Proposal for Determining the Electromagnetic Form Factor of the Pion*

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The possibility of measuring the electromagnetic form factor of the pion by extrapolation of the cross section for $e^-+p\rightarrow n+\pi^++e^-$ has been investigated. The method is based on the existence of a pole in the pion-electroproduction scattering amplitude as a function of the invariant momentum-transfer of the nucleon. The residue of this pole is the pion form factor multiplied by a known coefficient. Since the pole lies slightly outside the physical region of the invariant momentum transfer, an extrapolation of the experimental data is required. An approximate calculation of the pion electroproduction cross section has been made in order to estimate the experimental accuracy necessary for a significant extrapolation. Accuracy is required which is an order of magnitude better than that achieved at present in similar experiments.

I. INTRODUCTION

In recent years much attention has been devoted to the problem of the electromagnetic structure of the nucleon. Considerable experimental information on this subject has been provided by the experiments carried out by Hofstadter and his collaborators on the scattering of electrons by protons and deuterons.[1] Additional information was derived from the measurement of the cross section for electroproduction of pions ($e^-+p\rightarrow e^-+n+\pi^+$) by Panofsky and Allton.[2]

Among the many theoretical attempts which have been made to treat the nucleon-structure problem, the most successful have been based on the method of spectral representations.[3–5] In this method, however, one encounters the difficulty of requiring knowledge of the electromagnetic structure of the pion. It is qualitatively evident that the structure of the pion must contribute to the structure of the nucleon. If the spatial extension of the nucleon's charge and magnetic moment are visualized as due to the pion cloud of the nucleon, then the spatial extension of the pion will contribute to that of the nucleon. It has recently been shown[5] that if the pion structure is assumed to be sufficiently extended, the disagreement[6] between the spectral-representation theory and the nucleon structure may be removed.

Unfortunately, no experiments have been performed which probe the pion structure. The scattering of pions on electrons yields no information at available energies, because the relatively massive incident pion cannot transfer sufficient momentum to make anything but the outermost parts of the pion electromagnetic field effective. On the other hand, scattering of electrons on pions is not feasible because, of course, no way has been found for making targets from pions. The purpose of this paper is to describe a method of using the pion cloud of the proton as a "pion target."

The procedure to be described is an application of a general method suggested recently by Chew and Low.[7] In this case, their method enables one to measure the electromagnetic form factor of the pion by extrapolation of the cross section for $e^-+p\rightarrow n+\pi^++e^-$. The basic principle involved is the possibility of analytic continuation of the electroproduction scattering amplitude as a function of the square of the four-momentum transfer of the nucleon, Δ^2. If the initial nucleon has four-momentum p, and the final nucleon, p', then we have $\Delta^2=-(p-p')^2$. It is conjectured that at $\Delta^2=-\mu^2$, where μ is the mass of the pion, the scattering amplitude has an isolated pole whose residue is just the electromagnetic form factor of the pion multiplied by a known coefficient. It can easily be shown that negative values of Δ^2 are not physically attainable, so that an extrapolation of the measured cross section is necessary in order to reach the pole at $\Delta^2=-\mu^2$. The distance of extrapolation is, however, small compared to the physical range of Δ^2. The procedure for determining the electromagnetic form factor of the pion is, then, to extrapolate the cross section, with its singularities suitably removed, to the point $\Delta^2=-\mu^2$. This procedure will be described in detail in Sec. II.

Experimentally, it will be necessary to measure the electroproduction cross section as a function of at least two variables, Δ^2 and λ. If s is the incident electron four-momentum, and s' is the final, then we have $\lambda=-(s-s')^2$. One must know λ because the pion form factor is a function of this variable. In practice, one could determine these variables by measuring the energy and direction of the final electron, plus the direction of the final meson. No such measurements have yet been made. In their recent experiment,

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[1] Hofstadter, Bumiller, and Yearian, Revs. Modern Phys. **30**, 482 (1958).

[2] W. K. H. Panofsky and E. A. Allton, Phys. Rev. **110**, 1155 (1958).

[3] Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958).

[4] Federbush, Goldberger, and Treiman, Phys. Rev. **112**, 642 (1958).

[5] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).

[6] S. D. Drell, *1958 Annual International Conference on High-Energy Physics at* CERN edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958).

[7] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

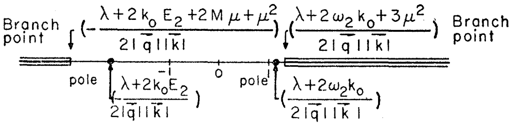

FIG. 1. Poles and branch points in the $\cos\theta$ plane. The scale of the figure is appropriate to the values $E=9.66$, $\lambda=10$. The right-hand pole is the one at $\Delta^2=-\mu^2$.

Panofsky and Allton[2] measured the electron variables but did not detect the meson directly.

The analytic properties of the electroproduction scattering amplitude, which are the basis of the method outlined above, have not been proved rigorously. They are, however, a very reasonable extension of properties which have been proved for other scattering problems. A plausibility argument will be given in Sec. II A. Such analytic properties of scattering amplitudes have been the object of much study. Their application in the form of spectral representations has been successful in correlating many experimental data in pion physics. Applications very similar to that proposed herein have already been made for the purpose of measuring the pion-nucleon coupling constant. Extrapolations of both the nucleon-nucleon[8,9] and pion-photoproduction[10] cross sections as functions of invariant momentum transfer yielded values of the coupling constant in reasonable agreement with values obtained by other means.

In principle, then, the analytic properties of the electroproduction scattering amplitude described above tell us that one can determine the pion form factor by an extrapolation procedure. In order to assess the practical difficulty involved in performing an extrapolation of a given set of experimental data, one must estimate the behavior of the electroproduction cross section as a function of Δ^2. If the electromagnetic interaction is treated in lowest order of perturbation theory, it is apparent that the electroproduction and photoproduction matrix elements are closely related. The dispersion-theoretical analysis of photoproduction by Chew, Goldberger, Low, and Nambu[11] (hereafter called CGLN) has been extended to electroproduction by Fubini, Nambu, and Wataghin[12] (hereafter FNW). The calculation in Sec. III is performed by the use of a modification of their theory: The Born terms in the matrix element are written in relativistic form and, correspondingly, certain recoil-correction terms are dropped.

In Sec. IV the results of this calculation are interpreted as implying that for a significant extrapolation experiments of great accuracy will be necessary, accuracy an order of magnitude better than that achieved in the electroproduction experiment of Panofsky and Allton.[2]

[8] Geoffrey F. Chew, Phys. Rev. **112**, 1380 (1959).

[9] P. Cziffra and M. J. Moravcsik, Phys. Rev. (to be published).

[10] Taylor, Moravcsik, and Uretsky, Phys. Rev. **113**, 689 (1959), quoted in the following as TMU.

[11] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 and 1345 (1957); quoted in the following as CGLN. See also Geoffrey F. Chew, *Encyclopedia of Physics* (Springer-Verlag, Berlin, to be published), second edition, Vol. 43.

