider first the unrenormalized photon commutator

$$iD_{\mu\nu}'(x-x')=\langle 0|\eta[A_\mu(x),A_\nu(x')]|0\rangle,$$

where the photon and current Heisenberg operators satisfy the relations

$$\Box A_\mu=J_\mu,\quad \partial_\mu J_\mu=0.$$

Under the following general assumptions:

(*a*) $D_{\mu\nu}'(x)|_{x_0=0}=D_{\mu\nu}(x)|_{x_0=0}=0;$

(*b*) $\dfrac{\partial}{\partial x_0}D_{\mu\nu}'(x)|_{x_0=0}=\dfrac{\partial}{\partial x_0}D_{\mu\nu}(x)|_{x_0=0}=\delta_{\mu\nu}\delta(\mathbf{r});$

(*c*) at equal times the photon and current operators commute;

(*d*) all matrix elements of $J_\mu(x)$ vanish outside of a sufficiently large volume in configuration space;

the theorem to be proved states that

$$\partial_\mu\partial_\nu D_{\mu\nu}'(x)=\partial_\mu\partial_\nu D_{\mu\nu}(x). \quad (1)$$

With the usual choice[5] of the free-field commutator, $D_{\mu\nu}=\delta_{\mu\nu}D(x)$, the right-hand side of Eq. (1) vanishes. The dressed commutator $D_{\mu\nu}'$ can therefore not have the form $D_{\mu\nu}'=\delta_{\mu\nu}D'+\partial_\mu\partial_\nu G$, where G is a gauge-variant function satisfying $\Box G=0$, since such form together with Eq. (1) implies that $\Box D'=0$. On the contrary, G must contain a (gauge-invariant) term, G_1, which obeys the relation $\Box D'+\Box\Box G_1=0$.

A similar statement holds for the unrenormalized photon propagator,

$$\partial_\mu\partial_\nu D_{F\mu\nu}'(x)=\partial_\mu\partial_\nu D_{F\mu\nu}(x), \quad (2)$$

where, with the usual choice[5] of the bare propagator, $D_{F\mu\nu}=\delta_{\mu\nu}D_F(x)$, the right-hand side of Eq. (2) does not vanish. This relation will be proved in a manner similar to that used in deriving Eq. (1), and also by direct construction in momentum space using the gauge-invariant form of the photon self-energy $\Pi_{\mu\nu}$.

The proof of Eq. (1) may be obtained by first calculating

$$\Box D_{\mu\nu}'(x)=-i\langle 0|\eta[J_\mu(x),A_\nu(0)]|0\rangle, \quad (3)$$

and then finding the solution of Eq. (3) which obeys the boundary conditions of assumptions (*a*) and (*b*). Writing

$$D_{\mu\nu}'(x)=D_{\mu\nu}(x)-i\int d^4y\,\mathfrak{D}(x,y)\times\langle 0|\eta[J_\mu(y),A_\nu(0)]|0\rangle, \quad (4)$$

Eq. (4) will be a unique solution for $D_{\mu\nu}'(x)$ if there exists a Green's function $\mathfrak{D}(x,y)$ with the properties

$$\Box_x\mathfrak{D}=\delta(x-y),\quad \mathfrak{D}|_{x_0=0}=0,\quad \partial\mathfrak{D}/\partial x_0|_{x_0=0}=0, \quad (5)$$

for all points y. [If there is a particular function satisfying Eq. (5) it must be unique.] Constructing $\mathfrak{D}$ out of the available Green's functions: it must contain one

[1] N. N. Bogoljubov and D. V. Sirkov, Nuovo cimento 3, 845 (1956).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[3] Umezawa, Tomozawa, Konuma, and Kamefuchi, Nuovo cimento 3, 772 (1956).

[4] S. Kamefuchi and H. Umezawa, Nuovo cimento 3, 1060 (1956).

