Geometry of Gravitation and Electromagnetism*

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(Received January 21, 1959)

An independent derivation is given of equations first derived by Rainich which show how, under certain circumstances, the combined theory of gravitation and electromagnetism of Einstein and Maxwell can be unified and described exclusively in terms of geometry. Some algebraic relations are presented between the Ricci tensor, the electromagnetic field tensor, and their principal null vectors. It is shown that in regions of space-time where the two invariants of the electromagnetic field both vanish, the unified theory cannot apply. Either such regions do not exist in nature or their description in terms of pure geometry has yet to be found. Advantage is taken of the correspondences between tensors and spinors to carry out most of the present calculations in spinor space.

I. INTRODUCTION

THE general theory of relativity relates gravitational effects to the curvature of space. The electromagnetic tensor has customarily been introduced apart from geometry, the electromagnetic stress energy tensor acting as a source of the gravitational field; the electromagnetic tensor itself satisfying the Maxwell equation generalized in conformity with the covariance requirements of general relativity. Thus while gravitation has been expressed as a geometric theory, electromagnetism has been coupled to geometry but with the inclusion of an additional nongeometric element, the electromagnetic tensor, for its description.

The desire to eliminate this dichotomy has been perhaps the primary motivation behind the attempts at producing a unified field theory. Most of these attempts have involved a generalization of geometry from the four-dimensional Riemannian geometry originally used by Einstein to describe gravitation. There is however a remarkable result first discovered over 30 years ago by Rainich[1] and rediscovered and discussed recently by Misner and Wheeler[2] which shows that such generalization is unnecessary. In regions where electromagnetism is the only contributor to the stress energy tensor and where the electromagnetic field itself is free of sources, one can replace the entire content of the combined Einstein-Maxwell theory by a theory which is purely geometrical. The new geometrical theory follows as a consequence of the old accepted theories of Einstein and Maxwell. If classical physics contains only regions of the aforementioned type or regions of this type together with line singularities in space-time, then classical physics can be said to be already unified and geometrized. Both the gravitational field and the electromagnetic field are entirely determined by the curvature of space-time.

The combined Einstein-Maxwell theory can be written in the form[3]

$$R_{ab}-\tfrac{1}{2}g_{ab}R=f_{ac}f_b{}^c-\tfrac{1}{4}g_{ab}f_{cd}f^{cd}, \tag{1.1}$$

$$f^{ab}{}_{,b}=0, \tag{1.2}$$

$$f_{ab,c}+f_{bc,a}+f_{ca,b}=0. \tag{1.3}$$

It has long been known and will be shown again explicitly in Sec. III that Eq. (1.1) above and the existence of the contracted Bianchi identity $R^{mn}{}_{,n}=0$ imply that only four of Maxwell's eight equations (1.2) and (1.3) can be independent.

The form of the theory represented by Eqs. (1.1), (1.2), and (1.3) can be called the nonunified form. The equations however can be shown to be satisfied if and only if the following set of equations are satisfied:

$$R=0, \tag{1.4}$$

$$R_{ab}R^{bc}=\delta_a{}^c(R_{mn}R^{mn})/4, \tag{1.5}$$

$$\alpha_{n,m}-\alpha_{m,n}=0, \tag{1.6}$$

where α_m is defined by[4]

$$\alpha_m\equiv-\frac{ig^{-\frac{1}{2}}\epsilon_{mnrs}R^{nb,r}R_b{}^s}{R_{ac}R^{ac}}. \tag{1.7}$$

In order to insure that the energy density is positive,

* This research was supported in part by the Aeronautical Research Laboratory.

[1] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[2] C. W. Misner and J. A. Wheeler, Ann. phys. **2**, 525 (1957).

[3] In the notation we are using, Latin letters take the four values 1, 2, 3, 0; the proper interval between neighboring points is given by $ds^2=g_{ab}dx^adx^b$. In normal coordinates at a point the metric tensor is diagonalized and takes the values -1, -1, -1, 1 at the point. (Our subsequent analysis will be carried out mostly in spinor space; the equivalence between tensors and spinors is written best with the above choice for the metric in normal coordinates.) g will stand for the determinant $|g_{ab}|$ of the metric tensor. R_{mn} represents the Ricci tensor and is symmetric in its two indices; R is the curvature invariant, $R=R^m{}_m$. A comma followed by a subscript, such as $_{,b}$, means covariant differentiation with respect to x^b. $f_{ab}\equiv[2G^{\frac{1}{2}}/c^2]F_{ab}$, where G is the gravitational constant, c the velocity of light, and F_{ab}, the usual antisymmetric electromagnetic tensor. In flat space and Cartesian coordinates, F_{12}= the z component of the magnetic field; F_{10}= the x component of the electric field multiplied by c; etc.

[4] ϵ_{mnrs} and ϵ^{mnrs} are covariant and contravariant totally antisymmetric tensor densities of weights -1 and $+1$, respectively. They take the numerical values $+1$ when $mnrs$ are an even permutation of 1, 2, 3, 0. They cannot be obtained from each other in the usual way by raising and lowering indices because of the difference in weights.

one must also impose the requirement

$$R_{00}<0. \tag{1.8}$$

The set of Eqs. (1.4), (1.5), (1.6), and (1.8) with the definition (1.7) contain only the metric tensor and its derivatives—solely geometric elements. If and only if these equations are satisfied can one find an electromagnetic tensor which, with the Ricci tensor, will satisfy the Einstein-Maxwell equations (1.1), (1.2), and (1.3). Equations (1.4), (1.5), (1.6), and (1.8) can be called the unified form of the existing theory of gravitation and electromagnetism.

In his derivation, Rainich[1] regarded the tensor $f^a{}_b$ as an operator for the transformation of vectors and used a four-dimensional vector algebra to investigate the two-dimensional elements which are transformed into themselves. Misner and Wheeler[2] have given a derivation of the unified form based on the introduction of a new tensor having the same symmetries as the Riemann curvature tensor. Misner and Wheeler have discussed a space defined by Eqs. (1.4), (1.5), (1.6), and (1.8) with the exclusion of all singularities but allowing a multiply connected space-time manifold. They have shown how the properties of such a space lead to the concepts of gravitation, electromagnetism, charge, and mass; but have not shown any natural origin for spinors and fundamental particles.

From the electromagnetic tensor, f_{ab}, one can deduce two invariants, $f_{ab}f^{ab}$ and $(-g)^{\frac{1}{2}}\epsilon_{mnst}f^{mn}f^{st}$. A field in which both invariants vanish we shall call a null field. If at least one does not vanish, the field is non-null. (In flat space with familiar notation, the invariants are $\mathbf{E}\cdot\mathbf{H}$ and $\mathbf{E}^2-\mathbf{H}^2$. A null field is one where both invariants vanish, i.e., $\mathbf{E}$ is perpendicular to $\mathbf{H}$ and $\mathbf{E}$ is equal to $\mathbf{H}$.) The derivations of the unified form of the Einstein-Maxwell theory are valid only in the case of a non-null field and break down in the case of a null field. Rainich has conjectured that a null field cannot be physically realized, Misner and Wheeler have suggested the possibility that the unified form may actually describe the null field case as well as the non-null case.

In this paper we present a third and independent and simpler derivation of the unified form of the Einstein-Maxwell theory. We introduce a spinor space at each point in space-time and exploit the relationships that then exist between tensors and tensor analysis and spinors and spinor analysis. The resultant derivation is simpler than the other derivations in showing how to construct the electromagnetic field f_{ab} from the contracted Riemann tensor R_{ab} when the equations of the unified form of the theory are satisfied. The proof holds only in the case of a non-null field just as do the others referred to. The null field case is also discussed. It is shown that this case is fundamentally different than the non-null field case and that Eqs. (1.4), (1.5), (1.6), and (1.8) are not adequate to describe the null electromagnetic field. It is shown that something new must enter in the case of the null field and that in any case a geometric theory will almost surely not be adequate to describe a null field uniquely. Interesting relationships between spinor equations and tensor equations are exhibited for the null field.

The equations of the theory stated in spinor form are of intrinsic interest. Moreover the spinor analysis facilitates the derivation of connections between f_{ab} and certain vectors in space-time which we make. Some of these relations have already been derived by Synge[5] and Ruse.[6] The connections between tensors and spinors have long been known, Maxwell's equations have been written in spinor forms, and Dirac's equations have been written in tensor form. However, in this paper we actually use these known connections to facilitate calculations.

The next section describes some relations between spinors and certain types of tensors. Section III derives the unified form of the Einstein-Maxwell theory for non-null fields. Section IV contains some geometric relations involving the contracted curvature tensor, R_{ab}, and the electromagnetic tensor, f_{ab}, derived by use of the spinor analysis. Section V contains a discussion of the null field and discusses the essential difference between this and the case where the field is non-null. Section VI contains some concluding considerations.

