PHYSICAL REVIEW VOLUME 115, NUMBER 6 SEPTEMBER 15, 1959

Construction of the Perturbation Series for Transition Amplitudes from their Analyticity and Unitarity Properties*

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(Received April 17, 1959)

The analyticity properties of transition amplitudes are used in conjunction with the unitarity requirements to generate successive terms in the perturbation series, without referring to a specific Lagrangian. In the sixth and higher orders, production is neglected in the unitarity condition; subject to this approximation, it is found that the series can be so constructed. No analyticity properties which have not been rigorously proved need be employed, and the terms are found to satisfy the double dispersion representation. By examining the connection between this method and the conventional calculation of the perturbation series, the types of spectral function corresponding to different Feynman diagrams can be found. Formulas are given for the regions in which the spectral functions are nonzero.

1. INTRODUCTION

A QUESTION of great current interest in quantum field theory is whether the dispersion relations, together with unitarity, can be used to calculate measurable quantities in terms of a small number of coupling constants and masses, without the introduction of a specific Lagrangian.[1] An approximation method for a calculation based on such principles has been outlined by the author,[2] but would meanwhile be of interest to see if one could generate the perturbation series term by term in this way. One can obtain much more definite answers in a perturbation than in a nonperturbation approach, and the results obtained should give one insight into the problem which may prove useful in a more general treatment.

We shall begin in the following section by considering the fourth-order terms. Previous attacks on the problem based on ordinary dispersion relations, have suffered from the difficulty that the result contained, as a "subtraction term," an unknown function of the momentum transfer. In the present paper, however, we shall put in more than the ordinary dispersion relations, and shall assume analytic properties of the transition amplitude as a function of both the energy and the momentum transfer. A representation which expresses these properties has been postulated by the author[2] and discussed in the previous paper[3]; if this representation is assumed to be correct, the fourth-order terms can be calculated in a form free from arbitrary *functions*. There will be an arbitrary *constant* only when all four particles are scalar. By comparing the new to the conventional calculations we shall be able to relate terms calculated by unitarity to particular Feynman diagrams. We shall also be able to see, without explicit calculation, which of the two-dimensional spectral functions corresponds to a given diagram, and to find the asymptotes of the region in which the spectral function is nonzero.

Though the calculation of the perturbation terms is most straightforward if the double dispersion representation is assumed to be correct, it is not necessary to assume as much as is implied by it. We shall, in fact, show, by re-examining our calculation, that it is sufficient to assume analyticity properties that have been proved rigorously by Lehmann,[4] Källén and Wightman,[5] and others. The result is then found to satisfy the double dispersion representation. We thus have another proof that the representation is true in fourth order.

In Sec. 3 we shall extend the calculation to higher orders. The unitarity condition would then involve production, so that the analyticity properties of multiparticles transition amplitudes would be required. If, however, one neglects the production terms in the unitarity condition, one can construct the perturbation series term by term to any order. As in the fourth-order case, it is unnecessary to assume any unproved analytic properties, and the result is found to satisfy the double dispersion relation in all orders.

The calculations of this and the preceding paper enable us to specify in more detail certain features of the double dispersion relation that were previously left undetermined. More specifically, the boundaries of the regions in which the spectral functions are nonzero can only be obtained with the aid of the unitarity condition. In Sec. 4 the equations of the boundaries are given for some frequently occurring cases, and the type of diagram corresponding to each spectral function is indicated.

A few final remarks are added in the last section with regard to the general validity of the double dispersion representation.

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* This research was supported by the U. S. Air Force under a contract monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

[1] This was first suggested by M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York), Sec. III, p. 30.

[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[3] S. Mandelstam, preceding paper [Phys. Rev. **115**, 1741 (1959)].

[4] H. Lehmann, Nuovo cimento **10**, 579 (1958).

[5] G. Källén and A. S. Wightman, Mat. Fys. Skrifter Danske Videnskab. Selskab **1**, No. 6 (1958). Only the general reasoning in Sec. III of their paper will be required, and this section is applicable to all Green's functions.

