Three-Pion Contribution to the Electromagnetic Structure of the Nucleon

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In this paper the contribution of three pions to the scalar electromagnetic form factors of the nucleon are computed by using the fixed source meson theory without rescattering corrections. The $(\gamma,3\pi)$ interaction is taken phenomenologically as a point interaction. It is shown that for those values of the $(\gamma,3\pi)$ coupling constant compatible with photoproduction experiments, the experimental charge distribution could be roughly fitted with a cutoff in the dispersion integral of the order of 7 pion masses.

1. INTRODUCTION

The electromagnetic structure of the nucleon has been in the last year the object of considerable attention by theorists. Recently Federbush, Goldberger, and Treiman,[1] and Chew, Karplus, Gasiorowicz, and Zachariasen,[2] have investigated the problem using dispersion theory. The agreement between theory and experiment is still rather unsatisfactory.[3] This is probably due to our still incomplete knowledge of the physics of the nucleon.

A very important and completely open question is the scalar part (in isotopic spin) of the nucleon form factors, which cannot be explained in terms of the simple model of a photon interacting with a single pion in the nucleon cloud. FGT have shown that the only reasonably possible way of interpreting the large scalar radius of the charge is in terms of three-pion states. This model is analogous to a previous one proposed by Tamm[4]: a photon produces three pions by passing through nucleon-antinucleon intermediate states. Those pions are then absorbed by the physical nucleon.

In this paper we want to consider in some detail the contribution from three-pion states.

Our point of view will be the following. We shall phenomenologically postulate the only possible local $(\gamma,3\pi)$ interaction,[5]

$$H_{\text{int}}=ie(\lambda/\mu^3)\epsilon_{\lambda\mu\nu\rho}\frac{\partial\phi_1}{\partial x_\mu}\frac{\partial\phi_2}{\partial x_\nu}\frac{\partial\phi_3}{\partial x_\rho}. \qquad (1)$$

$\phi_i(x)$ are the real pion fields and $\epsilon_{\lambda\mu\nu\rho}$ is the completely antisymmetrical fourth rank tensor. More complicated nonlocal interactions are not considered because of the high mass of the intermediate nucleon-antinucleon states responsible for the effect expressed by the interaction (1).

The constant λ is a new unknown parameter. It could in principle be obtained from very accurate measurements of the cross section for photoproduction of two pions on a nucleon. Chew and Low[6] have suggested a general extrapolation method which could be used to separate the effects of the interaction (1) from the remaining "nucleon" terms (see Fig. 1). For the moment we can only give a rough upper limit for λ by requiring the effect of (1) not to exceed the whole experimental cross section for $\gamma+N\rightarrow N+\pi+\pi$. We obtain $\lambda<\sim6$.

The expectation value of $H_{\text{int}}(x)$ in the physical nucleon state is computed in the static model neglecting the contribution of the $\frac{3}{2}$, $\frac{3}{2}$ resonance, which has been proved[7] to be unimportant in the two-pion contribution.

The calculation of the magnetic moment and of the charge is going to diverge very badly. However, we shall be able to express all observable quantities in term of a spectral representation which is the static counterpart of the relativistic dispersion relations,

$$G_1{}^S(q^2)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{g_1{}^S(\sigma^2)}{\sigma^2+q^2}d\sigma^2,$$

$$G_2{}^S(q^2)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{g_2{}^S(\sigma^2)}{\sigma^2+q^2}d\sigma^2.$$

The spectral functions will be finite and strongly increasing with the "mass" σ.

We are aware that our numerical results can only be taken as a rough indication. The main reason is that, as in the case of two pions, the recoil effects of the

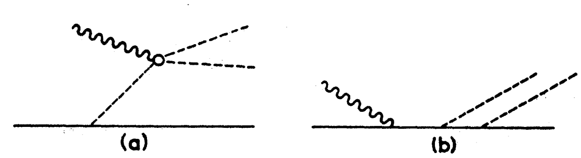

FIG. 1. Two possible graphs contributing to two-pion photoproduction; (a) shows the graph by which our interaction [Eq. (1)] contributes to two-pion photoproduction; (b) is a typical nucleon graph of the same process. Nucleon lines are solid, photon lines wavy, and meson lines broken.