II. CONNECTION BETWEEN SPINORS AND TENSORS OF CERTAIN CLASSES

Spinor fields have been treated in general relativity[7,8] from several different points of view. We shall use the notation of Corson[8] and shall introduce spinors by an analytic formalism. The spin matrices, $g^m{}_{\dot\alpha\beta}$, transform like vectors with respect to the index m and like spinors with respect to each index $\dot\alpha,\beta$ ($\dot\alpha=\dot 1, \dot 2$; $\beta=1, 2$). We use a representation in which the g matrices are Hermitean, $\bar g^m{}_{\dot\alpha\beta}=g^m{}_{\dot\beta\alpha}$ (bar denotes complex conjugate) and define[9] the spin matrices by

$$g^{m\dot\alpha\beta}g_{n\dot\alpha\mu}+g_n{}^{\dot\alpha\beta}g^m{}_{\dot\alpha\mu}=2\delta_n{}^m\delta_\mu{}^\beta. \tag{2.1}$$

The Latin index in $g^m{}_{\dot\alpha\beta}$ can be lowered or raised by using the metric tensor g_{mn} or g^{mn}. The Greek indices $\dot\alpha,\beta$ can be raised or lowered by using the antisymmetric

[5] J. L. Synge, *Principal Null-Directions Defined in Space Time by an Electromagnetic Field*, University of Toronto Studies, Applied Math. Series, No. 1 (University of Toronto Press, Toronto, 1935).

[6] H. S. Ruse, Proc. Math. Soc. London **41**, 302 (1936).

[7] W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953); this paper reviews spinor analysis and has references to the previous literature.

[8] E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations* (Hafner Publishing Company, New York, 1953). Chapter 2 discusses spinor algebra and uses it in special relatively. See page 32 ff. for the transcription from the dotted and undotted indices used below to the more familiar four-component spinors.

[9] For Hermitean g's, the definition (2.1) is equivalent to the more usual definition given in matrix notation by $g^{m*}g^n+g^ng^{m*}=-2g^{mn}$, where the asterisk denotes Hermitean conjugate matrix (transpose and complex conjugate).

TABLE I. The connection between certain types of tensors and spinors.

Tensors	Spinors	Connections
x^m (real, arbitrary)	$\phi_{\dot\alpha\beta}$ Hermitean	$x^m=\frac{1}{2}g^{m\dot\alpha\beta}\phi_{\dot\alpha\beta}$; $\phi_{\dot\alpha\beta}=g_{m\dot\alpha\beta}x^m$
y^m (real, null)	ϕ_α	$y^m=\frac{1}{2}g^{m\dot\alpha\beta}\phi_{\dot\alpha}\phi_\beta$; $\phi_{\dot\alpha}\phi_\beta=g_{m\dot\alpha\beta}y^m$
z^m (complex, null)	ϕ_α, ψ_α	$z^m=\frac{1}{2}g^{m\dot\alpha\beta}\phi_{\dot\alpha}\psi_\beta$; $\phi_{\dot\alpha}\psi_\beta=g_{m\dot\alpha\beta}z^m$
T_{mn} (real, arbitrary)	$\psi^{\dot\alpha\dot\mu\beta\nu}$; $\bar\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\beta\dot\nu\alpha\mu}$	
T_{mn} (real, symmetric)	$\psi^{\dot\alpha\dot\mu\beta\nu}$; $\bar\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\beta\dot\nu\alpha\mu}$, $\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\mu\dot\alpha\nu\beta}$	$T_{mn}=g_{m\dot\alpha\beta}g_{n\dot\mu\nu}\psi^{\dot\alpha\dot\mu\beta\nu}$
T_{mn} (real, symmetric, zero trace)	$\psi^{\dot\alpha\dot\mu\beta\nu}$; $\bar\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\beta\dot\nu\alpha\mu}$, $\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\mu\dot\alpha\beta\nu}$, $\psi^{\dot\alpha\dot\mu\beta\nu}=\psi^{\dot\alpha\dot\mu\nu\beta}$	$\psi^{\dot\alpha\dot\mu\beta\nu}=\frac{1}{4}g^{m\dot\alpha\beta}g^{n\dot\mu\nu}T_{mn}$
ω_{mn} (self-dual, antisymmetric)	$\phi^{\alpha\beta}$ symmetric	$\omega_{mn}=g_m{}^{\dot\mu}{}_\alpha g_{n\dot\mu\beta}\phi^{\alpha\beta}$, $\phi^{\alpha\beta}=\frac{1}{8}g^{m\dot\mu\alpha}g^n{}_{\dot\mu}{}^\beta\omega_{mn}$
ω_{mn} (self-dual, antisymmetric, null)	ϕ^α	$\omega_{mn}=g_m{}^{\dot\mu}{}_\alpha g_{n\dot\mu\beta}\phi^\alpha\phi^\beta$, $\phi^\alpha\phi^\beta=\frac{1}{8}g^{m\dot\mu\alpha}g^n{}_{\dot\mu}{}^\beta\phi_\alpha\phi_\beta$

fundamental spinors $\epsilon_{\dot\alpha\dot\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$, $\epsilon^{\alpha\beta}$ which are equal to 0 when $\alpha=\beta$, and equal to $+1$ if $\alpha=1$, $\beta=2$, and equal to -1 if $\alpha=2$, $\beta=1$.

By use of the spin matrices $g_{m\dot\alpha\beta}$ one can make correspondences between tensors and spinors. Sometimes these correspondences are one-to-one and sometimes not. Table I summarizes the correspondences we shall need in the rest of this paper. One sees in this table, for example, that a first rank spinor determines a null real vector uniquely but that the null vector determines the spinor only up to a phase factor.

Table I includes a correspondence for the self-dual antisymmetric tensor, ω_{mn}. If f_{cd} is antisymmetric, its dual $\tilde f^{cd}$ is defined by

$$\tilde f^{ab}\equiv\tfrac{1}{2}g^{-\frac{1}{2}}\epsilon^{abcd}f_{cd}. \tag{2.2}$$

The tensor $\omega_{ab}\equiv f_{ab}+\tilde f_{ab}$ is self-dual in that $\omega_{ab}=\tilde\omega_{ab}$.

The antisymmetric real tensor f_{ab} gives rise to two invariants $f_{ab}f^{ab}$ and $(-g)^{\frac{1}{2}}\epsilon_{mnst}f^{mn}f^{st}$; the self-dual antisymmetric tensor ω_{ab} constructed from f_{ab} also gives rise to two invariants $\omega_{ab}\omega^{ab}$ and $\bar\omega_{ab}\bar\omega^{ab}$. It is readily seen that $\omega_{ab}\bar\omega^{ab}=0$. If the invariants constructed from ω_{ab} both vanish, we call ω_{ab} a null self-dual antisymmetric tensor.

In order to permit covariant differentiation of spinors, it is necessary to introduce a spin connection, $\Gamma^\alpha{}_{\beta m}$, in addition to the Christoffel symbol $\Gamma^a{}_{bc}$. The Christoffel symbol is defined by

$$\Gamma^a{}_{bc}\equiv\tfrac{1}{2}g^{am}\left(\frac{\partial g_{bm}}{\partial x^c}+\frac{\partial g_{cm}}{\partial x^b}-\frac{\partial g_{bc}}{\partial x^m}\right). \tag{2.3}$$

The $\Gamma^\alpha{}_{\beta m}$ can be defined[10] by

$$\Gamma^\alpha{}_{\beta m}\equiv\tfrac{1}{2}g_n{}^{\dot\mu\alpha}\left(\frac{\partial g^n{}_{\dot\mu\beta}}{\partial x^m}+\Gamma^n{}_{sm}g^s{}_{\dot\mu\beta}\right). \tag{2.4}$$

This is not the most general spin connection that can be defined but it is good enough for our purposes. Covariant derivations of spinors and tensor-spinors are formed according to the usual rules: the Christoffel symbols are used with tensor indices, the spin connection $\Gamma^\alpha{}_{\beta m}$ with undotted indices, the complex conjugate $\bar\Gamma^\alpha{}_{\beta m}\equiv\Gamma^{\dot\alpha}{}_{\dot\beta m}$ with dotted indices; a plus sign goes with contravariant indices and a minus sign with covariant. Thus, to take a typical example, the covariant derivative of $g^m{}_{\dot\alpha\beta}$ is

$$g^m{}_{\dot\alpha\beta,\,n}=\partial g^m{}_{\dot\alpha\beta}/\partial x^n+\Gamma^m{}_{rn}\Gamma^r{}_{\dot\alpha\beta} -\Gamma^{\dot\mu}{}_{\dot\alpha n}g^m{}_{\dot\mu\beta}-\Gamma^\mu{}_{\beta n}g^m{}_{\dot\alpha\mu}. \tag{2.5}$$

With the above choices of connections $g^{mn}{}_{,r}=0$; $g^m{}_{\dot\alpha\beta,r}=0$; $\epsilon_{\alpha\beta,r}=0$; the covariant derivatives of all g tensors and spinors as well as of ϵ spinors vanish.