2. FOURTH-ORDER TERMS

Kinematical Preliminaries

The kinematical notation will be the same as in the previous paper. The momenta for the four particles A, B, C, and D are represented by p_1, p_2, p_3 and p_4 (Fig. 1), we then define the three invariants

$$\begin{aligned} s&=(p_1+p_2)^2,\\ u&=(p_1+p_4)^2,\\ t&=(p_1+p_3)^2. \end{aligned} \tag{2.1}$$

These are, respectively, the squares of the energies for the reactions

$$\begin{aligned} &A+B\rightarrow C+D, &&\text{(I)}\\ &A+D\rightarrow B+C, &&\text{(II)}\\ &A+C\rightarrow B+D. &&\text{(III)} \end{aligned}$$

They are related by the equation

$$s+t+u=M_1^2+M_2^2+M_3^2+M_4^2. \tag{2.2}$$

We continue to adopt the convention that, in an expression written as $A(s,t,u_1)$, s and t are regarded as independent and u is to be expressed in terms of them by (2.2). Expressions written as $A(s,t_1,u)$ or $A(s_1,t,u)$ are to be similarly understood. For future reference, we may quote one or two kinematical relations. The square of the momentum transfer t between two particles is connected with the cosine z of the angle of scattering by the equation

$$t=2q_\alpha q_\beta z-q_\alpha^2-q_\beta^2+[(M_\alpha^2+q_\alpha^2)^{\frac{1}{2}}+(M_\beta^2+q_\beta^2)^{\frac{1}{2}}]^2, \tag{2.3a}$$

where q_α and q_β, M_α, and M_β are the center-of-mass momenta and masses of the particles involved. If all masses are equal, the formula simplifies to

$$t=2q^2(z-1). \tag{2.3b}$$

The momentum q is given in terms of the corresponding square of the energy by the formula

$$q^2=[s^2-2s(M_1^2+M_2^2)+(M_1^2-M_2^2)^2]/4s, \tag{2.4a}$$

where M_1 and M_2 are the masses of the two particles forming the state. If $M_1=M_2$, the formula is simply

$$q^2=\tfrac{1}{4}s-M^2. \tag{2.4b}$$

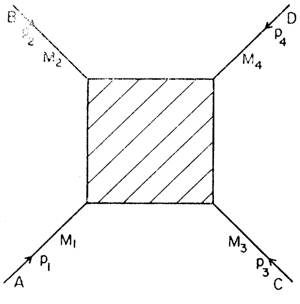

FIG. 1. Kinematics for the reactions.

On the assumption that the transition amplitude behaves like a constant at infinity, the double dispersion relation is

$$\begin{aligned} A={}&\frac{(s-s_0)(t-t_0)}{\pi^2}\int ds'dt'\frac{A_{13}(s',t')}{(s'-s_0)(s'-s)(t'-t_0)(t'-t)}\\ &+\frac{(t-t_0)(u-u_0)}{\pi}\int dt'du'\\ &\times\frac{A_{23}(t',u')}{(t'-t_0)(t'-t)(u'-u_0)(u'-u)}+\frac{(s-s_0)(u-u_0)}{\pi}\\ &\times\int ds'du'\frac{A_{12}(s',u')}{(s'-s_0)(s'-s)(u'-u_0)(u'-u)}\\ &+\frac{s-s_0}{\pi}\int ds'\frac{f_1(s')}{(s'-s_0)(s'-s)}\\ &+\frac{u-u_0}{\pi}\int du'\frac{f_2(u')}{(u'-u_0)(u'-u)}\\ &+\frac{t-t_0}{\pi}\int dt'\frac{f_3(t')}{(t'-t_0)(t'-t)}+\lambda. \end{aligned} \tag{2.5}$$

The scattering amplitude will satisfy dispersion relations (if we neglect subtraction terms)

$$A=\frac{1}{\pi}\int ds'\frac{A_1(s',t,u_1)}{s'-s}+\frac{1}{\pi}\int du'\frac{A_2(s_1,t,u')}{u'-u}, \tag{2.6a}$$

$$A=\frac{1}{\pi}\int ds'\frac{A_1(s',t_1,u)}{s'-s}+\frac{1}{\pi}\int dt'\frac{A_3(s_1,t',u)}{t'-t}, \tag{2.6b}$$

$$A=\frac{1}{\pi}\int du'\frac{A_2(s,t_1,u')}{u'-u}+\frac{1}{\pi}\int dt'\frac{A_3(s,t',u_1)}{t'-t}. \tag{2.6c}$$