[1] Federbush, Goldberger, and Treiman, Phys. Rev. **112**, 643 (1958), hereafter referred to as FGT.

[2] Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958).

[3] An excellent account of the present theoretical situation can be found in S. Drell, *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958), p. 20.

[4] See, e.g., I. Tamm, *1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958), p. 34.

[5] We use here $x_1=x$, $x_2=y$, $x_3=z$, $x_4=it$, and a system of units in which $\hbar=c=1$.

[6] G. F. Chew and F. E. Low (to be published).

[7] S. Fubini, Nuovo cimento **3**, 1425 (1956).

nucleon can be very important[8] and therefore a relativistic dispersion analysis of the problem is very desirable. However, we feel that this first field-theoretical exploration of the problem might be of some use since it gives the rigorous $1/M\rightarrow 0$ limit of any relativistic analysis without rescattering.

2. CHARGE AND MAGNETIC MOMENT DENSITY

In order to obtain charge and magnetic moment we have to evaluate $\langle\Psi_0|H_{\rm int}(x)|\Psi_0\rangle$, where $|\Psi_0\rangle$ is the physical nucleon state.

By expanding the pion fields $\phi_i(x)$ in the usual way,

$$\phi_i(x)=\sum\nolimits_{\mathbf{k}}\frac{1}{(2\omega_k)^{\frac{1}{2}}}\{a_{i\mathbf{k}}e^{ik\cdot x}+a_{i\mathbf{k}}{}^\dagger e^{-ik\cdot x}\},$$

$H_{\rm int}(x)$ can be cast into a linear combination of expectation values in the physical nucleon states of three pion creation or annihilation operators. To be more specific, the different kinds of expectation values are the following:

$$\langle\Psi_0|a_pa_qa_k|\Psi_0\rangle,\quad \langle\Psi_0|a_p{}^\dagger a_qa_k|\Psi_0\rangle, \tag{2}$$

and their complex conjugates.

We must now evaluate expressions (2) using the static model and neglecting the rescattering corrections, as previously stated. In doing this it is important to pay attention to the order of meson operators: all the creation operators must be placed to the left of the annihilation operators.[9]

Let us use the following identities:

$$\begin{aligned}\langle\Psi_0|[H,a_pa_qa_k]|\Psi_0\rangle&=0,\\ \langle\Psi_0|[H,a_p{}^\dagger a_qa_k]|\Psi_0\rangle&=0,\end{aligned} \tag{3}$$

where $H=H_0+H'$; H_0 is the free-pion field's Hamiltonian, and H' is the usual static pion-nucleon interaction,

$$\begin{aligned}H'&=\sum\nolimits_k(V_ka_k+V_k{}^\dagger a_k{}^\dagger),\\ V_k&=i(4\pi)^{\frac{1}{2}}(f^{(0)}/\mu)\boldsymbol{\sigma}\cdot\mathbf{k}\tau_kv(k)(2\omega_k)^{-\frac{1}{2}}.\end{aligned}$$

By explicit evaluation of the commutator one obtains:

$$\begin{aligned}-(\omega_p+\omega_q+\omega_k)&\langle\Psi_0|a_pa_qa_k|\Psi_0\rangle\\ &=\langle\Psi_0|V_p{}^\dagger a_qa_k|\Psi_0\rangle+\langle\Psi_0|V_q{}^\dagger a_ka_p|\Psi_0\rangle\\ &\qquad+\langle\Psi_0|V_k{}^\dagger a_pa_q|\Psi_0\rangle.\end{aligned} \tag{3'}$$

$$\begin{aligned}-(\omega_q+\omega_k-\omega_p)&\langle\Psi_0|a_p{}^\dagger a_qa_k|\Psi_0\rangle\\ &=\langle\Psi_0|V_k{}^\dagger a_p{}^\dagger a_q|\Psi_0\rangle+\langle\Psi_0|V_q{}^\dagger a_p{}^\dagger a_k|\Psi_0\rangle\\ &\qquad-\langle\Psi_0|V_pa_qa_k|\Psi_0\rangle.\end{aligned} \tag{3''}$$