[10] See, for example, H. S. Ruse, Proc. Roy. Soc. Edinburgh 57, 97 (1937).

III. DERIVATION OF UNIFIED FORM OF EINSTEIN-MAXWELL THEORY FOR NON-NULL FIELDS

In discussing the Einstein-Maxwell theory, we shall find it convenient to construct from the electromagnetic tensor f_{ab}, the self-dual antisymmetric tensor $\omega_{ab}=f_{ab}+\tilde f_{ab}$. The combined Einstein-Maxwell theory equations (1.1), (1.2), and (1.3) can be expressed by

$$R_{ab}-\tfrac{1}{2}g_{ab}R=\omega_{ac}\bar\omega_b{}^c, \tag{3.1}$$

$$\omega^{ab}{}_{,b}=0. \tag{3.2}$$

Equation (3.1) is the stress-momentum-energy equation of gravitation and (3.2) are Maxwell's equations.

Let us first consider Eq. (3.1) and see how it can be geometrized.

Theorem 1: $R_{ab}-\frac{1}{2}g_{ab}R=\omega_{ac}\bar\omega_b{}^c$ implies $R\equiv R^m{}_m=0$.
Proof: $\omega_{ac}\bar\omega^{ac}=0$; hence the theorem follows.

Theorem 2: $R_{ab}-\frac{1}{2}g_{ab}R=\omega_{ac}\bar\omega_b{}^c$ implies

$$R^a{}_bR^b{}_c=\delta^a{}_c(R_{mn}R^{mn})/4. \tag{3.3}$$

Proof: From Table I, we recall that there is a symmetric second-rank spinor $\phi^{\alpha\beta}$ such that

$$\omega_{ac}=g_a{}^{\dot\mu}{}_\alpha g_{c\dot\mu\beta}\phi^{\alpha\beta};\quad \bar\omega_b{}^c=g_{b\dot\mu}{}^\nu g^c{}_{\dot\alpha\nu}\phi^{\dot\mu\dot\alpha}. \tag{3.4}$$

For a symmetric second-rank spinor one has the relation

$$\phi^{\alpha\beta}\phi_{\beta}{}^{\mu}=\epsilon^{\alpha\mu}\phi^{\beta\nu}\phi_{\beta\nu}/2. \tag{3.5}$$

Inserting (3.3) into (3.1) with $R=0$ and calculating $R^a{}_bR^b{}_c$, the theorem follows immediately after the application of (3.5) and the definition (2.1).

Lemma 1: $R^a{}_b R^b{}_c = \delta^a{}_c R_{mn} R^{mn}/4$ and $R=0$ imply the existence of a $\phi^{\alpha\beta}$ such that $R_{ab} = 2g_a{}^{\dot\mu}{}_\alpha g_{b\dot\nu\beta}\phi^{\alpha\beta}\bar\phi^{\dot\nu}{}_{\dot\mu}$.

Proof: R_{mn} is a real symmetric tensor with vanishing trace ($R=0$), and can consequently (see Table I) be expressed by a fourth-rank spinor as

$$R_{mn} = g_{m\dot\alpha\beta} g_{n\dot\mu\nu} \psi^{\dot\alpha\dot\mu\beta\nu}, \tag{3.6}$$

with $\bar\psi^{\dot\alpha\dot\mu\beta\nu} = \psi^{\dot\beta\dot\nu\alpha\mu}$ and satisfying the symmetry requirements $\psi^{\dot\alpha\dot\mu\beta\nu} = \psi^{\dot\mu\dot\alpha\beta\nu} = \psi^{\dot\alpha\dot\mu\nu\beta}$. A fourth-rank spinor satisfying these conditions can always be expressed by two symmetric second-rank spinors,

$$\psi^{\dot\alpha\dot\mu\beta\nu} = \tfrac{1}{2}(\psi^{\dot\alpha\dot\mu}X^{\beta\nu} + X^{\dot\alpha\dot\mu}\psi^{\beta\nu}). \tag{3.7}$$

The lemma will be proved if it can be shown that relation (3.3) implies that $X^{\beta\mu}$ is proportional to $\psi^{\beta\mu}$. Using (3.6), (3.7), the relation

$$g^m{}_{\dot\alpha\beta} g_{m\dot\mu\nu} = 2\epsilon_{\dot\alpha\dot\mu}\epsilon_{\beta\nu}, \tag{3.8}$$

and the symmetry of $\psi^{\alpha\mu}$ and $X^{\beta\nu}$, one finds readily

$$\psi_{\dot\mu}{}^{\dot\alpha} X_\rho{}^\lambda X^{\dot\nu\dot\mu}\psi^{\sigma\rho} + X_{\dot\mu}{}^{\dot\alpha}\psi_\rho{}^\lambda \psi^{\dot\nu\dot\mu}X^{\sigma\rho} = \tfrac{1}{2}\epsilon^{\dot\nu\dot\alpha}\epsilon^{\sigma\lambda}\psi_{\dot\mu\dot\beta}X^{\dot\mu\dot\beta}X_{\rho\kappa}\psi^{\rho\kappa}. \tag{3.9}$$

By virtue of

$$\psi^{\alpha\beta}X^\mu{}_\beta + \psi^\mu{}_\beta X^{\alpha\beta} = \epsilon^{\alpha\mu}\psi^{\beta\rho}X_{\beta\rho},$$

(3.9) can be rewritten as

$$\psi_{\dot\mu}{}^{\dot\alpha}X^{\dot\nu\dot\mu}\epsilon^{\sigma\lambda}\psi_{\kappa\rho}X^{\kappa\rho} + X^{\sigma\kappa}\psi_\kappa{}^\lambda[X_{\dot\mu}{}^{\dot\alpha}\psi^{\dot\nu\dot\mu} - \psi_{\dot\mu}{}^{\dot\alpha}X^{\dot\nu\dot\mu}] = \tfrac{1}{2}\epsilon^{\dot\nu\dot\alpha}\epsilon^{\sigma\lambda}\psi_{\dot\beta\dot\mu}\psi^{\dot\beta\dot\mu}X_{\kappa\rho}X^{\kappa\rho}. \tag{3.10}$$

This relation can only be satisfied if the second term on the left-hand side vanishes or is antisymmetric in the indices σ and λ. Either requirement makes $X^{\alpha\beta}$ proportional to $\psi^{\alpha\beta}$ and this term must vanish. This proves the lemma. We are now in a position to state and prove Theorem 3.

Theorem 3: If $R=0$ and $R^a{}_b R^b{}_c = \delta^a{}_c R_{mn}R^{mn}/4$, one can give a prescription to find a self-dual antisymmetric tensor, ω_{mn}, that satisfies $R_{ab} - \frac{1}{2}g_{ab}R = \omega_{ac}\bar\omega_b{}^c$. ω_{mn} is not uniquely determined; from an ω_{mn} which satisfies the requirement of the theorem one can generate a family $\omega'_{mn} = e^{i\theta}\omega_{mn}$ which satisfies the requirements of the theorem, θ being an arbitrary real function of space and time. This lack of uniqueness is obviously necessary from the character of the energy equation, (3.1), which can only fix ω_{mn} up to an arbitrary phase factor.