[12] Fubini, Nambu, and Wataghin, Phys. Rev. **111**, 329 (1958), quoted in the following as FNW.

II. EXTRAPOLATION PROCEDURE

A. Location of Singularities of Scattering Amplitude

Let us consider in detail the analytic properties of the pion-electroproduction scattering amplitude on which the proposed extrapolation depends. As remarked in the introduction, no rigorous proof of these properties has been given. However, such properties have often been conjectured; for instance, in the two-dimensional spectral representation proposed by Mandelstam[13] and verified to fourth order in perturbation theory. Motivation for conjecturing them comes from two sources: analogy with proved properties of simpler scattering amplitudes and analysis of perturbation theory. Arguments of both types have been given by Chew[8] for nucleon-nucleon scattering, and by Taylor, Moravcsik, and Uretsky[10] (hereafter TMU) for photoproduction. The latter case can be extended very easily to electroproduction. To lowest order in the fine-structure constant, electroproduction is just photoproduction by a virtual photon. The only differences are that the "photon" has a nonzero, imaginary mass ($k^2=-\lambda$) in electroproduction, and that the matrix element contains longitudinal as well as transverse terms (i.e., $\mathbf{k}\cdot\boldsymbol{\varepsilon}\neq 0$). The existence of transverse terms will not affect the analytic properties, and the "photon" mass will only cause a shift in the position of the singularities.

The recipe used in the papers referred to above is the following: to get one part of the spectrum consider the intermediate states which could be reached if p and p' were incoming particles, with q and k outgoing ($k=s-s'$). The lowest of these is the discrete single pion state, which gives a pole at $\Delta^2=-\mu^2$ (on which the proposed extrapolation is based). The next state is that of two pions, which gives rise to a branch point at $\Delta^2=-4\mu^2$.

The other half of the spectrum, the crossed spectrum, is found in a similar way by considering the states which can be reached if p and q are incident. This leads to a pole at $(p-q)^2=M^2$, where M is the nucleon mass, and a branch point at $(p-q)^2=(M+\mu)^2$. Now since $p+k=p'+q$, we find that

$$(p-q)^2=\Delta^2-E^2+2M^2+\mu^2-\lambda. \qquad (2.1)$$

Here $E^2\equiv(p'+q)^2$; i.e., E is the total energy of the final nucleon and pion in their barycentric system (the system in which $\mathbf{p}'+\mathbf{q}=0$). Using (2.1), one finds that the crossed spectrum gives rise to a pole at

$$\Delta^2=E^2-M^2-\mu^2+\lambda,$$

and a branch point at

$$\Delta^2=E^2-M^2+2M\mu+\lambda.$$

[13] S. Mandelstam, Phys. Rev. **112**, 1344 (1959).

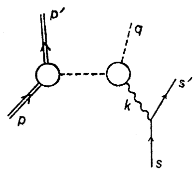

FIG. 2. The class of diagrams giving rise to the pole at $\Delta^2=-\mu^2$.

The spectrum of singularities in the Δ^2 plane can be re-expressed in terms of $\cos\theta$, where θ is the angle between $\mathbf{q}$ and $\mathbf{k}$ in the $\mathbf{p}'+\mathbf{q}=0$ system. With all symbols referring to this system, we have

$$\Delta^2=\lambda-\mu^2+2\omega_2k_0-2|\mathbf{q}||\mathbf{k}|\cos\theta, \qquad (2.2)$$

where

$$\omega_2\equiv(|\mathbf{q}|^2+\mu^2)^{\frac{1}{2}}.$$

Then in the $\cos\theta$ plane the analyticity region is the cut plane, with poles and branch points as shown in Fig. 1. The quantity E_2 is the energy of the final nucleon. Actually, the existence of such a large region of analyticity is not necessary for the proposed extrapolation. The method requires only that we have analyticity in some region containing the physical region $|\cos\theta|\leq 1$ and including the pole at $\cos\theta=(\lambda+2\omega k_0)/2|\mathbf{q}||\mathbf{k}|$ as an isolated singularity.

B. The Extrapolation Formula

If one accepts the viewpoint that perturbation theory can yield information on analytic behavior, then the existence of the pole at $\Delta^2=-\mu^2$ can be demonstrated and its residue computed. It is evident that the class of diagrams shown in Fig. 2 gives rise to the pole. It is easy to show that no other type of diagram can contain this pole. The contribution of this diagram[14] to the cross section gives, in covariant form,

$$\frac{d\sigma}{dP}=\frac{4g^2\alpha^2\Delta^2F_\pi^2(\lambda)}{\lambda^2\pi^2I(\Delta^2+\mu^2)^2}[4(q\cdot s)(q\cdot s')-\mu^2\lambda] +\text{other terms}, \qquad (2.3)$$

where $I\equiv[(p\cdot s)^2-M^2m^2]^{\frac{1}{2}}=Ms_L$, and where dP is the phase space factor:

$$dP\equiv\frac{d^3p'd^3s'd^3q}{8\omega_2E_2\epsilon_2}\delta^4(p+s-p'-s'-q). \qquad (2.4)$$

The symbols ϵ_1, ϵ_2 are defined as the initial and final electron energies. One finds that

$$dP=\frac{\pi|\mathbf{q}|^2d\lambda dE^2d\Omega_q}{16Ms_L\omega_2|(d/d|\mathbf{q}|)q\cdot(p+s-s')|}, \qquad (2.5)$$

where $d\Omega_q$ refers to the outgoing meson. Since dP as defined by Eq. (2.4) is a Lorentz invariant, this expression must be valid in any coordinate system. In the lab system, we have

$$dP_L=\frac{\pi|\mathbf{q}|^2d\lambda dE^2d\Omega_q}{16Ms_L[|\mathbf{q}|(M+k_0)-\omega_2|\mathbf{k}|\cos\theta]}. \qquad (2.6)$$

In Eq. (2.3) two quantities in the numerator, g and $F_\pi(\lambda)$, are not the most general expressions corresponding to the diagram of Fig. 2. They have been given the value appropriate to $\Delta^2=-\mu^2$ in anticipation of the extrapolation to that point. The pion form factor, $F_\pi(\lambda)$, is defined by considering the pion-photon vertex with both pions on the mass shell. If we write the contribution of this vertex as $j_\mu(q_1,q_2)\epsilon^\mu$, the most general form of j consistent with Lorentz invariance is

$$j(q_1,q_2)=-eF_\pi(\lambda)q_1-eG_\pi(\lambda)q_2,$$

where $\lambda=-(q_2-q_1)^2$. The continuity equation imposes the further requirement $(q_2-q_1)\cdot j=0$, giving $F_\pi(\lambda)=G_\pi(\lambda)$, or

$$j(q_1,q_2)=-(q_1+q_2)eF_\pi(\lambda).$$

This definition of the pion form factor is normalized so that $F_\pi(0)=1$.