Let us now remember the identities:

$$\begin{aligned}a_k|\Psi_0\rangle&=-(\omega_k+H)^{-1}V_k{}^\dagger|\Psi_0\rangle,\\ a_ka_{k'}|\Psi_0\rangle&=(\omega_k+\omega_{k'}+H)^{-1}V_k{}^\dagger(\omega_{k'}+H)^{-1}V_{k'}{}^\dagger|\Psi_0\rangle\\ &\quad+(\omega_k+\omega_{k'}+H)^{-1}V_{k'}{}^\dagger(\omega_k+H)^{-1}V_k{}^\dagger|\Psi_0\rangle.\end{aligned} \tag{4}$$

From (3′) and (3″), taking in account (4), we finally obtain, in the zero-meson approximation, for the needed expectation values,

$$\begin{aligned}\langle\Psi_0|a_pa_qa_k|\Psi_0\rangle&\\ =\frac{-i(f/\mu)^3(4\pi)^{\frac{3}{2}}v(p)v(q)v(k)}{(\omega_p+\omega_q+\omega_k)(8\omega_p\omega_q\omega_k)^{\frac{1}{2}}}&\times\bigg\{\frac{(\sigma\tau)_p(\sigma\tau)_q(\sigma\tau)_k}{(\omega_q+\omega_k)\omega_k}+\frac{(\sigma\tau)_p(\sigma\tau)_k(\sigma\tau)_q}{(\omega_q+\omega_k)\omega_q}\\ &+\frac{(\sigma\tau)_q(\sigma\tau)_p(\sigma\tau)_k}{(\omega_p+\omega_k)\omega_k}+\frac{(\sigma\tau)_q(\sigma\tau)_k(\sigma\tau)_p}{(\omega_p+\omega_k)\omega_p}\\ &+\frac{(\sigma\tau)_k(\sigma\tau)_p(\sigma\tau)_q}{(\omega_p+\omega_q)\omega_q}+\frac{(\sigma\tau)_k(\sigma\tau)_q(\sigma\tau)_p}{(\omega_p+\omega_q)\omega_p}\bigg\},\end{aligned} \tag{5}$$

$$\begin{aligned}\langle\Psi_0|a_p{}^\dagger a_qa_k|\Psi_0\rangle&\\ =\frac{i(f/\mu)^3(4\pi)^{\frac{3}{2}}}{(8\omega_p\omega_q\omega_k)^{\frac{1}{2}}}v(p)v(q)v(k)&\times\bigg\{\frac{(\sigma\tau)_p(\sigma\tau)_k(\sigma\tau)_q}{\omega_q\omega_p(\omega_q+\omega_k)}+\frac{(\sigma\tau)_p(\sigma\tau)_q(\sigma\tau)_k}{\omega_p\omega_k(\omega_q+\omega_k)}\bigg\}.\end{aligned} \tag{6}$$

In (5) and (6) f is the renormalized pion-nucleon coupling constant and $(\sigma\tau)_p$ means $\boldsymbol{\sigma}\cdot\mathbf{p}\tau_p$. Expressions (5), (6), and their complex conjugates, after quite simple but rather tedious calculation allow us to write down the expression for the charge and current density:

$$\begin{aligned}\rho(\mathbf{x})=-\frac{e(\lambda/\mu^3)(f/\mu)^3(4\pi)^{\frac{3}{2}}}{(2\pi)^9}&\times\int\int\int\frac{(\mathbf{k}_1\cdot\mathbf{k}_2\times\mathbf{k}_3)^2}{\omega_{k_1}{}^2\omega_{k_2}{}^2\omega_{k_3}{}^2}v(k_1)v(k_2)v(k_3)\\ &\times\exp\{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{x}\}d^3k_1d^3k_2d^3k_3,\end{aligned} \tag{7}$$