Proof of Theorem 2: From Lemma 1, there exists a symmetric spinor $\phi^{\alpha\beta}$ such that

$$R_{ab} = -2g_{a\dot\alpha\rho}g_{b\dot\mu\sigma}\bar\phi^{\dot\alpha\dot\mu}\phi^{\rho\sigma}. \tag{3.11}$$

This can be inverted by multiplying by $g^{a\dot\beta\kappa}g^{b\dot\nu\lambda}$,

$$\phi^{\kappa\lambda}\bar\phi^{\dot\beta\dot\nu} = -\tfrac{1}{8}g^{a\dot\beta\kappa}g^{b\dot\nu\lambda}R_{ab}. \tag{3.12}$$

From Eq. (3.12), by choosing $\kappa\lambda = \beta\nu$ one can get the square of the moduli, $|\phi^{11}|^2$, $|\phi^{12}|^2$, $|\phi^{22}|^2$; by choosing $\kappa\lambda \neq \beta\nu$, one can get relative phases. (3.12) can obviously only determine $\phi^{\alpha\beta}$ up to a phase factor. $\phi^{\alpha\beta}$ being so determined, one has a family of tensors ω_{mn} determined up to a phase factor. The general tensor of this type can be written

$$\omega_{mn} = g_m{}^{\dot\mu}{}_\alpha g_{n\dot\mu\beta}\phi^{\alpha\beta}e^{i\theta}. \tag{3.13}$$

There is one ambiguity which must be cleared up. The equations $R=0$ and $R^a{}_b R^b{}_c = \delta^a{}_c R_{mn}R^{mn}/4$ clearly do not determine the sign of the various components of R_{mn}. If R_{mn} satisfies the above relations, so also will $-R_{mn}$. Clearly however only one choice of sign will be consistent with Eq. (3.13). ($\phi^{11}\bar\phi^{11}$ must be positive, for example.) A criterion to choose the proper sign can be deduced from the stress-momentum-energy equation (3.1). The value of R_{00} at a point is equal to minus the energy density at the point. Consequently R_{00} must be negative. This can be seen also from Eq. (3.11). In normal coordinates ($g_{11}=g_{22}=g_{33}=-1$; $g_{00}=1$; $g_{ab}=0$, $a\neq b$) at a point, a representation of $g_{a\dot\alpha\beta}$ can be given by the four matrices

$$g_{1\dot\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_{3\dot\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$g_{2\dot\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad g_{0\dot\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.14}$$

With this representation, Eq. (3.11) gives R_{00} as

$$R_{00} = -2[\phi^{11}\bar\phi^{11} + \phi^{22}\bar\phi^{22} + 2\phi^{12}\bar\phi^{12}]. \tag{3.15}$$

Consequently,

$$R_{00} < 0. \tag{3.16}$$

Two points summarize the results found so far: (1) If $R_{ab} - \frac{1}{2}g_{ab}R = \omega_{ac}\bar\omega_b{}^c$, then $R=0$, $R^a{}_b R^b{}_c = \delta^a{}_c R^{mn}R_{mn}/4$, and $R_{00}<0$; and (2) if the latter conditions are satisfied, one can find a family of ω_{mn}' of the form $\omega_{mn}e^{i\theta}$, where θ is an arbitrary real function of space and time, which satisfies $R_{ab} - \frac{1}{2}g_{ab}R = \omega'_{ac}\bar\omega'_b{}^c$. The way to find this family of tensors has been explicitly exhibited.

So far no mention has been made of Maxwell's equations (3.2). It is obvious to ask now whether one can choose θ so that these equations are satisfied. From (3.16) the question is whether one can find a θ such that

$$\omega'^{mn}{}_{,n} = g^{m\dot\mu}{}_\alpha g^n{}_{\dot\mu\beta}\phi^{\alpha\beta}{}_{,n}e^{i\theta} + ig^{m\dot\mu}{}_\alpha g^n{}_{\dot\mu\beta}\phi^{\alpha\beta}e^{i\theta}\theta_{,n} = 0.$$

Multiply this expression by $g_{m\dot\nu}{}^\rho g^{r\dot\nu}{}_\lambda\phi_\kappa{}^\lambda$ and use (2.1) to get

$$ig^{r\dot\alpha}{}_\mu g^n{}_{\dot\alpha\beta}\theta_{,n}\phi^{\nu\beta}\phi_\nu{}^\mu + g^{r\dot\alpha}{}_\mu g^n{}_{\dot\alpha\beta}\phi^{\nu\beta}{}_{,n}\phi_\nu{}^\mu = 0.$$

With the aid of (3.5) and (2.1), this can be solved to yield

$$\theta_{,r} = \frac{ig_{r\dot\alpha\rho}g^n{}_{\dot\beta\sigma}\phi^{\lambda\sigma}{}_{,n}\phi_\lambda{}^\rho\bar\phi^{\dot\mu\dot\beta}\bar\phi_{\dot\beta}{}^{\dot\alpha}}{2\phi_{\rho\sigma}\phi^{\rho\sigma}\bar\phi_{\dot\mu\dot\nu}\bar\phi^{\dot\mu\dot\nu}} \equiv \beta_r. \tag{3.17}$$

This can be satisfied for real θ, if and only if $\beta_{n,m} - \beta_{m,n} = 0$ and $\beta_m = \bar\beta_n$. The question to be settled is whether or not there is a geometric relationship which is satisfied if and only if Eq. (3.17) is satisfied.

Consider the vector

$$\alpha_m \equiv -\frac{i g^{-\frac{1}{2}} \epsilon_{mnrs} R^{na,r} R_a{}^s}{R_{cf} R^{cf}}. \tag{3.18}$$

An easy calculation shows the two relations

$$R_{cf} R^{cf} = 16 \phi_{\rho\sigma} \phi^{\rho\sigma} \phi_{\dot{\alpha}\dot{\beta}} \phi^{\dot{\alpha}\dot{\beta}}, \tag{3.19}$$

$$R^{na,r} R_a{}^s = 4 g^{n\dot{\alpha}}{}_{\kappa} g^s{}_{\dot{\alpha}}{}^{\rho} \phi^{\kappa\lambda,r} \phi_{\lambda\rho} \phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}} + 4 g^n{}_{\dot{\alpha}}{}^{\rho} g^s{}_{\dot{\beta}\rho} \phi^{\dot{\mu}\dot{\alpha},r} \phi_{\dot{\mu}}{}^{\dot{\beta}} \phi_{\kappa\lambda} \phi^{\kappa\lambda}. \tag{3.20}$$

Recall the following two spinor relations (see Corson,[8] p. 20):

$$g_m{}^{\dot{\beta}}{}_{\alpha} g_{n\dot{\beta}\mu} = g_{mn} \epsilon_{\alpha\mu} + \tfrac{1}{2} g^{-\frac{1}{2}} \epsilon_{mnab} g^{a\dot{\beta}}{}_{\alpha} g^b{}_{\dot{\beta}\mu}, \tag{3.21}$$

$$g_{m\dot{\alpha}}{}^{\beta} g_{n\dot{\mu}\beta} = g_{mn} \epsilon_{\dot{\alpha}\dot{\mu}} - \tfrac{1}{2} g^{-\frac{1}{2}} \epsilon_{mnab} g^a{}_{\dot{\alpha}}{}^{\beta} g^b{}_{\dot{\mu}\beta}. \tag{3.22}$$

Use these to find

$$N \equiv -i g^{-\frac{1}{2}} \epsilon_{mnrs} R^{na,r} R_a{}^s = -8 i g_m{}^{\dot{\alpha}}{}_{\kappa} g^r{}_{\dot{\alpha}\lambda} \phi^{\kappa\rho}{}_{,r} \phi_{\rho}{}^{\lambda} \phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}} + 8 i \phi^{\alpha\beta}{}_{,m} \phi_{\alpha\beta} \phi^{\dot{\mu}\dot{\nu}} \phi_{\dot{\mu}\dot{\nu}} + 8 i g_{m\dot{\alpha}}{}^{\nu} g^r{}_{\dot{\beta}\nu} \phi^{\dot{\mu}\dot{\alpha}}{}_{,r} \phi_{\dot{\mu}}{}^{\dot{\beta}} \phi_{\kappa\lambda} \phi^{\kappa\lambda} - 8 i \phi^{\dot{\alpha}\dot{\beta}}{}_{,m} \phi_{\dot{\alpha}\dot{\beta}} \phi_{\kappa\lambda} \phi^{\kappa\lambda}. \tag{3.23}$$

Differentiate (3.5):

$$\phi^{\kappa\lambda}{}_{,r} \phi_{\kappa}{}^{\sigma} + \phi^{\kappa\lambda} \phi_{\kappa}{}^{\sigma}{}_{,r} = \epsilon^{\lambda\sigma} \phi^{\alpha\beta}{}_{,r} \phi_{\alpha\beta}. \tag{3.24}$$

Use this in (3.23) to obtain

$$N = -8 i g_m{}^{\dot{\alpha}}{}_{\rho} g^r{}_{\dot{\alpha}\sigma} \phi^{\rho\beta} \phi_{\beta}{}^{\sigma}{}_{,r} \phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}} - 8 i \phi^{\alpha\beta}{}_{,m} \phi_{\alpha\beta} \phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}} - 8 i g_{m\dot{\alpha}}{}^{\sigma} g_{r\dot{\beta}\sigma} \phi^{\dot{\mu}\dot{\alpha}} \phi_{\dot{\mu}}{}^{\dot{\beta},r} \phi_{\kappa\lambda} \phi^{\kappa\lambda} + 8 i \phi^{\dot{\alpha}\dot{\beta}}{}_{,m} \phi_{\dot{\alpha}\dot{\beta}} \phi_{\kappa\lambda} \phi^{\kappa\lambda}. \tag{3.25}$$