Equation (2.3) reveals an additional singularity which must be removed before an extrapolation can be performed. The factor in brackets, which results from taking the trace of matrix factors, has the form, as a function of $\cos\theta$: $P_1(\cos\theta)+\sin\theta P_2(\cos\theta)$, where P_1 and P_2 are polynomials. Now since $\sin\theta=(1-\cos^2\theta)^{\frac{1}{2}}$, $\sin\theta$ has a branch point at $\cos\theta=1$, preventing extrapolation. We cannot get rid of this singularity by division by the factor in brackets, since it can vanish in the region of extrapolation and does vanish in the case to be discussed in Sec. III. We can, however, eliminate this undesirable $\sin\theta$ by defining a "symmetrized" cross section. Define $\sigma(\theta)=d\sigma/dP$. The quantity $\sigma(\theta)$ does, of course, depend upon variables other than θ. Then define the symmetrized quantity

$$\langle\sigma(\cos\theta)\rangle_s=\sigma(\theta)+\sigma(-\theta). \qquad (2.7)$$

In Eq. (3.19) an equivalent definition of this symmetrization will be given. The quantity $\langle\sigma(\cos\theta)\rangle_s$, will, of course, be free from the branch point at $\cos\theta=1$, and we can at last write the extrapolation formula:

$$(\Delta^2+\mu^2)^2\langle\sigma\rangle_s|_{\Delta^2=-\mu^2}=-\frac{4g^2\alpha^2\mu^2F_\pi^2(\lambda)}{\pi^2Ms_L\lambda^2} \times\langle 4(q\cdot s)(q\cdot s')-\mu^2\lambda\rangle_s|_{\Delta^2=-\mu^2}. \qquad (2.8)$$

The above discussion has shown that if one knew the value of the differential cross section $d\sigma/dP$ over some portion of the physical region, one could obtain the value of $F_\pi(\lambda)$ by analytic continuation of the function $(\Delta^2+\mu^2)\langle\sigma(\cos\theta)\rangle_s$. The relation between the value of this function at $\Delta^2=-\mu^2$, and the pion form factor is given by Eq. (2.8). This is an idealization, of course, since in practice one can know the function only to within a certain error and at a finite number of points.

[14] The coupling constants used are $g^2=g_r^2/4\pi$, $f=g\mu/2M$, and $f^2\approx 0.08$.

One practical procedure that can be employed is to plot the experimental value of $(\Delta^2+\mu^2)^2\langle\sigma\rangle_s$ and fit a polynomial in $\cos\theta$ to these points by the method of least squares.[9,15] The residue is then given by the value of this polynomial at the pole. Some consideration will be given in the next section to the error involved in this method of extrapolation.

C. Kinematical Considerations

In order to perform the proposed extrapolation, it is necessary to know the cross section as a function of both the extrapolation variable Δ^2 and the variable λ on which the form factor $F_\pi(\lambda)$ depends. Since an N-particle (incoming plus outgoing) scattering problem is a function of $3N-10$ variables (neglecting spins), electroproduction is a function of five variables. A convenient choice for the other three variables is E, the total energy of the pion and nucleon in the $\mathbf{p}'+\mathbf{q}=0$ system; T_L, the laboratory kinetic energy of the incident electron; and ϕ, defined in the $\mathbf{p}'+\mathbf{q}=0$ system by

$$\cos\phi=(\mathbf{k}\times\mathbf{s})\cdot(\mathbf{k}\times\mathbf{q})/|\mathbf{k}\times\mathbf{s}||\mathbf{k}\times\mathbf{q}|. \quad (2.9)$$

In principle these three variables could be integrated out and the cross section measured as a function of Δ^2 and λ only. In practice, it may be most convenient to determine all five variables; for instance, by knowing the incident electron energy and measuring the distribution of mesons as a function of direction, in coincidence with final electrons of given direction and energy. Then in performing the extrapolation all variables except Δ^2 must be held fixed. The question then arises: what values of E, T_L, and ϕ are most favorable to the extrapolation procedure?

To answer this question we must be able to estimate the error associated with extrapolation. Let us assume that the extrapolation will be done by fitting a polynomial in $\cos\theta$ to the function $(\Delta^2+\mu^2)^2\langle\sigma\rangle_s$ by the method of least squares. Then we can use a well-known formula to calculate the error.[15]

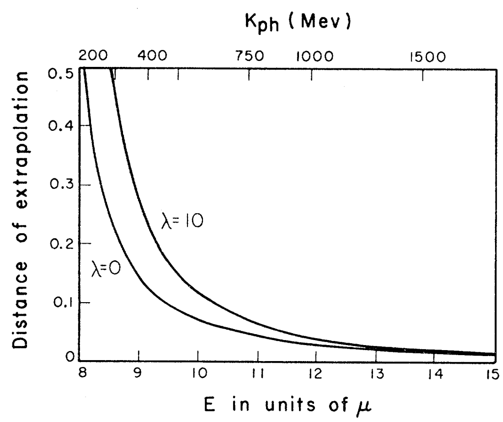

FIG. 3. The distance of extrapolation as a function of E for two values of λ. Also shown is k_{ph}, the equivalent laboratory energy in photoproduction $(E^2=M^2+2Mk_{\text{ph}})$.

[15] P. Cziffra and M. J. Moravcsik, University of California Radiation Laboratory Report UCRL-8523, October 17, 1958 (unpublished).

Let x_0 be the position of the pole as a function of $\cos\theta$. Then define

$$x\equiv x_0-\cos\theta.$$

In the $\mathbf{p}'+\mathbf{q}=0$ system, we find

$$\Delta^2+\mu^2=2|\mathbf{q}||\mathbf{k}|x.$$

We wish to extrapolate the function

$$f(x)\equiv(2|\mathbf{q}||\mathbf{k}|)^2x^2\langle\sigma\rangle_s \quad (2.10)$$

to the point $x=0$, the position of the pole of $\langle\sigma\rangle_s$. The error in the least-squares polynomial at the point $x=0$ is given by

$$\Delta a_0=\rho[(h^{-1})_{00}]^{\frac{1}{2}}, \quad (2.11)$$

where

$$h_{mn}=\sum_{i=1}^{N}\frac{x_i^{m+n}}{\zeta_i^2}, \quad (2.11')$$

and where ρ, which depends upon the goodness of fit, is ≈ 1.[15] The summation in Eq. (2.11′) extends over the points at which $f(x)$ is known, and ζ_i is the experimentally determined standard deviation of $f(x)$ at x_i. In order to use Eq. (2.11) to determine the dependence of Δa_0 on the experimental parameters, we must make some assumption about the behavior of the ζ_i's. Let us assume that there is a constant standard deviation ξ in the cross section $\langle\sigma\rangle_s$ for all values of x. This seems to be the most reasonable assumption to make in the absence of detailed knowledge of the behavior of $\langle\sigma\rangle_s$. This assumption implies that the standard deviation ζ_i of $f(x)$ is

$$\zeta_i=(2|\mathbf{q}||\mathbf{k}|)^2x_i^2\xi.$$

Then if $\rho=1$, Eq. (2.11) becomes

$$\Delta a_0=\xi(2|\mathbf{q}||\mathbf{k}|)^2[(H^{-1})_{00}]^{\frac{1}{2}}, \quad (2.12)$$

where

$$H_{mn}\equiv\sum_{i=1}^{N}x_i^{m+n-4}. \quad (2.12')$$

Note that the error Δa_0 is proportional to the absolute error in the measured cross section, and to the factor $[(H^{-1})_{00}]^{\frac{1}{2}}$ which depends only upon the distance of extrapolation and the distribution of points in the physical region.