$$\begin{aligned}\mathbf{j}(\mathbf{x})=\frac{3ie(\lambda/\mu^3)(f/\mu)^3(4\pi)^{\frac{3}{2}}}{(2\pi)^9}&\times\int\int\int\frac{(\boldsymbol{\sigma}\cdot\mathbf{k}_1)(\mathbf{k}_2\cdot\mathbf{k}_3)(\mathbf{k}_1\times\mathbf{k}_3)}{\omega_{k_1}{}^2\omega_{k_2}{}^2\omega_{k_3}}\\ &\times\left(1+\frac{\omega_{k_2}-\omega_{k_3}}{\omega_{k_2}+\omega_{k_3}}\right)v(k_1)v(k_2)v(k_3)\\ &\times\exp\{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{x}\}d^3k_1d^3k_2d^3k_3.\end{aligned} \tag{8}$$

3. SPECTRAL REPRESENTATION OF CHARGE AND MAGNETIC MOMENT

Our purpose is now to obtain a spectral representation for charge and magnetic moment corresponding to (7) and (8).

[8] J. D. Walecka (to be published); and reference 1 and 2.

[9] This point is discussed in detail in Bosco, Fubini, and Stanghellini, Nuclear Phys. 10, 663 (1959).

TABLE I. Values of α, magnetic moment, and λ for different $\sigma_{\max}$, the charge being normalized to one, in units $e/2$.

$\sigma_{\max}$	Charge	α	Magnetic moment	λ
5	1	0.047	1.7	130
7	1	0.026	1.1	5.4
11	1	0.011	0.70	0.19
15	1	0.0063	0.48	0.025

From here on we neglect the cutoff functions, since they affect only the high-energy behavior of our integrands.

From (7), by performing the integration, one obtains

$$\rho(r)=-\frac{3}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{\pi^{\frac{3}{2}}}(2/r^9+6\mu/r^8+7\mu^2/r^7+4\mu^3/r^6+\mu^4/r^5)e^{-3\mu r}. \quad (9)$$

Our final step is to put $\rho(r)$ in the form

$$\rho(r)=\int_{3\mu}^{\infty}F(\sigma)\frac{e^{-r\sigma}}{r}d\sigma. \quad (10)$$

With an obvious change of variable we get

$$\rho(r)=e^{-3\mu r}\int^{\infty}F(\xi+3\mu)\frac{e^{-\xi r}}{r}d\xi.$$

We can immediately obtain $G(\xi)\equiv F(\xi+3\mu)$ as the inverse Laplace transform of $r\rho(r)e^{3\mu r}$. Furthermore, since $r\rho(r)e^{3\mu r}$ is of the form $\sum_{n=4}^{8}(c_n/r^n)$, $F(\sigma)\equiv G(\sigma-3\mu)$ is of the form $\sum_n[(\sigma-3\mu)^{n-1}/(n-1)!]$ which insures that the spectral function vanishes at the lower limit of integration.

In this way, from (9) one obtains

$$\rho(r)=-\frac{1}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{840\pi^{\frac{3}{2}}}\int_{3\mu}^{\infty}d\sigma\frac{e^{-r\sigma}}{r}(\sigma-3\mu)^3\times(\sigma^4+9\sigma^3\mu+12\sigma^2\mu^2-3\sigma\mu^3-3\mu^4). \quad (11)$$

If, as usual, we consider the Fourier transform $G_1^S(q^2)$, we get

$$G_1^S(q^2)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{g_1^S(\sigma^2)}{\sigma^2+q^2}d\sigma^2,$$

where

$$g_1^S(\sigma^2)=\frac{\pi^{\frac{1}{2}}e\lambda f^3}{1680}\left(\frac{\sigma}{\mu}-3\right)^3\left[\left(\frac{\sigma}{\mu}\right)^3+9\left(\frac{\sigma}{\mu}\right)^2+12\frac{\sigma}{\mu}-3-3\left(\frac{\sigma}{\mu}\right)^{-1}\right]. \quad (12)$$