The Bianchi identity $R^{mn}{}_{,n} = 0$ in spinor notation takes the form

$$g_{m\dot{\beta}\alpha} g^r{}_{\dot{\mu}\rho} (\phi^{\alpha\rho} \phi^{\dot{\beta}\dot{\mu}})_{,r} = 0. \tag{3.26}$$

Differentiate, multiply by $g^{m\dot{\nu}\sigma}$, to get

$$g^r{}_{\dot{\mu}\rho} \phi^{\sigma\rho}{}_{,r} \phi^{\dot{\nu}\dot{\mu}} = -g^r{}_{\dot{\mu}\rho} \phi^{\sigma\rho} \phi^{\dot{\nu}\dot{\mu}}{}_{,r}. \tag{3.27}$$

Using (3.6) and the Bianchi identity (3.32) one can now readily show that

$$-8 i g_{m\dot{\alpha}}{}^{\rho} g^r{}_{\dot{\beta}\rho} \phi^{\dot{\mu}\dot{\alpha}} \phi_{\dot{\mu}}{}^{\dot{\beta}}{}_{,r} \phi_{\kappa\lambda} \phi^{\kappa\lambda} = 8 i g_m{}^{\dot{\alpha}}{}_{\kappa} g^r{}_{\dot{\alpha}\rho} \phi^{\kappa\lambda} \phi_{\lambda}{}^{\rho}{}_{,r} \phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}}. \tag{3.28}$$

Now from (3.25), (3.28) and the definitions of α_m and β_m follows immediately

$$\alpha_m = \beta_m + (i/4)[\ln(\phi^{\kappa\lambda} \phi_{\kappa\lambda} / \phi^{\dot{\mu}\dot{\nu}} \phi_{\dot{\mu}\dot{\nu}})]_{,m}. \tag{3.29}$$

So $\beta_{m,m} - \beta_{m,n} = 0$ if and only if

$$\alpha_{n,m} - \alpha_{m,n} = 0. \tag{3.30}$$

This proves theorem 4, which is now stated.

Theorem 4: If the Maxwell-Einstein theory [Eqs. (3.1) and (3.2)] is valid, then

$$R = 0; \quad R^a{}_b R^b{}_c = \delta^a{}_c R^{mn} R_{mn} / 4; \quad R_{00} < 0;$$

and

$$\alpha_{n,m} - \alpha_{m,n} = 0 \quad (\alpha_n \equiv -i g^{-\frac{1}{2}} \epsilon_{mnrs} R^{na,r} R_a{}^s / R_{cf} R^{cf}).$$

If the latter relations are satisfied, then there exists a self-dual antisymmetric tensor, ω_{mn}, such that the Maxwell-Einstein theory is satisfied.

We have already shown how to find ω_{mn} from the energy equation up to a phase factor θ which is a function of space and time. Equations (3.17) and (3.29) tell us that

$$\theta_{,n} = \alpha_n - \tfrac{1}{4}[\ln(\phi^{\kappa\lambda} \phi_{\kappa\lambda} / \phi^{\dot{\mu}\dot{\nu}} \phi_{\dot{\mu}\dot{\nu}})]_{,n}. \tag{3.31}$$

Consequently if θ is known at one point, P, in space-time, its value at any other point, P', in space time is given by a line integral

$$\theta_{P'} = \int_P^{P'} \alpha_m dx^m - \tfrac{1}{4} i \ln\left[\frac{\phi^{\kappa\lambda} \phi_{\kappa\lambda}}{\phi_{\dot{\mu}\dot{\nu}} \phi^{\dot{\mu}\dot{\nu}}}\right]\Bigg|_P^{P'} + \theta_P. \tag{3.32}$$

The geometry thus determines Maxwell's field almost uniquely; the only arbitrariness is the value of θ at an arbitrary point in space-time which can be arbitrarily given. It is worth noting that this arbitrariness already exists in the nonunified description of space and time. For if ω^{mn} and R^{mn} are two tensors that satisfy Eqs. (3.1) and (3.2), one can create from this solution a family of solutions given by $\omega^{mn} e^{i\theta}$ and R^{mn}. Here θ is a constant, independent of space-time.

The analysis leading to theorem 4 does not hold in the case of a null field for which $R^a{}_b R^b{}_a = 0$. For a null field the self-dual antisymmetric tensor, ω_{mn}, can be represented by Eq. (3.4) with $\phi^{\alpha\beta}$ reduced to the bilinear product of a first rank spinor $\phi^{\alpha\beta} = \phi^{\alpha} \phi^{\beta}$. Consequently Eq. (3.5) does not hold since $\phi^{\alpha\beta} \phi_{\beta}{}^{\mu} = \phi^{\alpha} \phi^{\beta} \phi^{\mu} \phi_{\beta} = 0$ by virtue of the fact that $\phi^{\beta} \phi_{\beta} = 0$. The null field will be discussed in greater detail later.

No use has been made in the above derivation of the fact that by virtue of the contracted Bianchi identity, only four of Maxwell's eight equations are independent. However, the remark is so readily proved in spinor notation that it is presented here. Maxwell's equations take the form

$$\omega^{mn}{}_{,n} = g^{m\dot{\alpha}}{}_{\kappa} g^n{}_{\dot{\alpha}\lambda} \phi^{\kappa\lambda}{}_{,n} = 0. \tag{3.33}$$

Multiply by $g_{m\dot{\beta}\rho} \phi^{\dot{\mu}\dot{\beta}}$, giving

$$g^n{}_{\dot{\beta}\lambda} \phi^{\rho\lambda}{}_{,n} \phi^{\dot{\mu}\dot{\beta}} = 0. \tag{3.34}$$

The set of Eqs. (3.34) and (3.33) are each four complex or eight real relations; moreover they are equivalent to one another. To see this equivalence, note that it is already shown that (3.33) implies (3.34); that (3.34) implies (3.33) can be seen by multiplying (3.34) by $g_{m\dot{\nu}\kappa} \phi^{\dot{\nu}}{}_{\dot{\mu}}$ and using the antisymmetry of $\phi^{\dot{\mu}\dot{\beta}} \phi^{\dot{\nu}}{}_{\dot{\mu}}$ in $\dot{\beta}$ and $\dot{\nu}$. Note that this equivalence is not true for the null field for which $\phi^{\dot{\mu}\dot{\beta}}$ can be represented by $\phi^{\dot{\mu}} \phi^{\dot{\beta}}$ and consequently $\phi^{\dot{\mu}} \phi^{\dot{\beta}} \phi^{\dot{\nu}} \phi_{\dot{\mu}} = 0$. The contracted Bianchi identity has the form

$$R^{mn}{}_{,n} = 0 = g^m{}_{\dot{\alpha}\kappa} g^n{}_{\dot{\beta}\lambda} (\phi^{\dot{\beta}\dot{\alpha}} \phi^{\kappa\lambda})_{,n}.$$

Multiply by $g_m{}^{\dot{\mu}\rho}$, to get

$$0 = g^n{}_{\dot{\beta}\lambda} \phi^{\rho\lambda}{}_{,n} \phi^{\dot{\mu}\dot{\beta}} + g^n{}_{\dot{\beta}\lambda} \phi^{\dot{\mu}\dot{\beta}}{}_{,n} \phi^{\rho\lambda}. \tag{3.35}$$

When $\rho = 1$, $\dot{\mu} = \dot{1}$; or $\rho = 2$, $\dot{\mu} = \dot{2}$, the Eqs. (3.34) are purely complex by virtue of (3.35). $\rho = 1$, $\dot{\mu} = \dot{2}$ gives the same real pair of equations as $\rho=2$, $\dot{\mu}=\dot{1}$, again by virtue of (3.35). Hence Eqs. (3.34), the Maxwell equations, contain only four independent equations because of the contracted Bianchi identity. This proof is not complete for the null-field case by virtue of the remark that (3.34) does not imply (3.33) for this case.