It is qualitatively evident that the error will rise with the distance of extrapolation. Let us calculate the distance of extrapolation as a function of E and λ. Equation (2.2) shows that the pole occurs at

$$\cos\theta=(\lambda+2\omega_2k_0)/2|\mathbf{q}||\mathbf{k}|\equiv x_0. \quad (2.13)$$

Holding λ fixed, one finds that all quantities on the right-hand side are known functions of E. One can easily show that in the $\mathbf{p}'+\mathbf{q}=0$ system the following rela-

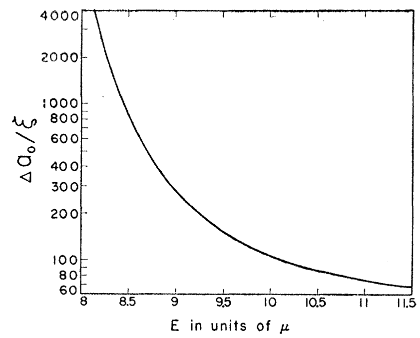

FIG. 4. The dependence of the extrapolation error $\Delta a_0/\xi$ on E.

tions hold:

$$2\omega_2 = E-(M^2-\mu^2)/E,$$
$$2k_0 = E-(M^2+\lambda)/E,$$
$$E_2 = E-\omega_2 = (|\mathbf{q}|^2+M^2)^{\frac{1}{2}},$$
$$E_1 = E-k_0 = (|\mathbf{k}|^2+M^2)^{\frac{1}{2}}.$$

The distance of extrapolation as a function of E, for two values of λ is shown in Fig. 3. Unless otherwise specified, pion mass units are used throughout this paper.

In order to evaluate the dependence of Δa_0 on the distance of extrapolation, we must calculate $(H^{-1})_{00}$. If we assume that we can fit $f(x)$ with a polynomial of the fourth order (S and P waves only), then H_{mn} is a five-by-five matrix. The inversion of this matrix was performed by a machine calculation for various distances of extrapolation for the arbitrary case of eleven points x_i spread evenly over the physical region. The results are summarized in Fig. 4, where $\Delta a_0/\xi$ is shown as a function of E for $\lambda=10$. If ξ does not vary much with E, higher values of E are clearly favored by this consideration.

A second consideration is the size of the residue at the pole. For a given Δa_0, the size of the residue determines the percent error. From Eq. (2.8) one sees that for a given value of λ and T_L the residue is proportional to

$$\langle 4(q\cdot s)(q\cdot s')-\mu^2\lambda\rangle_s.$$

In Fig. 5 this factor is plotted against E for two extreme values of ϕ. If one chooses $\phi=0$ or π, the size of the residue decreases with E. This decrease is, however, greatly outweighed by the more rapid decrease of $\Delta a_0/\xi$ with E shown in Fig. 4. The conclusion indicated is that unless experimental conditions create very strong variations with E in the absolute accuracy attainable, high values of E are desirable.

On the other hand, if E is too high, one will be forced to use a polynomial of higher than fourth order to fit $f(x)$; i.e., D waves will become important. The machine calculation of $(H^{-1})_{00}$ showed that the error increases markedly with the order of the polynomial. The error was calculated in the same manner as for Fig. 4 for the point $E=9.66$ for fifth- and sixth-order polynomials. For the fourth-order polynomial, Fig. 4 shows $\Delta a_0/\xi=134$. For the fifth-order, we found $\Delta a_0/\xi=345$, and for the sixth-order, $\Delta a_0/\xi=952$.

By machine calculation of the error for several specific distributions of points x_i, some qualitative conclusions were drawn concerning the most favorable distribution. First, it seems desirable to have measurements spread over as wide a region as possible, preferably the entire range of $\cos\theta$. Second, it seems desirable to concentrate most of the points close to $\cos\theta=1$. For example, at $E=9.66$ we saw that for eleven evenly spaced points $\Delta a_0/\xi=134$. For nine points at $\cos\theta=1$, 0.98, 0.94, 0.87, 0.71, 0.26, 0, -0.71, and -1, we found $\Delta a_0/\xi=95$. Third, it is of course desirable to have as many points as possible. For 6, 11, 14, and 21 evenly spaced points ($E=9.66$, fourth-order polynomial); $\Delta a_0/\xi=423$, 134, 100, and 75, respectively.

It is beyond the scope of this work to provide a definitive answer to the experimental question of the optimum values of the parameters for the purpose of the extrapolation. An attempt has been made in this section to discuss the most important factors. In Sec. III D a calculation will be made of the electroproduction cross section in order to translate Eq. (2.12) into an estimate of the maximum percent error compatible with a significant extrapolation. It is desirable, of course, to carry out this calculation at the most favorable value of the parameters. In order to consider a more familiar parameter, let us introduce the equivalent photoproduction energy k_{ph}, defined by

$$k_{\mathrm{ph}} = (E^2-M^2)/2M. \tag{2.14}$$

The quantity k_{ph} is the energy a photon must have in the laboratory in order to produce a pion-nucleon final state having total energy E in the barycentric system. On the basis of the considerations discussed in this section, a reasonable guess for the most promising value of k_{ph} is ≈ 500 Mev. The corresponding value of $E=9.66$ will be used in the calculation in Sec. III D.

As a final kinematical consideration let us determine the possible range of λ for a given E and T_L. In order to derive this range, note the relationship of λ, E, and Δ^2

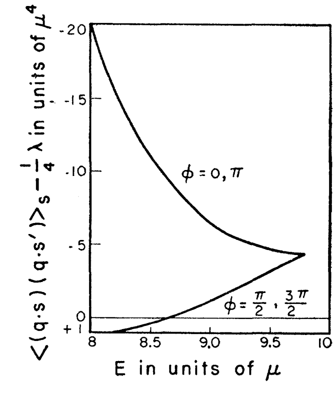

FIG. 5. The variation of the symmetrized residue at the pole as a function of E for various ϕ at $T_L=700$ Mev and $\lambda=10$.