The analogous calculation for the magnetic moment yields

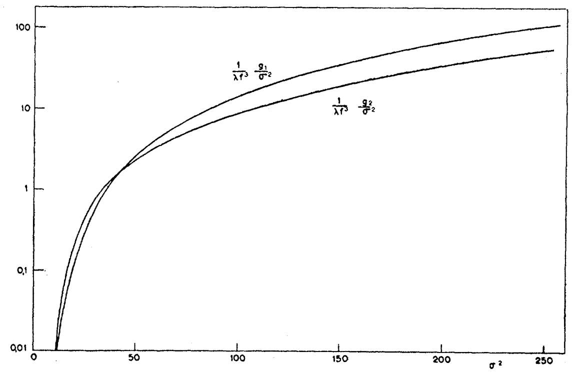

FIG. 2. Spectral functions $(1/\lambda f^3)[g_1^S(\sigma^2)/\sigma^2]$ and $(1/\lambda f^3)[g_2^S(\sigma^2)/\sigma^2]$ *versus* σ^2.

$$G_2^S(q^2)=\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{g_2^S(\sigma^2)}{\sigma^2+q^2}d\sigma^2,$$

where

$$\begin{aligned}g_2^S(\sigma^2)=\frac{3e\lambda f^3}{2\pi^{\frac{1}{2}}\mu}\Bigg\{&(\beta^2-1)^{\frac{1}{2}}\Bigg[\frac{1}{5040}\left(\frac{\sigma}{\mu}\right)^{-3}\times(200-105\beta+112\beta^2-70\beta^3-140\beta^5-216\beta^7)\\&-\frac{1}{720}\left(\frac{\sigma}{\mu}\right)^{-1}(112+15\beta+80\beta^2+10\beta^3-16\beta^5)\\&+\frac{1}{18}(1+2\beta^2)+\frac{1}{96}\frac{\sigma}{\mu}(16+3\beta+2\beta^3)\\&-\frac{1}{45}\left(\frac{\sigma}{\mu}\right)^2(5+\beta^2)+\frac{1}{210}\left(\frac{\sigma}{\mu}\right)^4\Bigg]-\frac{1}{96}\Bigg[2\left(\frac{\sigma}{\mu}\right)^{-3}\\&+2\left(\frac{\sigma}{\mu}\right)^{-1}-3\frac{\sigma}{\mu}\Bigg]\ln[\beta+(\beta^2-1)^{\frac{1}{2}}]\Bigg\}, \quad (13)\end{aligned}$$

and $\beta=(\sigma-\mu)/2\mu$. Its derivation is similar to the previous one, though much more cumbersome. The details are given in the Appendix.

It is evident from the expressions just obtained for $g_1^S(\sigma^2)$ and $g_2^S(\sigma^2)$, that our integrals are strongly divergent. In order to obtain some indication we shall limit our integrals to some values $\sigma_{\max}$ of the "mass."

In Fig. 2 are shown the curves representing $(1/\lambda f^3)[g_1^S(\sigma^2)/\sigma^2]$ in units $e/2$, and $(1/\lambda f^3)[g_2^S(\sigma^2)/\sigma^2]$ in units $e/2M$.

We give in Table I the values of the scalar magnetic moment and of $\alpha=\frac{1}{6}\langle r_{1S}^2\rangle$ which experimentally is between $0.03/\mu^2$ and $0.05/\mu^2$, for different values of $\sigma_{\max}$. These values are obtained by normalizing the charge to one (in units $e/2$). In the last column the corresponding values of λ are given. α is in units $1/\mu^2$.

4. CONCLUSIONS

It has been shown in the preceding sections that the three-pion contribution to the electromagnetic structure of the nucleon can be simply computed in the static no-rescattering approximation.

From this preliminary investigation it is still difficult to draw any conclusion whether the three-pion effects could reasonably account for the scalar part of the charge and magnetic moment of the nucleon.

From our calculation it follows that for those values of λ compatible with photoproduction experiments, the experimental charge distribution could be roughly fitted with a cutoff in the dispersion integrals of the order of 7μ.