IV. SOME GEOMETRIC RELATIONS INVOLVING THE RICCI TENSOR R_{mn} AND THE ELECTROMAGNETIC TENSOR ω_{mn}

By use of spinor analysis, one can readily derive some geometric relations involving the tensors R_{mn} and ω_{mn} and their expressions in terms of null vectors and of principal vectors. Some of these relations will now be derived. Proceeding in this direction, one can also readily obtain the connections already found by Rainich,[1] Synge,[11] and Ruse.[6]

The key to these possibilities is that one can express the symmetric spinor $\phi^{\alpha\beta}$ of the second rank, which determines ω_{mn} and R_{mn}, by means of two first rank spinors, χ^α and ψ^α.

$$\phi^{\alpha\beta}=\tfrac{1}{2}(\chi^\alpha\psi^\beta+\psi^\alpha\chi^\beta). \qquad (4.1)$$

$\phi^{\alpha\beta}$ has three arbitrary complex components; χ^α and ψ^β have four, two from each spinor, so that one of the four components can be arbitrarily chosen. From χ^α and ψ^β four vectors can be formed:

$$\begin{aligned} k_m&=\tfrac{1}{2}g_{m\dot{\alpha}\kappa}\chi^{\dot{\alpha}}\chi^\kappa,\\ l_m&=\tfrac{1}{2}g_{m\dot{\alpha}\kappa}\psi^{\dot{\alpha}}\psi^\kappa,\\ p_m&=\tfrac{1}{2}g_{m\dot{\alpha}\kappa}\psi^{\dot{\alpha}}\chi^\kappa,\\ \bar{p}_m&=\tfrac{1}{2}g_{m\dot{\alpha}\kappa}\chi^{\dot{\alpha}}\psi^\kappa. \end{aligned} \qquad (4.2)$$

k_m and l_m are real vectors, p_m and $\bar{p}_m$ are complex and the complex conjugates of each other. All four vectors are null and each is orthogonal to two other vectors. The only nonvanishing scalar products that can be obtained from these vectors are

$$l^m k_m=-p^m\bar{p}_m=\tfrac{1}{2}A\bar{A}\,;\quad A\equiv\psi_\alpha\chi^\alpha. \qquad (4.3)$$

The rays determined by the four vectors (4.2) are uniquely determined by (4.1). A new choice of the free component will merely multiply all components of any one vector by a factor and may change the naming of the vectors.

ω_{mn} can now be expressed as

$$\begin{aligned}\omega_{mn}&=\tfrac{1}{2}g_m{}^{\dot{\mu}}{}_\kappa g_{n\dot{\mu}\lambda}(\chi^\kappa\psi^\lambda+\chi^\lambda\psi^\kappa)\\ &=\tfrac{1}{2}g_{m\dot{\mu}\kappa}g_{n\dot{\nu}\lambda}\epsilon^{\dot{\mu}\dot{\nu}}(\psi^\lambda\psi^\kappa+\psi^\kappa\chi^\lambda). \end{aligned}\qquad (4.4)$$

But $\epsilon^{\dot{\mu}\dot{\nu}}=(\chi^{\dot{\mu}}\psi^{\dot{\nu}}-\chi^{\dot{\nu}}\psi^{\dot{\mu}})/\chi_{\dot{\alpha}}\psi^{\dot{\alpha}}$. Consequently,

$$2\bar{A}\omega_{mn}=g_{m\dot{\mu}\kappa}g_{n\dot{\nu}\lambda}(\chi^{\dot{\nu}}\psi^{\dot{\mu}}-\chi^{\dot{\mu}}\psi^{\dot{\nu}})(\chi^\kappa\psi^\lambda+\chi^\lambda\psi^\kappa). \qquad (4.5)$$

This can be expanded, using (4.2), as

$$\omega_{mn}=2(l_\mu k_\nu-k_\mu l_\nu+p_\mu\bar{p}_\nu-\bar{p}_\mu p_\nu)/\bar{A}. \qquad (4.6)$$

[11] See reference 5. See also J. L. Synge, *Relativity, The Special Theory* (Interscience Publishers, Inc., New York, 1956), Chap. IX.

In this expression ω_{mn} is a complex tensor; l_m, k_m are real null vectors, p_m is a complex null vector with $\bar{p}_m$ its complex conjugate; and A is a complex scalar.

One can now show that

$$R_{mn}=\omega_{ma}\bar{\omega}_n{}^a=-2[p_m\bar{p}_n+\bar{p}_m p_n+l_m k_n+k_m l_n]. \qquad (4.7)$$

Equations (4.6) and (4.7) show that *the contracted curvature tensor, R_{mn}, is completely determined by four null vectors, two real and two complex; and that the electromagnetic tensor, ω_{mn}, is completely determined by these same null vectors and a complex scalar*. The work of the preceding sections shows how to find these null vectors and the scalar from R_{mn} and ω_{mn}. Define k_m to be a principal vector of ω_{mn} if $\omega^{mn}k_n$ is a scalar multiple, β, of k^m. Then the four vectors (4.2) are principal vectors of R_{mn} and ω_{mn}. Using (4.15), (4.16), (4.3), (4.4), (4.5), and (4.6), one gets immediately

$$\begin{aligned} \omega^{mn}k_n&=-Ak^m, & R^{mn}k_n&=-A\bar{A}k^m,\\ \omega^{mn}l_n&=Al^m, & R^{mn}l_n&=-A\bar{A}l^m,\\ \omega^{mn}p_n&=-Ap^m, & R^{mn}p_n&=A\bar{A}p^m,\\ \omega^{mn}\bar{p}_n&=A\bar{p}^m, & R^{mn}\bar{p}_n&=A\bar{A}p^m. \end{aligned} \qquad (4.8)$$

Moreover the complex scalar A is related to one of the complex invariants of the electromagnetic field and $\bar{A}$ to the other,

$$\omega_{mn}\omega^{mn}=-2A^2. \qquad (4.9)$$

In the case of a null field, $A=0$, and the above analysis breaks down. The null field will be discussed again later in Sec. V.

The relationships between the null vectors, the complex scalar A, R^{mn}, and ω^{mn} can be rewritten by use of a readily established spinor relation [multiply Eq. (9.1g) of Corson[8] by $g_{n\dot{\mu}\rho}$]:

$$\begin{aligned}2g^m{}_{\dot{\mu}\sigma}g^l{}_{\dot{\nu}\rho}=g^m{}_{\dot{\mu}\rho}g^l{}_{\dot{\nu}\sigma}-2g^{al}\epsilon_{\dot{\nu}\dot{\mu}}\epsilon_{\sigma\rho}+g^l{}_{\dot{\mu}\rho}g^m{}_{\dot{\nu}\sigma}\\ -g^{-\frac{1}{2}}\epsilon^{mnlk}g_{k\dot{\nu}\sigma}g_{n\dot{\mu}\rho}.\end{aligned} \qquad (4.10)$$

This and (4.2) lead to

$$2p^m\bar{p}^n=l^mk^n+k^nl^n-\tfrac{1}{2}g^{mn}A\bar{A}-g^{-\frac{1}{2}}\epsilon^{mlnk}k_kl_l. \qquad (4.11)$$

Consequently,

$$\omega^{mn}=2(l^mk^n-k^nl^m-g^{-\frac{1}{2}}\epsilon^{mlnk}k_kl_l)/\bar{A}, \qquad (4.12)$$

$$R^{mn}=A\bar{A}g^{mn}-4(l^mk^n+k^ml^n). \qquad (4.13)$$

This means that ω^{mn} and R^{mn} are completely determined by two real null vectors and a complex scalar; moreover by the previous work *the two real vectors and the complex scalar are completely determined by R_{mn} and ω_{mn}*. Again, in the case of a null field, $A=0$; $\chi^\alpha=\psi^\alpha$; and all four null vectors (4.2) degenerate into a single real null vector.

For the non-null case, ω^{mn} and R^{mn} can be represented by means of the complex null vector p^m and the complex scalar A. From (4.11) and (4.2).

$$2k^ml^n=\bar{p}^mp^n+2g^{mn}A\bar{A}+p^m\bar{p}^n-\tfrac{1}{2}g^{-\frac{1}{2}}\epsilon^{mlnk}p_k\bar{p}_l. \qquad (4.15)$$

Consequently

$$\omega^{mn}=2(p_m\bar{p}_n-\bar{p}_m p_n+g^{-\frac{1}{2}}\epsilon^{mlnk}p_k\bar{p}_l)/\bar{A}, \qquad (4.16)$$

$$R^{mn}=-4(g^{mn}A\bar{A}+p^m\bar{p}^n+\bar{p}^m p^n). \qquad (4.17)$$

V. A DISCUSSION OF THE NULL FIELD

A null field has been defined as one in which the two invariants of the electromagnetic field (often represented in flat space as $\mathbf{E}^2-\mathbf{H}^2$ and $\mathbf{E}\cdot\mathbf{H}$) both vanish. In the notation of this paper a null field is one in which the complex scalar $\psi_{mn}\omega^{mn}=\bar{\omega}_{mn}\bar{\omega}^{mn}=0$. We recall from Table I that for a null field

$$\omega_{mn}=g_m{}^{\dot{\alpha}}{}_{\kappa}g_{n\dot{\alpha}\lambda}\phi^\kappa\phi^\lambda. \qquad (5.1)$$

A null field is the special case of a non-null field in which the spinor $\phi^{\kappa\lambda}$ can be written as $\phi^\kappa\phi^\lambda$. For a null field, Eq. (3.11) still holds for R_{ab} with the understanding that $\phi^{\alpha\beta}=\phi^\alpha\phi^\beta$:

$$R_{ab}=2q_a{}^{\dot{\alpha}}{}_{\kappa}g_{b\dot{\beta}\lambda}\phi^\kappa\phi^\lambda\phi^{\dot{\beta}}\phi_{\dot{\alpha}}. \qquad (5.2)$$

Consequently for a null field, $R^a{}_bR^b{}_c=0$, since $\phi_\alpha\phi^\alpha=0$. For a null field, the theorems and lemmas of Sec. III which still apply take the following forms:

Theorem 1: $R_{ab}-\frac{1}{2}g_{ab}R=\omega_{ac}\bar{\omega}_b{}^c$ implies $R=0$.