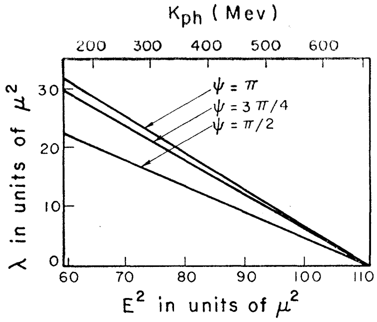

FIG. 6. E^2 vs λ for $T_L=700$ Mev for various values of the laboratory scattering angle ψ of the final electron. Points above the line $\psi=\pi$ are not kinematically possible.

to laboratory quantities, denoted by the subscript L:

$$\lambda=2T_L\epsilon_{2L}(1-\cos\psi), \tag{2.15}$$

$$E^2=M^2-\lambda+2M(T_L-\epsilon_{2L}), \tag{2.16}$$

$$\Delta^2=2M(E_{2L}-M). \tag{2.17}$$

Here ψ is the laboratory angle between the initial and final electron directions. In Eq. (2.15) the mass of the electron has been neglected compared to its momentum. From Eqs. (2.15) and (2.16) we find

$$E^2=M^2+2MT_L-\lambda\left[1+\frac{M}{T_L(1-\cos\psi)}\right]. \tag{2.18}$$

For a given λ, the maximum value of E^2 is obtained for backward scattering, $\psi=\pi$. Conversely, $\psi=\pi$ also gives the maximum value of λ permitted for a given E and T_L (see Fig. 6). By increasing T_L one can obtain higher values of λ.

III. ESTIMATE OF ELECTROPRODUCTION CROSS SECTION

A. General

In order to assess the difficulty of carrying out the proposed extrapolation, one must estimate the electroproduction cross section as a function of $\cos\theta$, with the parameters λ, E, T_L, and ϕ held fixed. A general treatment of the electroproduction problem has been given by Dalitz and Yennie (hereafter DY).[16] The most recent calculation, based on the photoproduction theory of Chew, Goldberger, Low, and Nambu,[11] was made by FNW. In this section a treatment will be given which relies heavily on the aforementioned papers, but which treats somewhat differently the corrections due to the finite mass of the nucleon.

Define the T matrix

$$S_{fi}=\delta_{fi}-i\delta^4(p'+q+s'-p-s)\left(\frac{m^2M^2}{\omega_2E_1E_2\epsilon_1\epsilon_2}\right)^{\frac{1}{2}}T_{fi}. \tag{3.1}$$

The T matrix element can be expressed in terms of the current j associated with the transition from nucleon to final pion-nucleon state:

$$T=[eg_r/(2\pi)^{7/2}]j\cdot\epsilon. \tag{3.2}$$

This form applies to both electroproduction and photoproduction. In the latter case, ϵ is proportional to the polarization vector; in the former, ϵ has to lowest order in e the value

$$\epsilon_\mu=e\bar{u}(s')\gamma_\mu u(s)/\lambda. \tag{3.3}$$

As pointed out in DY, the time components of j and ϵ can be expressed in terms of their space components by means of the continuity equation. This is useful in adapting photoproduction results to electroproduction.

The cross section can be written

$$\frac{d\sigma}{dP}=\frac{\alpha^2g^2}{\pi^2Ms_L\lambda}\left(\frac{E}{M}\right)^2\Phi, \tag{3.4}$$

where the dimensionless quantity Φ has the value

$$\Phi=\frac{(2M)^2}{\lambda}\left(\frac{M}{E}\right)^2\sum j_\mu^\dagger j_\nu(2s^\mu s^\nu-s^\mu k^\nu-s^\nu k^\mu-\tfrac{1}{2}\lambda g^{\mu\nu}). \tag{3.5}$$

We must now develop an expression for the current j.

B. Dispersion Theory of Photoproduction and Electroproduction

The most successful calculation of photoproduction, given in CGLN, is based on the method of spectral representations. This calculation was extended to electroproduction by FNW. At the values of the experimental parameters suggested in Sec. II C, namely $T_L=700$ Mev and $k_{ph}=500$ Mev, the approximate solutions of the dispersion equations given in CGLN cannot be expected to be very accurate. The solutions are based on neglecting terms of higher than first order in μ/M, ω/M, and $|\mathbf{k}|/M$. When $k_{ph}=500$ Mev and $\lambda=10$, $|\mathbf{k}|/M=0.56$. An indication of the accuracy of the CGLN formulas can be obtained by comparison with photoproduction data. Although photoproduction of neutral pions from protons is in excellent agreement with the CGLN theory up to 450-Mev lab energy,[17] positive-pion production shows poor agreement above 300 Mev. A summary of the comparison of theory and experiment in positive-pion production is given in Fig. 8 of the paper by Lazarus, Panofsky, and Tangherlini.[18] The lack of quantitative success of the photoproduction theory at high energies need not discourage us much here. The theory gives a semiquantitative fit, which should be quite adequate for the purpose of estimating the difficulty of the proposed extrapolation.

Let us now consider those aspects of the dispersion theory of electroproduction which are most important for our calculation, referring the reader to FNW for a

[16] R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1957), quoted in the following as DY.

[17] McDonald, Peterson, and Corson, Phys. Rev. **107**, 577 (1957).

[18] Lazarus, Panofsky, and Tangherlini, Phys. Rev. **113**, 1330 (1959).

more detailed account. The dominant terms in the amplitude for the production of positive mesons are the renormalized electron-charge Born approximation j_B and the resonant magnetic-dipole amplitude $\mathbf{j}_M$. The j_B term is

$$j_B{}^\nu=\bar{u}(p')\left\{i\gamma_5\frac{F_\pi(\lambda)(2q-k)^\nu}{\Delta^2+\mu^2}-i\gamma_5\frac{\gamma\cdot(p+k)+M}{E^2-M^2}\gamma^\nu F_1{}^p(\lambda)-\gamma^\nu F_1{}^n(\lambda)\frac{\gamma\cdot(p-q)+M}{(p-q)^2-M^2}i\gamma_5\right\}. \quad (3.6)$$

The first term, arising from the diagram of Fig. 2, contains the pole on which the extrapolation is based. Note the presence of the pion form factor, which was set equal to unity in FNW. The proton form factor $F_1{}^p(\lambda)$ is quite well known from the electron-proton scattering experiments by Hofstadter and collaborators. For $F_1{}^n(\lambda)$, the assumption $F_1{}^n(\lambda)=0$ will be made. At present experimental results indicate that no large error is made by this assumption at the value $\lambda=10\mu^2$.[19]

The resonant magnetic dipole term is

$$\mathbf{j}_M=-\frac{E}{M}\frac{D}{2M}\langle f|[2\mathbf{q}\times\mathbf{k}-i\mathbf{q}(\boldsymbol{\sigma}\cdot\mathbf{k})+i\boldsymbol{\sigma}(\mathbf{q}\cdot\mathbf{k})]|i\rangle, \quad (3.7)$$

$$D=\frac{\mu^V(\lambda)}{6f^2|\mathbf{q}|^3}e^{i\delta_{33}}\sin\delta_{33}. \quad (3.7')$$

Additional resonant terms arising from the meson current will be neglected here, along with corrections to the Born-approximation values of the S-wave and small P-wave amplitudes. In this approximation the complete expression found by FNW for the current j consists of $\mathbf{j}_M$, of the static limit of j_B, and of recoil corrections of order μ/M.