The corresponding dispersion relations for the absorptive parts A_1, A_2 and A_3, associated with the reactions I, II, and III, will be

$$A_1=\frac{1}{\pi}\int dt'\frac{A_{13}(s,t')}{t'-t}+\frac{1}{\pi}\int du'\frac{A_{12}(s,u')}{u'-u}, \tag{2.7a}$$

$$A_2=\frac{1}{\pi}\int dt'\frac{A_{23}(t',u)}{t'-t}+\frac{1}{\pi}\int ds'\frac{A_{12}(s',u)}{s'-s}, \tag{2.7b}$$

$$A_3=\frac{1}{\pi}\int ds'\frac{A_{13}(s',t)}{s'-s}+\frac{1}{\pi}\int du'\frac{A_{23}(t,u')}{u'-u}, \tag{2.7c}$$

or, with subtractions,

$$A_1 = f_1(s) + \frac{t-t_0}{\pi}\int dt' \frac{A_{13}(s,t')}{(t'-t_0)(t'-t)} + \frac{u-u_0}{\pi}\int du' \frac{A_{12}(s,u')}{(u'-u_0)(u'-u)}, \tag{2.7d}$$

$$A_2 = f_2(u) + \frac{t-t_0}{\pi}\int dt' \frac{A_{23}(t',u)}{(t'-t_0)(t'-t)} + \frac{s-s_0}{\pi}\int ds' \frac{A_{12}(s',u)}{(s'-s_0)(s'-s)}, \tag{2.7e}$$

$$A_3 = f_3(t) + \frac{s-s_0}{\pi}\int ds' \frac{A_{13}(s',t)}{(s'-s_0)(s'-s)} + \frac{u-u_0}{\pi}\int du' \frac{A_{23}(t,u')}{(u'-u_0)(u'-u)}. \tag{2.7f}$$

According to the procedure outlined in the introduction, we shall begin by assuming all these formulas to be true and shall then return to re-examine the assumptions which it is really necessary to introduce.

Calculation without Subtraction Terms

As in the last paper, the calculation will initially be done with all the masses equal and with scalar particles. Besides reducing the amount of algebra, this second simplification has the effect that all dispersion relations can be written down without subtraction terms, so that the functions satisfying them can be determined from their imaginary parts along the real axis. We shall return later to consider how the calculation can be carried out when there are subtraction terms.

Our method of approach will be to determine the absorptive parts by unitarity and to bring them into the form of the dispersion relations (2.7). The spectral functions A_{13}, A_{23}, and A_{12} to be inserted into the fourth-order expression for (2.5) are thereby determined. If states with more than two particles are neglected, as they may be in fourth order, the unitarity equation for the absorptive part A_1 takes the form

$$A_1\{s,t(z),u_1\} = [q/(32\pi^2 w)]\int d^2\mathbf{n}_i A_{ei}^*\{s,t(z_{ie}),u_1\} \times A_{io}\{s,t(z_{io}),u_1\}. \tag{2.8}$$

A_{ei} and A_{io} are transition amplitudes from the initial state of the reaction I to a two-particle intermediate state, and from the intermediate state to the final state.[6] There are similar equations for A_2 and A_3. $\mathbf{n}_i$ denotes a unit vector, z the cosine of the angle of scattering, z_{ie} the cosine of the angle between the direction of the incoming particles and $\mathbf{n}_i$, and z_{io} the cosine of the angle between the direction of the outgoing particles and $\mathbf{n}_i$. The expression $A_1\{s,t(z),u_1\}$ indicates that t is to be expressed as a function of z by (2.3), so that A_1 is a function of s and z. z_{io} is connected with z_{ie} and ϕ, the azimuthal angle between $\mathbf{n}_i$ and the plane of scattering, by the formula

$$z_{io} = z z_{ie} + (1-z^2)^{\frac{1}{2}}(1-z_{ie}^2)^{\frac{1}{2}}\cos\phi. \tag{2.9}$$

Since we require A_1 in fourth-order perturbation theory, it will be sufficient to take the second-order terms for A^* and A. As an example, we shall suppose that

$$A_{ei}^{(2)*} = g^2/(t_{ie}-M^2), \quad A_{io}^{(2)} = g^2/(t_{io}-M^2). \tag{2.10}$$

We exclude for the moment the case where there may be terms of the form $g^2/(s-M^2)$ or constant terms, since all dispersion relation would not then be free of subtractions. As we shall see later, the terms excluded correspond to reducible graphs in the scattering amplitude, which we know to have the form of single dispersion integrals.