Theorem 2': $R_{ab}-\frac{1}{2}g_{ab}R=\omega_{ac}\bar{\omega}_b{}^c$ for a null field implies $R^a{}_bR^b{}_c=0$.

Lemma 1': $R^a{}_bR^b{}_c=0$ and $R=0$ imply the existence of ϕ^α such that $R_{ab}=2g_a{}^{\dot{\mu}}{}_{\kappa}g_{b\dot{\nu}\lambda}\phi^\kappa\phi^\lambda\phi^{\dot{\nu}}\phi_{\dot{\mu}}$.

Proof: By Lemma 1, $R_{ab}=2g_a{}^{\dot{\mu}}{}_{\kappa}g_{b\dot{\nu}\lambda}\phi^{\kappa\lambda}\phi^{\dot{\nu}}{}_{\dot{\mu}}$. Any second-rank symmetric tensor can be written in the form of Eq. (4.1), $\phi^{\alpha\beta}=\frac{1}{2}(\phi^\alpha\psi^\beta+\psi^\alpha\phi^\beta)$. Consequently

$$R_{ab}=\tfrac{1}{2}g_a{}^{\dot{\mu}}{}_{\kappa}g_{b\dot{\nu}\lambda}(\phi^\kappa\phi^\lambda+\psi^\kappa\phi^\lambda)(\phi^{\dot{\nu}}\psi_{\dot{\mu}}+\psi^{\dot{\nu}}\phi_{\dot{\mu}}). \qquad (5.3)$$

The lemma is proved if it can be shown that ϕ^α must be proportional to ψ^α. A straightforward calculation shows that

$$R^a{}_bR^b{}_c=0=g^a{}_{\dot{\mu}\kappa}g_{c\dot{\nu}\lambda}(\phi_{\dot{\alpha}}\phi^{\dot{\nu}}\psi_\rho\psi^\lambda\psi^{\dot{\mu}}\psi^{\dot{\alpha}}\phi^\kappa\phi^\rho +\psi_{\dot{\alpha}}\psi^{\dot{\nu}}\phi_\rho\phi^\lambda\phi^{\dot{\mu}}\phi^{\dot{\alpha}}\psi^\kappa\psi^\rho).$$

When $a=c$, the expression can be equal to zero only if ϕ^α is proportional to ψ^α. This proves the lemma.

Theorem 4': If $R=0$ and $R^a{}_bR^b{}_c=0$, one can give a prescription to find a self-dual antisymmetric tensor, ω_{mn}, that satisfies $R_{ab}-\frac{1}{2}g_{ab}R=\omega_{ac}\bar{\omega}_b{}^c$. ω_{mn} is not uniquely determined; from an ω_{mn} which satisfies the requirements of the theorem one can generate a family $\omega'_{mn}=e^{i\theta}\omega_{mn}$ which satisfy the requirements, θ being an arbitrary function of space and time.

Proof: From Lemma 1', $R_{ab}=-2g_{a\dot{\mu}\kappa}g_{b\dot{\nu}\lambda}\phi^\kappa\phi^\lambda\phi^{\dot{\nu}}\phi^{\dot{\mu}}$. Inverting this gives

$$\phi^\kappa\phi^\lambda\phi^{\dot{\mu}}\phi^{\dot{\nu}}=-\tfrac{1}{8}g^{a\dot{\mu}\kappa}g^{b\dot{\nu}\lambda}R_{ab}. \qquad (5.4)$$

This will determine ϕ^α up to a phase factor exactly as in the non-null case. ω_{mn} is then given by Eq. (5.1); again however $R_{00}<0$ is necessary. So far the results are in direct correspondence with the results of the non-null case. However as was pointed out in Sec. III, the proof given for Theorem 4 completely breaks down in the case of a null field. We will now give some considerations that show conclusively that Theorem 4 does not hold for the null field.

The question is whether the arbitrary θ alluded to after Theorem 1' can be picked so that ω_{mn} satisfies Maxwell's equations, $\omega^{mn}{}_{,n}=0$. From Theorem 1', a ϕ^α has been found such that

$$\omega^{mn}=g^{m\dot{\mu}}{}_{\kappa}g^n{}_{\dot{\mu}\lambda}\phi^\kappa\phi^\lambda e^{i\theta}.$$

Can θ be chosen so that

$$\omega^{mn}{}_{,n}=g^{m\dot{\mu}}{}_{\kappa}g^n{}_{\dot{\mu}\lambda}[(\phi^\kappa\phi^\lambda)_{,n}+i\phi^\kappa\phi^\lambda\theta_{,n}]=0.$$

Multiply by $g_{m\dot{\nu}\kappa}$ to obtain

$$0=g^{n\dot{\mu}}{}_{\lambda}[(\phi_\kappa\phi^\lambda)_{,n}+i\phi_\kappa\phi^\lambda\theta_{,n}]. \qquad (5.5)$$

Equations (5.5) are a set of four linear algebraic equations for the four unknown variables $\theta_{,n}$. In order that a set of linear algebraic equations be consistent it is necessary and sufficient that the rank of the determinant of coefficients be equal to the rank of the augmented matrix. If the set of equations is to yield a unique solution for the four variables $\theta_{,n}$, it is necessary and sufficient that the rank of these matrices be four. Examining Eq. (5.5) at a point in space-time using the normal coordinate system with the special spin representation at this point given by (3.15), one sees readily that the determinant of coefficients is of rank 2 but the rank of the augmented matrix is 3. The rank of the augmented matrix can be reduced to 2 if and only if the following spinor relationship (5.6) holds:

$$\phi^\kappa g^{a\dot{\mu}}{}_{\kappa}\phi_\lambda\phi^\lambda{}_{,a}=0. \qquad (5.6)$$

Consequently it has been proved that the $\theta_{,n}$ can be described by Eq. (5.5) in an algebraically consistent way if and only if $\phi^\kappa g^{a\dot{\mu}}{}_{\kappa}\phi_\lambda\phi^\lambda{}_{,a}=0$. In this case $\theta_{,n}$ is described not uniquely but with two arbitrary parameters.

Contrast this with the case for the non-null field where Eq. (5.5) is replaced by

$$0=g^{n\dot{\mu}}{}_{\lambda}(\phi_\kappa{}^\lambda{}_{,n}+i\phi_\kappa{}^\lambda\theta_{,n})$$

and $\phi^{\alpha\beta}$ cannot be represented by $\phi^\alpha\phi^\beta$. In this case it is readily shown that the rank of the determinant of coefficients of $\theta_{,n}$ is equal to four. So $\theta_{,n}$ is consistently and uniquely determined by the algebraic equations.

Thus Theorem 4 cannot be applied to the null field for at least two reasons. Firstly Eq. (5.6) is a necessary and sufficient condition for the existence of $\theta_{,n}$ in the null case; but (5.6) cannot even be given meaning in the non-null case. Secondly, if Eq. (5.6) holds; $\theta_{,n}$ is still not uniquely determined, by (5.5); there remain two arbitrary parameters. The null field is thus fundamentally different from the non-null field.

We have not succeeded in finding whether a geometric condition exists which is a necessary and sufficient condition for (5.6). Nor have we shown, if (5.6) is satisfied, that Eqs. (5.5) looked on as differential equations for $\theta_{,n}$ are integrable.