An alternative procedure, which will be employed here for including recoil corrections, is to use the complete covariant expression j_B instead of its static limit; i.e., set

$$\mathbf{j}=\mathbf{j}_B+\mathbf{j}_M. \quad (3.8)$$

Except for the small part of μ^S that arises from the anomalous moment, this expression includes all the μ/M terms in FNW. In addition, by avoiding the expansion of j_B in powers of $1/M$, it includes some of the higher-order corrections which have been dropped in FNW. One might hope that these corrections which have been included are the more important ones. The magnetic dipole amplitude $\mathbf{j}_M$ already includes recoil corrections to some extent when one uses experimental values from pion-nucleon scattering for the phase shift. In fact, the CGLN formula for photoproduction of neutral pions, which is dominated by $\mathbf{j}_M$, has been shown to agree well with the experimental data[17] up to 450-Mev photon laboratory energy.

In order to gain further confidence in the use of the amplitude $\mathbf{j}_M$ at energies well above resonance and at values of λ up to 10, which is to be used here, let us examine how the specific form of $\mathbf{j}_M$ arises in the CGLN theory. In their notation M_{1+} is essentially our $\mathbf{j}_M$. They found [CGLN Eq. (13.1)] a simple proportionality accurate to order μ/M, of $M_{1\pm}$ and the corresponding pion-nucleon scattering amplitude $f_{1\pm}$. If we wish to use this result at energies well above the (3, 3) resonance energy, the important question is: How large a deviation from Eq. (13.1) of CGLN occurs when higher-order terms in μ/M become appreciable? A partial answer to this question can be obtained by considering the Born-term contributions to M_{1+} and f_{1+}. By a straightforward but lengthy projection, one obtains, neglecting D waves and higher waves and setting $\mu=1$,

$$f_{1+}{}^{B(\pm)}=\pm\frac{2M(E-M)g^2|\mathbf{q}|^2V_2{}^2}{3E(2E_2\omega_2-1)^2}, \quad (3.9)$$

where $V_{1,2}=[(M+E_{1,2})/2M]^{\frac{1}{2}}$. For electroproduction, which reduces easily to photoproduction when $\lambda=0$, the nucleon-current part of the electric Born terms gives

$$\frac{M_{1+,e}{}^{B(\pm)}}{|\mathbf{q}||\mathbf{k}|}=\pm\frac{e^Vf}{3}F_{1+}, \quad (3.10)$$

where

$$F_{1+}=\frac{M^2V_1V_2}{E}\left[\frac{E-M}{(2E_2\omega_2-1)^2}+\frac{1}{(E_1+M)(2E_1\omega_2-1)}\right]. \quad (3.10')$$

The anomalous-moment Born term gives

$$\frac{1}{|\mathbf{q}||\mathbf{k}|}M_{1+,\mu}{}^{B(\pm)}=\pm\frac{\mu'^VgMV_1V_2}{3E(2E_1\omega_2-1)}\times\left[1+\frac{k_0}{2(E_1+M)}+\frac{k_0\omega_2}{2(2E_1\omega_2-1)}\right]. \quad (3.11)$$

Recalling that $\mu^V(\lambda)=\mu'^V(\lambda)+e^V(\lambda)/2M$, we find that the simple proportionality,

$$\frac{1}{|\mathbf{q}||\mathbf{k}|}(M_{1+,e}{}^{B(\pm)}+M_{1+,\mu}{}^{B(\pm)})\equiv\frac{1}{|\mathbf{q}||\mathbf{k}|}M_{1+}{}^{B(\pm)}=r\frac{\mu^V(\lambda)}{2f|\mathbf{q}|^2}f_{1+}{}^{B(\pm)}, \quad (3.12)$$

[19] Wolfgang K. H. Panofsky, *1958 Annual International Conference on High-Energy Physics at* CERN, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958). See especially p. 18.

is exact to order μ/M with $r=1$. Calculating the exact r from Eqs. (3.9), (3.10), (3.11), and (3.12), one finds for photoproduction at 260- and 400-Mev laboratory energies that $r=0.990$ and 1.007, respectively, whereas for electroproduction at $k_{\mathrm{ph}}=500$ Mev and $\lambda=10$, $r=0.932$. Thus in photoproduction the simple proportionality of the Born terms is retained well above the (3, 3) resonance energy; even for electroproduction, up to at least the stated values of k_{ph} and λ, the value of r can be set equal to unity with sufficient accuracy for our purposes. The expression (3.7) for $\mathbf{j}_M$ will therefore be used as it stands.

One further comment should be made about Eq. (3.8) which will be used for $\mathbf{j}$, namely, that the inclusion of both $\mathbf{j}_B$ and $\mathbf{j}_M$ results in counting one term twice. We have seen above in Eq. (3.10) that $\mathbf{j}_B$ contributes to the amplitude $M_{1+,e}{}^{B(\pm)}$. This contribution is included in $\mathbf{j}_M$ and should therefore be subtracted out of $\mathbf{j}_B$. Equation (3.10) shows that the quantity that should be subtracted from $\mathbf{j}$ is

$$-\frac{e^V E}{6M^2}F_{1+}\langle f|[2\mathbf{q}\times\mathbf{k}-i(\boldsymbol{\sigma}\cdot\mathbf{k})\mathbf{q}+i(\mathbf{q}\cdot\mathbf{k})\boldsymbol{\sigma}]|i\rangle.$$

Numerically this term proves to be no more important in the case to be considered than many terms already neglected and will therefore be neglected also.

C. Calculation of the Photoproduction Cross Section

The formula (3.8) for $\mathbf{j}$ developed in the previous section avoids expanding the electric Born terms in powers of $1/M$ and may therefore include nucleon-recoil effects more accurately than the CGLN formula. Since definite disagreement has been observed[18] in positive-pion production between the CGLN formula and experiment,[20] a calculation has been made at a laboratory energy of 400 Mev to see if Formula (3.8) leads to better agreement. Unfortunately this does not seem to be the case. As shown in Fig. 7, the calculation led to an angular distribution almost identical to that predicted by the CGLN formula. Nevertheless the formula developed in this paper may be significantly more accurate in the electroproduction case, where recoil corrections can be more important.

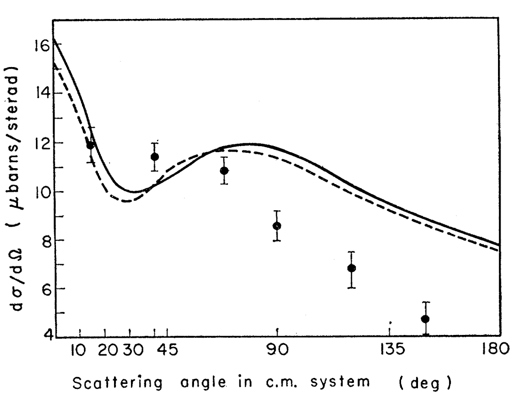

FIG. 7. Photoproduction at 400 Mev. Solid line: prediction of modified CGLN formula, Eq. (3.8). Dashed line: prediction of CGLN formula. Experimental points are those of Walker *et al.*[20]

[20] Walker, Teasdale, Peterson, and Vette, Phys. Rev. **99**, 210 (1955).