The magnetic moment (whose experimental value is -0.06 nuclear magneton) comes out rather large and with the wrong (positive) sign.

It is difficult to predict whether accounting for the recoil effects would remove such discrepancy.

One trivial $1/M$ correction is to define $G_2{}^S(0)=\mu_S+\frac{1}{2}$ (in units $e/2M$), because the expectation value of the current in the physical nucleon is related to the total magnetic moment. Such a correction goes in the right direction but is still insufficient.

Therefore, we should finally like to point out that a relativistic computation of the three-pion effects would be very desirable in order to see whether a large charge effect is compatible with a small magnetic moment.

ACKNOWLEDGMENT

We are deeply indebted to Professor S. Fubini for suggesting this investigation, and for constant advice and illuminating discussions.

APPENDIX

We start from (8) and perform the angular integration. We get

$$\mathbf{j}(\mathbf{x})=-\frac{3e(\lambda/\mu^3)(f/\mu)^3}{\pi^{9/2}}\boldsymbol{\sigma}\times\frac{\mathbf{x}}{r}\int_0^\infty j_1(pr)\frac{p^3dp}{\omega_p{}^2}\times\Big\{r^{-2}\int_0^\infty j_1(kr)\frac{k^3dk}{\omega_k}\int_0^\infty j_1(qr)\frac{q^3dq}{\omega_q{}^2}-r^{-1}\int_0^\infty j_1(kr)\frac{k^3dk}{\omega_k}\int_0^\infty j_2(qr)\frac{q^4dq}{\omega_q{}^2}-r^{-1}\int\int\frac{\omega_q-\omega_k}{\omega_q+\omega_k}j_1(kr)[qj_2(qr)-r^{-1}j_1(qr)]\frac{k^3q^3}{\omega_k\omega_q{}^2}dkdq\Big\},\quad\text{(A1)}$$

where $j_l(kr)$ is the spherical Bessel function.

By using the explicit expressions for j_1 and j_2, the integrals $\int_0^\infty j_1(kr)(k^3/\omega_k{}^2)dk$ and $\int_0^\infty j_2(kr)(k^4/\omega_k{}^2)dk$ can be evaluated by elementary methods.

The double integral also can be transformed, by multiplying and dividing by $\omega_q-\omega_k$ and using the relation

$$P\int_0^\infty\frac{\sin(qr)}{q^2-k^2}qdq=\tfrac{1}{2}\pi\cos(kr)\quad(\text{for } r>0).$$

Then (A1) can be written in the form

$$\mathbf{j}(\mathbf{x})=\frac{3(4\pi)^{\frac{1}{2}}e(\lambda/\mu^3)(f/\mu)^3}{(2\pi)^4}\boldsymbol{\sigma}\times\frac{\mathbf{x}}{r}\left(\frac{\mu}{r^2}+\frac{1}{r^3}\right)e^{-\mu r}\times\Big\{2r^{-5}\int_0^\infty\sin2kr\frac{kdk}{\omega_k}-4r^{-4}\int_0^\infty\cos2kr\frac{k^2dk}{\omega_k}-3r^{-3}\int_0^\infty\sin2kr\frac{k^3dk}{\omega_k}+r^{-2}\int_0^\infty\cos2kr\frac{k^4dk}{\omega_k}\Big\},$$

or, in equivalent form,

$$\mathbf{j}(\mathbf{x})=\frac{3(4\pi)^{\frac{1}{2}}e(\lambda/\mu^3)(f/\mu)^3}{(2\pi)^4}\boldsymbol{\sigma}\times\frac{\mathbf{x}}{r}\left(\frac{\mu}{r^2}+\frac{1}{r^3}\right)e^{-\mu r}\times\left(-2r^{-4}\frac{d}{dr}+\frac{3}{4}r^{-3}\frac{d^2}{dr^2}-\frac{1}{8}r^{-2}\frac{d^3}{dr^3}+2r^{-5}\right)\times\int_0^\infty\sin2kr\frac{kdk}{\omega_k}.\quad\text{(A2)}$$

For our purpose it is useful now to write the integral $\int_0^\infty(k/\omega_k)\sin2krdk$ in the form

$$\int_0^\infty\frac{k\sin2kr}{\omega_k}dk=\int_\mu^\infty\frac{\alpha e^{-2\alpha r}}{(\alpha^2-\mu^2)^{\frac{1}{2}}}d\alpha.\quad\text{(A3)}$$

This identity follows from the fact that the integrand has the branching points at $k=\pm i\mu$. The integration path is shown in Fig. 3.