[5] This assumption will be held until the discussion following Eq. (12). As the remarks following Eqs. (6) and (10) indicate, both Eqs. (1) and (2) hold in the stronger sense: $\partial_\mu D_{\mu\nu}'=\partial_\mu D_{\mu\nu}$, $\partial_\mu D_{F\mu\nu}'=\partial_\mu D_{F\mu\nu}$.

member which is a solution of the inhomogeneous wave equation with a $\delta(x-y)$ source; further, any of the Green's functions obeying the homogeneous wave equation may be multiplied by a factor $\epsilon(\pm y_0) = \pm y_0/|y_0|$ with a resulting change in the form of their boundary conditions. If one uses the convention of defining all Green's functions in terms of the usual contour integrals, each having the common multiplicative factor $-(2\pi)^{-4}$, that function which satisfies the conditions of Eq. (5) can be written as

$$\mathfrak{D}(x,y) = D_F(x-y) - \tfrac{1}{2}D^{(1)}(x-y) + \tfrac{1}{2}\epsilon(-y_0)D(x-y) = \tfrac{1}{2}\{\epsilon(-y_0) - \epsilon(x_0-y_0)\}D(x-y),$$

and Eq. (4) then becomes[6]

$$D_{\mu\nu}'(x) = D_{\mu\nu}(x) - \frac{i}{2}\int d^4y\{\epsilon(-y) - \epsilon(x-y)\} \times D(x-y)\langle 0|\eta[J_\mu(y), A_\nu(0)]|0\rangle. \quad (6)$$

The remainder of the proof then consists of applying ∂_μ to both sides of Eq. (6), and using current conservation to obtain

$$\left(\frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial y_\mu}\right)\{\epsilon(-y) - \epsilon(x-y)\}D(x-y) \times \langle 0|\eta[J_\mu(y), A_\nu(0)]|0\rangle = 2i\delta(y_0)D(x-y)\langle 0|\eta[J_4(y), A_\nu(0)]|0\rangle.$$

By assumption (c), the equal time commutator of J_4 and A_ν vanishes; what remains is the relation

$$\partial_\mu D_{\mu\nu}'(x) = \partial_\mu D_{\mu\nu}(x) + \frac{i}{2}\int d\sigma_\mu\{\epsilon(-y) - \epsilon(x-y)\} \times D(x-y)\langle 0|\eta[J_\mu(y), A_\nu(0)]|0\rangle.$$

By assumption (d) this surface integral reduces to (the difference of) two volume integrals as $y_0 \to \pm\infty$. Finally, since the limit $\{\epsilon(-y) - \epsilon(x-y)\}_{y_0\to\pm\infty} = 0$, each such volume integral multiples a zero coefficient. [One needs the reasonable implicit assumption that all matrix elements of $J_\mu(y)$ yield well-behaved volume integrals as $y_0 \to \pm\infty$.] Operation with ∂_ν [necessary if Eq. (3) is written symmetrically in J_μ and A_ν] then yields Eq. (1). This relation is satisfied by the parametric representations of the commutators as given by Källén[7] and Wightman.[8]

The corresponding relation between the propagators, Eq. (2), can be obtain by writing

$$D_{F\mu\nu}'(x-x') = \langle 0|\eta T_{x,x'}(A_\mu(x)A_\nu(x'))|0\rangle,$$

which, using the previous assumptions (a) and (b), satisfies the equation

$$\Box D_{F\mu\nu}'(x) = \Box D_{F\mu\nu}(x) + \langle 0|\eta T_{x,0}(J_\mu(x)A_\nu(0))|0\rangle. \quad (7)$$

The boundary conditions on $D_{F\mu\nu}'$ suggest a different approach, at this point, from that employed above. Applying the operator $\partial_\mu\partial_\nu$ (or just ∂_μ alone) to both sides of Eq. (7), using current conservation and assumption (c), one obtains

$$\partial_\mu\partial_\nu\Box(D_{F\mu\nu}'(x) - D_{F\mu\nu}(x)) = 0,$$

which has the solution

$$\partial_\mu\partial_\nu(D_{F\mu\nu}' - D_{F\mu\nu}) = aD(x) + bD^{(1)}(x), \quad (8)$$

where a and b are constants.[9] The boundary conditions imposed on $D_{F\mu\nu}'$, as well as on $D_{F\mu\nu}$, are such that the left-hand side of Eq. (8) has positive frequency dependence for $x_0 > 0$ and negative frequency dependence for $x_0 < 0$; the right-hand side of Eq. (8), however, has a frequency dependence proportional to

$$a\sin kx_0 - ib\cos kx_0 = -\tfrac{1}{2}i[(b+a)e^{ikx_0} + (b-a)e^{-ikx_0}], \quad k \geq 0.$$

For $x_0 > 0$ one must require $b+a=0$, while for $x_0 < 0$ the requirement is $b-a=0$; therefore $a=b=0$, and Eq. (8) reduces to Eq. (2).