D. Calculation of the Electroproduction Cross Section

Having developed and investigated a formula for $\mathbf{j}$, we are ready to proceed to a calculation of the electroproduction cross section. Since the machinery for the calculation has been set up in Sec. III A, we need only insert Formula (3.8) for $\mathbf{j}$ into Eq. (3.5). Separating Φ into the part coming from $\mathbf{j}_B$, from $\mathbf{j}_M$, and from the cross term, we write

$$\Phi=\Phi_B+\Phi_M+\Phi_C. \tag{3.13}$$

For the magnetic-dipole term Φ_M, one finds

$$\Phi_M=|D|^2\{5|\mathbf{q}|^2|\mathbf{k}|^2-3(\mathbf{q}\cdot\mathbf{k})^2 + (4/\lambda)[3(\mathbf{q}\cdot\mathbf{k}\times\mathbf{s})^2+|\mathbf{q}|^2(\mathbf{s}\times\mathbf{k})^2]\}, \tag{3.14}$$

where D is defined by Eq. (3.7′).

The Born-term part of Φ can be evaluated covariantly by substituting Eq. (3.6) into (3.5). The result is:

$$\begin{aligned}\frac{1}{4}\left(\frac{E}{M}\right)^2\Phi_B=&\frac{\Delta^2}{\lambda}\frac{F_\pi^2}{(\Delta^2+1)^2}[4(q\cdot s)(q\cdot s')-\lambda]\\&+\frac{F_N^2}{(E^2-M^2)^2}\Big\{-M^2(\lambda+\Delta^2)+(p\cdot k)(\Delta^2+2M^2)\\&-2M^2(p'\cdot k)+4\frac{\Delta^2}{\lambda}(p\cdot s)(p\cdot s')+2(p\cdot s)(p'\cdot s)\\&+2(p\cdot s')(p'\cdot s')+\frac{4}{\lambda}[2(p'\cdot k)(p\cdot s)(p\cdot s')\\&-(p\cdot k)(p\cdot s)(p'\cdot s')-(p\cdot k)(p\cdot s')(p'\cdot s)]\Big\}\\&-\frac{2F_NF_\pi}{\lambda(E^2-M^2)(\Delta^2+1)}[(\Delta^2+p'\cdot k)h(p)\\&-(p\cdot k)h(p')].\end{aligned} \tag{3.15}$$

In this formula the assumption has been made that $F_1{}^n(\lambda)=0$, and $F_1{}^p(\lambda)=F_2{}^p(\lambda)=F_2{}^n(\lambda)\equiv F_N(\lambda)$, and the abbreviation

$$h(p)\equiv(2q-k)_\mu p_\nu(2s^\mu s^\nu-s^\mu k^\nu-s^\nu k^\mu-\tfrac{1}{2}\lambda g^{\mu\nu}) \tag{3.15′}$$

has been introduced.

Finally, the cross term is

$$\frac{E\Phi_C}{2MV_1V_2\,\mathrm{Re}D} = -\frac{2F_N}{1+\dfrac{\omega^*}{2M}}\left(\mathbf{q}\cdot\mathbf{k}+\frac{2}{\lambda}C_1+\frac{2\mathbf{s}\cdot\mathbf{k}-\lambda}{\lambda k_0^2}C_2\right)$$
$$+\frac{F_\pi}{V_1^2(\Delta^2+1)}\left\{2|\mathbf{q}|^2|\mathbf{k}|^2\sin^2\theta+\frac{2C_2}{\lambda}\times\left[2(2\mathbf{q}-\mathbf{k})\cdot\mathbf{s}-\frac{(2\mathbf{s}\cdot\mathbf{k}-\lambda)(2\mathbf{q}-\mathbf{k})\cdot\mathbf{k}}{k_0^2}\right]\right\}$$
$$-\frac{F_N}{2M\omega^*V_1^2V_2^2}\left\{|\mathbf{q}|^2|\mathbf{k}|^2(3-\cos^2\theta)+\frac{4}{\lambda}[2(\mathbf{q}\cdot\mathbf{k}\times\mathbf{s})^2+C_1\mathbf{q}\cdot\mathbf{k}+C_2\mathbf{q}\cdot\mathbf{s}]\right\}$$
$$-\frac{2(2\mathbf{s}\cdot\mathbf{k}-|\mathbf{k}|^2)}{\lambda k_0^2}(F_\pi-F_N)\frac{C_2}{V_1^2} \qquad (3.16)$$

The following symbols were introduced:

$$\omega^*\equiv E-M,$$
$$C_1\equiv(\mathbf{s}\times\mathbf{q})\cdot(\mathbf{s}\times\mathbf{k}), \qquad (3.16')$$
$$C_2\equiv(\mathbf{k}\times\mathbf{q})\cdot(\mathbf{k}\times\mathbf{s}).$$

The last term in Eq. (3.16) is the contribution of a term that, following FNW, was added to j_B to restore gauge invariance.

A numerical calculation of these formulas as a function of θ has been made for the case discussed in Sec. II C, namely

$$k_{\mathrm{ph}}=500 \text{ Mev}, \quad \text{or} \quad E=9.657,$$
$$T_L=5.039\approx 700 \text{ Mev}.$$

Two values of the angle ϕ [defined in Eq. (2.9)] were used: $\phi=0°$ and $\phi=180°$. From electron-nucleon scattering experiments the value of F_N was chosen to be $F_N(10)=0.62$.[1]

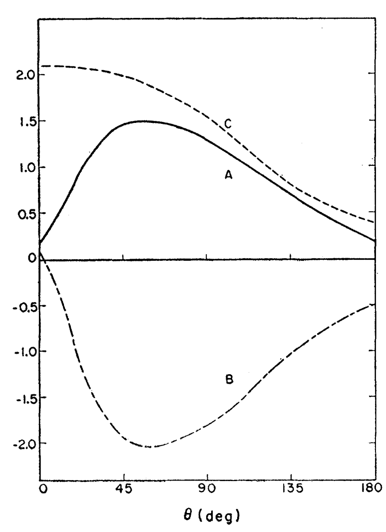

FIG. 8. The dimensionless functions $A(\theta)$, $B(\theta)$, and $C(\theta)$, defined by $\Phi=AF_\pi^2+BF_\pi F_N+CF_N^2$, for $\phi=180°$.