The expressions for A_{ei} and A_{io} in (2.10) have now to be substituted into (2.8). When this is done, however, we arrive precisely at Eq. (3.5) of the previous paper. The expression can be evaluated by expressing z_{io} in terms of z and z_{ie} by (2.9), integrating over ϕ and z_{ie} (which is the same as integrating over $\mathbf{n}_i$), and finally expressing z in terms of t by (2.3b). We then find that the resulting value of A_1 can be written as a dispersion integral (2.7a) and that the spectral function is given by

$$\begin{aligned} A_{13}^{(4)}(s,t) &= -1/\{8[\kappa(s,t)]^{\frac{1}{2}}\}, \\ &\qquad \kappa>0,\ t>0,\ \text{and}\ s>4M^2, \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{2.11}$$

where

$$\kappa(s,t) = 4st[st-4M^2(s+t)+12M^4]. \tag{2.12}$$

We omit the details of calculation, which can be found in the portion of the previous paper following Eq. (3.5).

If $A_{ei}^{(2)*}$ or $A_{io}^{(2)}$ in (2.10) contains a term $g^2/(u-M^2)$, we need not repeat the calculation of the corresponding term in A_1, for we can find the result from the calculation already performed by making use of the symmetry of the problem. On interchanging the

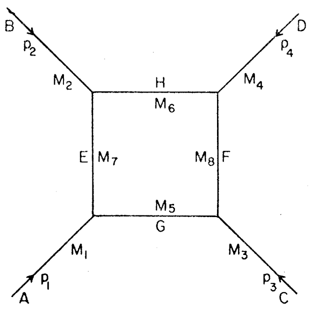

FIG. 2. Fourth-order diagram for the reaction $A+B \to C+D$.

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[6] We do not assume that the particles taking part in the reaction are necessarily identical, so that A_{ei} and A_{io} may be different from A.

two particles in one of the states, initial, intermediate or final, we make *two* of the interchanges $t \leftrightarrow u$, $t_{ie} \leftrightarrow u_{ie}$, $t_{io} \leftrightarrow u_{io}$. We should therefore expect the results to be unchanged by such a transformation. Hence, if both t_{ie} and t_{io} in (2.10) are replaced by u_{io} and u_{io}, we should get the same contribution to A_{13} as before. If only one of these replacements is made, t will also have to be replaced by u; the term therefore contributes to A_{12} instead of A_{13},[7] and the contribution is obtained by substituting u for t in (2.11).

By doing analogous unitarity calculations for A_2 and A_3, one can obtain expressions for A_{23} and A_{12}, and A_{13} and A_{23}, respectively. Each of the spectral functions can therefore be calculated from either of two unitarity equations. These calculations must give the same result, otherwise there is an inconsistency in the theory or in the assumptions made. We observe that (2.11) is a symmetric function of s and t, so that the same result would indeed be obtained whether one used the unitarity equation for A_1 or for A_3. It is necessary to use only two of the three unitarity equations in calculating the spectral functions; the third is, in this case, completely redundant.

It has previously been pointed out by Grisaru[8] that the imaginary part of a transition amplitude, calculated by conventional methods, is equal to the corresponding absorptive part calculated by unitarity. As we shall make further use of the unitarity condition in the next section, we have written it out explicitly.

Relation between the Two Methods

We shall now examine again the connection between the present and the conventional method of calculating perturbation terms, and we shall see how we can relate a term calculated in this way to a particular Feynman diagram. By doing so, we shall gain more insight into such questions as to whether a unitarity calculation of a particular term in A_1, say, will yield a spectral function A_{13} or A_{12}, and what asymptotes the region in which it is nonzero will approach as either variable becomes infinite. In the simple case we have been considering, the answer to this last question is more or less obvious, as the only threshold is at $4M^2$, but in more complicated cases where there may be several thresholds the answer is not quite so clear.

We saw in the previous paper that the most convenient way, for our purposes, of evaluating the conventional integral corresponding to Fig. 2 was to prove that it satisfied a dispersion relation (2.6a). It was then shown that the imaginary part in the physical region for the reaction I could be calculated by regarding G and H as a real intermediate state for the reaction I, and performing an integration over the variables necessary to specify that intermediate state. In the calculation from dispersion relations and unitarity it was found that the absorptive part could be written as an integral over variables corresponding to a real intermediate state through which the scattering could take place. The integrand was found to be the same as in the first method of calculation—it consisted of two factors, representing the second-order contribution for the scattering from the initial to the intermediate, and the intermediate to the final state respectively. In Fig. 2, the left and right half of the diagram represent these two factors.