By inserting (A3) into (A2) with some manipulations one finally obtains $\mathbf{j}(\mathbf{x})$ in the form

$$\mathbf{j}(\mathbf{x})=\frac{3}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{\pi^{5/2}}\boldsymbol{\sigma}\times\frac{\mathbf{x}}{r}\int_{3\mu}^\infty\Big\{\mu\left(\frac{\gamma-\mu}{2}\right)^4\frac{1}{r^4}+\left[3\mu\left(\frac{\gamma-\mu}{2}\right)^3+\left(\frac{\gamma-\mu}{2}\right)^4\right]\frac{1}{r^5}+\left[4\mu\left(\frac{\gamma-\mu}{2}\right)^2+3\left(\frac{\gamma-\mu}{2}\right)^3\right]\frac{1}{r^6}+\gamma\frac{\gamma-\mu}{r^7}+\frac{\gamma-\mu}{r^8}\Big\}\frac{e^{-\gamma r}d\gamma}{\{[(\gamma-\mu)/2]^2-\mu^2\}^{\frac{1}{2}}}.\quad\text{(A4)}$$

Let us remember the relation between current and magnetic moment density,

$$-\mathbf{j}(\mathbf{x})=\boldsymbol{\sigma}\times\frac{\mathbf{x}}{r}\frac{d\mu(r)}{dr};$$

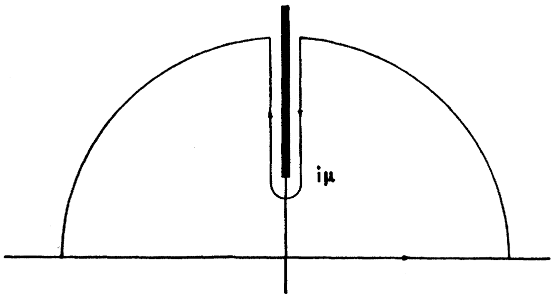

FIG. 3. Path of integration to be used in order to obtain formula (A3).

then

$$\frac{d\mu(r)}{dr}=-\frac{3}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{\pi^{5/2}}\int_{3\mu}^{\infty}\left\{\mu\left(\frac{\gamma-\mu}{2}\right)^4\frac{1}{r^4}+\left[3\mu\left(\frac{\gamma-\mu}{2}\right)^3+\left(\frac{\gamma-\mu}{2}\right)^4\right]\frac{1}{r^5}+\left[4\mu\left(\frac{\gamma-\mu}{2}\right)^2+3\left(\frac{\gamma-\mu}{2}\right)^3\right]\frac{1}{r^6}+\gamma\left(\frac{\gamma-\mu}{r^7}\right)+\frac{\gamma-\mu}{r^8}\right\}\frac{e^{-\gamma r}d\gamma}{\{[(\gamma-\mu)/2]^2-\mu^2\}^{\frac{1}{2}}}.$$

We write now $d\mu(r)/dr$ in the form

$$\frac{d\mu(r)}{dr}=\int_{3\mu}^{\infty}d\gamma\int_{\gamma}^{\infty}F(\sigma)e^{-\sigma r}d\sigma,$$

then

$$\mu(r)=-\int_{3\mu}^{\infty}d\gamma\int_{\gamma}^{\infty}\frac{F(\sigma)}{\sigma}e^{-\sigma r}d\sigma,$$

and, by integration by parts, we get

$$\mu(r)=-\int_{3\mu}^{\infty}d\gamma\frac{F(\gamma)}{\gamma}\frac{e^{-\gamma r}}{r}-\int_{3\mu}^{\infty}d\gamma\int_{\gamma}^{\infty}\frac{d[F(\sigma)/\sigma]}{d\sigma}\frac{e^{-\sigma r}}{r}d\sigma. \quad \text{(A5)}$$