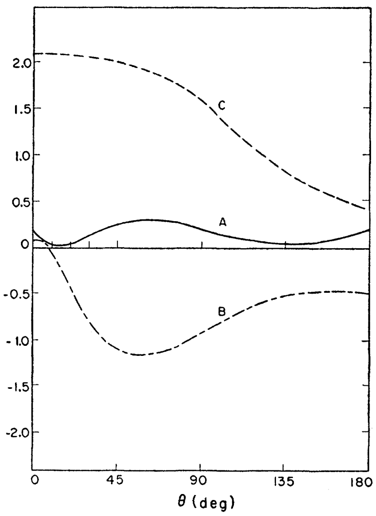

FIG. 9. The functions $A(\theta)$, $B(\theta)$, and $C(\theta)$ for $\phi=0°$.

In order to carry out the calculation it was necessary to estimate the value of the phase shift δ_{33} at $E=9.66$, which corresponds to pion-nucleon scattering at 350 Mev. One cannot use the Chew-Low effective-range formula at such a high energy. At 3 7 Mev, Chiu and Lomon[21] find a significant deviation. In order to estimate the phase shift at 350 Mev, the points of Chiu and Lomon and one point by Willis[22] have been joined by a smooth curve and the value $\delta_{33}=145°$ read off at 350 Mev. This crude estimate should be quite adequate for our purposes.

To make the dependence on the unknown form factor F_π explicit, let us write

$$\Phi=A(\theta,\phi)F_\pi^2+B(\theta,\phi)F_\pi F_N+C(\theta,\phi)F_N^2. \qquad (3.17)$$

The calculated dependence of A, B, and C on θ is shown in Fig. 8 for $\phi=180°$ and in Fig. 9 for $\phi=0°$. The physical meaning of the two values of ϕ is illustrated in Fig. 10. Comparison of the two cases reveals the fact that A depends very strongly on ϕ. The origin of this strong dependence is the term in brackets in the

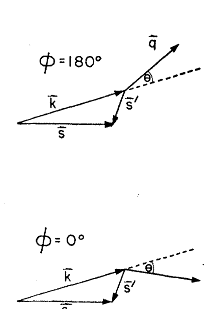

FIG. 10. Illustration of the two cases $\phi=180°$ and $\phi=0°$ in terms of momenta in the system in which $\mathbf{p}'+\mathbf{q}=0$.

[21] H. Y. Chiu and E. L. Lomon, Ann. Phys. **6**, 50 (1959).

[22] Reported by Oreste Piccioni, in *1958 Annual International Conference on High-Energy Physics at* CERN, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 67.

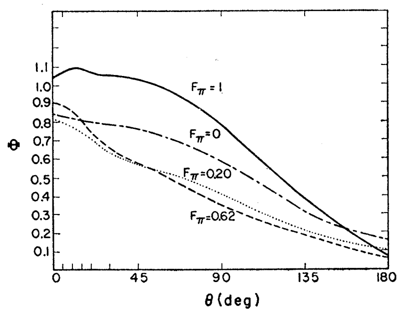

FIG. 11. A plot of the dimensionless function Φ, proportional to the electroproduction cross section, for $E=9.66$, $\lambda=10$, $T_L=700$ Mev, $\phi=\pi$, and various values of the pion form factor.

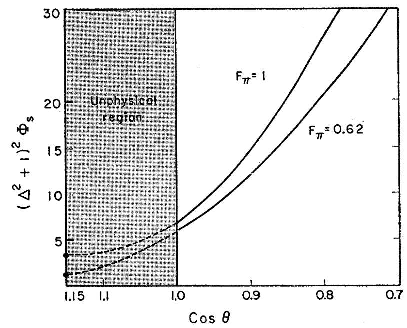

FIG. 12. The calculated behavior of the extrapolation function $(\Delta^2+1)^2\Phi_s$ at the end of the physical region for the case discussed.

following expression for A:

$$A=4\left(\frac{M}{E}\right)^2\frac{\Delta^2}{\lambda(\Delta^2+1)^2}[4(q\cdot s)(q\cdot s')-\lambda]. \quad (3.18)$$

The term A is of course the pole term which has been discussed, but the extreme ϕ dependence noted above will be smoothed by the extrapolation procedure suggested in Sec. II B. Recall that it is necessary to symmetrize the cross section as prescribed by Eq. (2.7) in order to eliminate a branch point. The symmetrized Φ can equivalently be defined as

$$\Phi_s(\theta,\phi)\equiv\Phi(\theta,\phi)+\Phi(\theta,\phi+\pi). \quad (3.19)$$

This symmetrized Φ_s is then the average of the $\phi=\pi$ case of Fig. 8 and the $\phi=0$ case of Fig. 9.

Before going onto the consideration of Φ_s and its extrapolation, let us note the dependence of the electroproduction cross section on the parameter $F_\pi(10)$, with $F_N(10)=0.62$. Several cases are plotted in Fig. 11. One interesting feature of this graph is that for any curve $F_\pi(10)=\beta$, where $\beta\lesssim 0.85$, there is another very similar curve corresponding to a different value of $F_\pi(10)$. This is illustrated in Fig. 11 by the curves for $F_\pi(10)=F_N(10)=0.62$ and for $F_\pi=0.20$.

Finally, Fig. 12 shows the calculated behavior of the extrapolation function $(\Delta^2+1)^2\Phi_s$ at the end of the physical region.

IV. CONCLUSIONS

We are now able to apply the error estimates developed in Sec. II C. For example, recall that for one case in which the cross section was measured at nine specific points, the error was found to be $\Delta a_0\approx 100\xi$. If Δa_0 is the error in $(\Delta^2+1)^2\Phi_s$ at the point $\Delta^2=-1$, then ξ is the error (assumed constant) in Φ_s at each measured point. From Fig. 12 we see that if $\Delta a_0\gtrsim 2$ the extrapolation yields no useful information. This requires a constant absolute error in Φ_s less than 0.02. Figure 11 shows that if $F_\pi=0.62$, this means a percent error ranging from 2% at $\theta=0$ to 6% at $\theta=90°$ and 20% at $\theta=180°$. Even this estimated upper limit of accuracy is considerably better than that achieved by Panofsky and Allton[2] in an electroproduction experiment in which only E and λ (not Δ^2) were determined. Thus it appears that the proposed extrapolation, although possible in principle, will require experiments of great accuracy.

ACKNOWLEDGMENTS

This problem was suggested by Professor Geoffrey F. Chew, whose valuable advice throughout the course of the work I greatly appreciate. The analysis of error in a least-squares extrapolation was done with the help of Dr. Michael J. Moravcsik. I also benefited greatly from very frequent discussions with James S. Ball and Peter Cziffra.

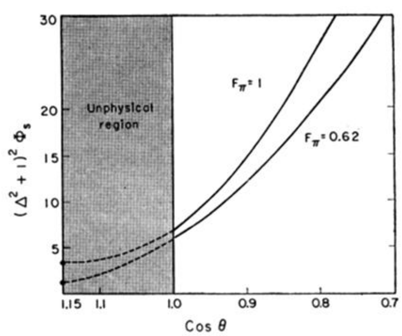

FIG. 12. The calculated behavior of the extrapolation function $(\Delta^2+1)^2\Phi_s$ at the end of the physical region for the case discussed.