It has already been pointed out that for a null field the four vectors (4.2) degenerate into a single null vector, k^m. From (4.8) one can say that for a null field,

$$R_{mn}=-8k_mk_n. \tag{5.7}$$

This also follows immediately from Lemma 1′ with $k^m=\frac{1}{2}g^m{}_{\dot{\alpha}\beta}\phi^{\dot{\alpha}}\phi^\beta$. Equation (4.6), in the case of a null field, approaches an indeterminate form in which the numerator and denominator both vanish. One can, however, find an expression for a null Maxwell field reminescent of (4.6):

$$\omega^{mn}=g^{m\dot{\mu}}{}_\kappa g^n{}_{\dot{\mu}\lambda}\phi^\kappa\phi^\lambda=g^m{}_{\dot{\mu}\kappa}g^n{}_{\dot{\nu}\lambda}\epsilon^{\dot{\mu}\dot{\nu}}\phi^\kappa\phi^\lambda.$$

Let $\psi^{\dot{\alpha}}$ be an arbitrary spinor not proportional to $\phi^{\dot{\alpha}}$; then

$$\epsilon^{\dot{\mu}\dot{\nu}}=(\phi^{\dot{\mu}}\psi^{\dot{\nu}}-\psi^{\dot{\mu}}\phi^{\dot{\nu}})/\phi_{\dot{\alpha}}\psi^{\dot{\alpha}},$$

$$\omega^{mn}=g^m{}_{\dot{\nu}k}g^n{}_{\dot{\mu}\lambda}(\phi^\kappa\phi^\lambda\phi^{\dot{\nu}}\psi^{\dot{\mu}}-\psi^{\dot{\nu}}\phi^{\dot{\mu}}\phi^\kappa\phi^\lambda)/(\phi_{\dot{\alpha}}\psi^{\dot{\alpha}}).$$

Now call

$$l^n=g^n{}_{\dot{\mu}\lambda}\psi^{\dot{\mu}}\phi^\lambda/2^{\frac{1}{2}}i\phi_{\dot{\alpha}}\psi^{\dot{\alpha}}, \tag{5.8}$$

$$\omega^{mn}=(1/i2^{\frac{3}{2}})(k^ml^n-k^nl^m). \tag{5.9}$$

The vector l^n is complex, null, and orthogonal to the real, null vector, k_n. Moreover $l^m\bar{l}_m=-1$. For a null field there exists a real, null, uniquely determined vector k^m, and a family of complex null vectors l^m, with the properties $l^ml_m=0$, $l^m\bar{l}_m=-1$, $l^mk_m=0$, such that $\omega^{mn}=i2^{-\frac{3}{2}}(l^mk^n-l^mk^m)$.

There is a curious relationship involving the null field, ω_{mn}, the two-component spinor, ϕ^α, which it determines, and a geometric equation involving R_{mn}. Recall first the two-component spinor equation which has recently been given prominence in regard to neutrino theory,

$$g^m{}_{\dot{\alpha}\beta}\phi^\beta{}_{,m}=0. \tag{5.10}$$

Differentiating the expression $\omega_m{}^n=g_m{}^{\dot{\mu}}{}_\kappa g^n{}_{\dot{\mu}\lambda}\phi^\kappa\phi^\lambda$, and using the definition of spinors (2.1) will yield

$$\omega^{mn}{}_{,n}=2g^{m\dot{\mu}}{}_\kappa\phi^\kappa g^n{}_{\dot{\mu}\lambda}\phi^\lambda{}_{,n}+2\phi_\alpha\phi^{\alpha,m}. \tag{5.11}$$

Now we proceed to prove that $\phi_\alpha\phi^\alpha{}_{,m}=0$ if and only if the following geometric relation holds:

$$R^a{}_{n,m}R^{rn}{}_{,s}=0. \tag{5.12}$$

Define the vector $H_m\equiv\phi_\alpha\phi^\alpha{}_{,m}$. From (5.4) one can deduce

$$R^a{}_{n,m}R^{rn}{}_{,s}=-8R^{ar}(k_{n,m}k^n{}_{,s}). \tag{5.13}$$

Moreover from the definitions of k_n and H_n,

$$k_{n,m}k^n{}_{,s}=-\tfrac{1}{2}(\bar{H}_mH_s+\bar{H}_sH_m). \tag{5.14}$$

Consequently

$$R^a{}_{n,m}R^{rn}{}_{,s}=4(\bar{H}_mH_s+\bar{H}_sH_m)R^{ar}. \tag{5.15}$$

If $H=0$ it follows that $R^a{}_{n,m}R^{rn}{}_{,s}=0$. Conversely, if $\bar{H}_mH_s+\bar{H}_sH_m=0$ for all m and s, it is easily seen that every individual component H_m must vanish. Consequently these statements follow:

1. The validity of Maxwell's equations, $\omega^{mn}{}_{,n}=0$, and of the two-component spinor equation, $g^n{}_{\dot{\mu}\lambda}\phi^\lambda{}_{,n}=0$, implies the validity of the geometric relation

$$R^a{}_{n,m}R^{rn}{}_{,s}=0.$$

The same relation follows if the two-component spinor equation is replaced by the less restrictive equation $g_m{}^{\dot{\mu}}{}_\alpha\phi^\alpha g^n{}_{\dot{\mu}\lambda}\phi^\lambda{}_{,n}=0$.

2. The validity of the two-component spinor equation, $g^n{}_{\dot{\mu}\lambda}\phi^\lambda{}_{,n}=0$, and of the geometric relation, $R^a{}_{n,m}R^{rn}{}_{,s}=0$, implies the validity of Maxwell's equations, $\omega^{mn}{}_{,n}=0$. Again it is sufficient to satisfy

$$g_m{}^{\dot{\mu}}{}_\alpha\phi^\alpha g^m{}_{\dot{\mu}\lambda}\phi^\lambda{}_{,n}=0$$

instead of the two-component spinor equation.

3. The validity of the geometric relation

$$R^a{}_{n,m}R^{rn}{}_{,s}=0$$

and of Maxwell's equations, $\omega^{mn}{}_{,n}=0$, implies the validity of $g_m{}^{\dot{\mu}}{}_\lambda\phi^\lambda g^n{}_{\dot{\mu}\kappa}\phi^\kappa{}_{,n}=0$.

VI. CONCLUDING REMARKS

A considerable effort has been expended in recent years in developing a quantized theory of general relativity and in developing a classical unified field theory. It has been shown here, as has already been known, that, if the only fields of nature are gravitational and non-null electromagnetic, the existing theory can already be described in a unified geometric way. Consequently, attempts at finding a different unified theory may be superfluous. On the other hand almost all the work that has been done towards quantizing gravitational theory has dealt with gravitation in the absence of other fields. One would guess however that the interesting features to be learned from the quantization, if indeed there are any, would arise from the interaction between the gravitational field and other fields. It is consequently crucial that this interaction be treated in the proper way. In considering the interplay between the gravitational and electromagnetic fields, one might suppose that the proper theory is a quantized version of the unified field theory described in this paper rather than an independent quantization of the gravitational and electromagnetic fields. However, one does not know how to proceed, even in the most vague way, to quantize the unified field. This is because the theory has not been derived from a variational principle. A Lagrangian function for the theory has not been found, nor has a Hamiltonian density function. The known techniques for quantization are thus not applicable to this theory as its present stage of development.

Another interesting remark is that the unified field contains partial differential equations of the fourth order. With few exceptions, one of the latest of which is that of Lanczos,[12] all of the attempts at developing unified field theories have dealt with second-order theories; this was because the basic field equations of mathematical physics apparently did not surpass the order two. This constraint should not, apparently, be taken so seriously as it has been in the past. A comparatively simple Lagrangian might be found, from which a variational principle will lead to a fourth-order theory very close to the unified theory here presented.

In addition to the major unsolved problem of quantization some other issues remain in the already unified theory. One is the question of null fields. It may be that the existence of physically interesting null fields is altogether denied by the Einstein-Maxwell theory. If so one should like to see this remark precisely stated and proved. It may be however that the possibility of having null fields, which are regions in space-time where the already unified theory breaks down, is trying to tell us something very important. The unified theory breaks down at particle-like singularities and also at null fields; can there be some still mysterious connections between these two regions of breakdown?

It is also unexplained why no magnetic charges are seen in nature. If one charge is arbitrarily called "electric," do the purely geometrical equations of already unified theory automatically guarantee that all charges are electric? Or is this a separate postulate to be added to the theory?

It is probably true that in no four-dimensional region of space-time is the electromagnetic energy density identically zero. It is therefore conceivable that in principle one should always deal with the equations of the unified field and not with the case of pure gravitation. Should not one allow only those solutions of Einstein's equations which are limiting cases of solutions of the full set of the unified Einstein-Maxwell equations? Does this requirement limit in any way the solutions of Einstein's equations of pure gravitation which are physically meaningful? Or might it not assist in reaching an understanding of gravitational radiation?

[12] C. Lanczos, Revs. Modern Phys. **24**, 337 (1957).