With considerations analogous to those on formula (10) one can see that $G(\xi)=F(\xi+\gamma)$ is the inverse Laplace transform of the function

$$-\frac{3}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{\pi^{5/2}}\left\{\mu\left(\frac{\gamma-\mu}{2}\right)^4\frac{1}{r^4}+\left[3\mu\left(\frac{\gamma-\mu}{2}\right)^3+\left(\frac{\gamma-\mu}{2}\right)^4\right]\frac{1}{r^5}+\left[4\mu\left(\frac{\gamma-\mu}{2}\right)^2+3\left(\frac{\gamma-\mu}{2}\right)^3\right]\frac{1}{r^6}+\gamma\left(\frac{\gamma-\mu}{r^7}\right)+\frac{\gamma-\mu}{r^8}\right\}\left[\left(\frac{\gamma-\mu}{2}\right)^2-\mu^2\right]^{-\frac{1}{2}}$$

The first integrand in (A5) then clearly vanishes, and

$$\mu(r)=-\int_{3\mu}^{\infty}d\gamma\int_{\gamma}^{\infty}\frac{d[F(\sigma)/\sigma]}{d\sigma}\frac{e^{-\sigma r}}{r}d\sigma.$$

One obtains in this way, by reversing the order of integration,

$$\mu(r)=-\frac{3}{4}\frac{e(\lambda/\mu^3)(f/\mu)^3}{\pi^{5/2}}\int_{3\mu}^{\infty}\frac{e^{-\sigma r}}{r}d\sigma\times\int_{3\mu}^{\sigma}\frac{\sum_{n=-1}^{6}nC_n(\gamma)\sigma^{n-1}d\gamma}{\{[(\gamma-\mu)/2]^2-\mu^2\}^{\frac{1}{2}}}. \quad \text{(A6)}$$

The C_n are

$$C_6=(\gamma-\mu)/5040,$$

$$C_5=0,$$

$$C_4=\frac{\gamma-\mu}{960}(-\gamma^2+2\mu\gamma-5\mu^2),$$

$$C_3=\frac{\gamma-\mu}{1152}(\gamma^3-3\mu\gamma^2+3\mu^2\gamma+15\mu^3),$$

$$C_2=\frac{\gamma-\mu}{96}(\mu^2\gamma^2-2\mu^3\gamma-\mu^4),$$

$$C_1=\frac{\gamma-\mu}{960}(\gamma^5-5\mu\gamma^4+5\mu^2\gamma^3-15\mu^3\gamma^2+30\mu^4\gamma).$$

C_0 does not contribute to (A6).

$$C_{-1}=\frac{\gamma-\mu}{13.440}(9\gamma^7-63\mu\gamma^6+175\mu^2\gamma^5-245\mu^3\gamma^4+140\mu^4\gamma^3).$$

The Fourier transform of $\mu(r)$ yields

$$F_2{}^S(q^2)=\frac{3e(\lambda/\mu^3)(f/\mu)^3}{2\pi^{\frac{3}{2}}}\int_{(3\mu)^2}^{\infty}\frac{d\sigma^2}{\sigma^2+q^2}\sum_{n=-1}^{6}\sigma^{n-2}\phi_n(\sigma), \quad \text{(A7)}$$

where

$$\phi_n(\sigma)=n\int_{3\mu}^{\sigma}\frac{C_n(\gamma)d\gamma}{\{[(\gamma-\mu)/2]^2-\mu^2\}^{\frac{1}{2}}}. \quad \text{(A8)}$$

The $\phi_n(\sigma)$ can be evaluated by quite elementary, if rather lengthy, methods. When the results are inserted into (A8), we obtain for $g_2{}^S(\sigma^2)$ the expression (13) of the text.