3. DISCUSSION

In momentum space Eq. (2) reads

$$k_\mu k_\nu D_{F\mu\nu}'(k) = k_\mu k_\nu D_{F\mu\nu}(k), \quad (9)$$

and may be obtained in a direct way by the use of the relation

$$D_{F\mu\nu}'(k) = D_{F\mu\nu}(k) + D_{F\mu\lambda}(k)\Pi_{\lambda\sigma}(k)D_{F\sigma\nu}'(k), \quad (10)$$

where the function $\Pi_{\mu\nu}(k)$ is the sum of all proper photon self-energy parts, assumed to have the gauge-invariant form

$$\Pi_{\mu\nu}(k) = (\delta_{\mu\nu} - k_\mu k_\nu/k^2)\Pi(k^2).$$

[6] Performing the y integration of Eq. (6) will yield a covariant result on the indices μ,ν because the function multiplying $\mathfrak{D}(x,y)$ has the space-time dependence of a commutator; that is, the quantity

$$\int_0^\infty d\kappa^2\,\pi(\kappa^2)\int d^4\kappa\,\epsilon(k)\delta(k^2+\kappa^2)\int d^4y\,\mathfrak{D}(x,y)e^{ik\cdot y}$$

is an invariant function of x. An extension of this procedure permits one to attempt a spectral representation for $D_{\mu\nu}'$ without assuming relations between the one-photon matrix elements of A_μ and its asymptotic form, as used in reference 7.

[7] G. Källén, Helv. Phys. Acta **25**, 417 (1952). Both Eqs. (1) and (2) are satisfied as relations between the renormalized fields A_μ and $A_\mu^{(0)}$, even through the corresponding commutators obey different spacelike commutation relations; this is because Källén's formalism is just an operator gauge transformation away from the more conventional equation for the renormalized fields.

[8] A. S. Wightman (private conversation).

[9] This argument is shorthand for setting Eq. (8) equal to

$$\int d^4y\{D(x-y)f(y) + D^{(1)}(x-y)g(y)\},$$

where f and g are even functions of y but are otherwise arbitrary.

With $D_{F\mu\nu}=\delta_{\mu\nu}D_F(k^2)$ and $D_F'=D_F+D_F\Pi D_F'$, Eq. (10) requires that $D_{F\mu\nu}'$ have the form

$$D_{F\mu\nu}'(k)=(\delta_{\mu\nu}-k_\mu k_\nu/k^2)D_F'(k^2)+(k_\mu k_\nu/k^2)D_F(k^2),$$

which evidently satisfies Eq. (9). One sees that if *only* the gauge-invariant part of $D_{F\mu\nu}'$ is renormalized,

$$D_F'=Z_3 D_{Fc}', \tag{11}$$

then one cannot simultaneously write

$$A_\mu=Z_3^{\frac{1}{2}}A_{\mu c}, \quad D_{F\mu\nu}'=Z_3 D_{F\mu\nu c}', \tag{12}$$

since the latter statement, together with Eqs. (9) and (11), leads to the requirement that $Z_3=1$.

If one begins with the more general bare propagator

$$D_{F\mu\nu}(k)=(\delta_{\mu\nu}-\lambda k_\mu k_\nu/k^2)D_F(k^2),$$

Eq. (10) then requires the dressed propagator to have the form

$$D_{F\mu\nu}'=(\delta_{\mu\nu}-k_\mu k_\nu/k^2)D_F'+(1-\lambda)(k_\mu k_\nu/k^2)D_F.$$

In order for the usual multiplicative renormalization of Eqs. (11) and (12) to hold, one must choose

$$D_{F\mu\nu c}'=(\delta_{\mu\nu}-k_\mu k_\nu/k^2)D_{Fc}'+[(1-\lambda)/Z_3](k_\mu k_\nu/k^2)D_F.$$

Since Z_3 is probably zero, in the interest of keeping a finite function finite one should probably choose $\lambda=1$; the propagators then have manifestly gauge-invariant forms, with Eqs. (2) and (9) vanishing identically.

Aside from this desirable property, it is interesting to note that in this special gauge, calculations to first order in e^2 yield a convergent result for the renormalization constants Z_1 and Z_2; that is, the only divergence present in the electron's self-energy, Σ, is that corresponding to a mass renormalization, δm.[10] Subtracting the latter, the quantity $\Sigma-\delta m$ is finite, as, by virtue of Ward's identity, is the corresponding vertex operator.

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[10] H. M. Fried and D. R. Yennie, Phys. Rev. **112**, 1391 (1958). Set $\zeta=-1$ in Eq. (2.13) and compare Eqs. (2.14) and (2.11).