However, we can equally well do the evaluation of the Feynman-Dyson integral for Fig. 2 by writing it as a dispersion relation (2.6c). We then have to calculate the imaginary part in the physical region for the reaction III, and we find that we must now regard the particles E and F as a real intermediate state. Alternatively, if we are doing the calculation from dispersion relations and unitarity, we can calculate the absorptive part A_3 for the third reaction. The first factor of the equation corresponding to (2.8) will then be the transition amplitude for the process $A+C \rightarrow E+F$, and the second the amplitude for the process $E+F \rightarrow B+D$. We observe that these terms can be represented by the bottom and top halves of Fig. 2, connected by the real lines E and F. Thus, as before, the present and conventional methods of calculation correspond very closely, and give the same result for $A_{13}^{(4)}$.

From the two methods of evaluating the Feynman-Dyson integral for Fig. 2, we observe that it gives rise to a contribution to A_1 with threshold at $4M^2$ and a contribution to A_3 with threshold at $4M^2$. There is no corresponding way of dividing up Fig. 2 so as to represent the reaction II going through a real intermediate state, and the calculation will give no contribution to A_2. The term in the scattering amplitude corresponding to Fig. 2 therefore yields a spectral function which provides contributions to A_1 and A_3 but not A_2; according to (2.7), this spectral function must be A_{13}. Also, the thresholds for $A_1^{(4,\mathrm{i})}$ and $A_3^{(4,\mathrm{i})}$ are $s=4M^2$ and $t=4M^2$, so that we should expect the region in which $A_{13}^{(4)}$ is non-zero to approach these lines asymptotically.

It is thus a simple matter to identify the Feynman diagram corresponding to a term of the transition amplitude calculated by analyticity and unitarity, and, having identified the diagram, to determine which of the spectral functions A_{13}, A_{23}, and A_{12} it yields and to find the asymptotes of the region in which the spectral function is nonzero. The Feynman diagram is obtained from the diagrams for A_{ei}^* and A_{io} which have been used in (2.8) by joining the lines representing the intermediate state. The diagram can then be divided up in the way it was constructed, and perhaps in alternative ways, to represent one of the reactions going through a real intermediate state. Corresponding

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[7] Either of the dispersion integrals in (2.7a) could be written in terms of t and u, but, once a function has been brought to this form, we can decide whether the spectral function is A_{13} or A_{12} from the range of values of the primed variable. A_{13} involves positive t' (and negative u'), A_{12} positive u' (and negative t').

[8] M. T. Grisaru, Phys. Rev. **111**, 1719 (1958).

to each of these methods of division there will be a contribution to the absorptive part A_1, A_2, or A_3 of the reaction concerned, beginning at a threshold given by the lowest mass of the intermediate state. One can thereby determine which of the spectral functions A_{13}, A_{23}, or A_{12} is involved, as well as the asymptotes of the region in which it is nonzero.

The analysis of this section can also be carried out if the masses are unrestricted. Suppose that M_1 and M_2, M_3 and M_4, M_5 and M_6 are the masses of the particles in the initial, final and intermediate states, and M_7 and M_8 the masses in the denominators of (2.10) (Fig. 2). As in the conventional calculation, it is then easiest to work in terms of cosines of angles rather than in terms of momentum transfers. (2.10) is written in the form

$$A_{ei}{}^{(2)*}=g^2/[q_eq_i(z_{ie}-z_{ie}')],\qquad A_{io}{}^{(2)}=g^2/[q_iq_o(z_{io}-z_{io}')].\tag{2.13}$$

q_e, q_i, and q_o are the center-of-mass momenta of the initial, intermediate and final states, given by (2.4a). z_{ie}' and z_{io}' are the cosines of the angles associated with squares of the momentum transfer $M_7{}^2$ and $M_8{}^2$ between the initial and intermediate, and the intermediate and final, states. They are given by Eq. (2.3a). Following the procedure of the equal-mass case, we now substitute (2.13) into (2.8), express z_{io} by (2.9), and perform the integral over z_{ie} and ϕ. The result can be expressed as a dispersion integral in z (corresponding to a dispersion integral in t), with weight function given by

$$\begin{aligned}A_{13}{}^{(4)}&=-1/\{8q_eq_oq_iw[k(z,z_{ie}',z_{io}')]^{\frac{1}{2}}\},\quad (z>z_1)\\&=0,\quad (z<z_1)\end{aligned}\tag{2.14}$$

where

$$k(z,z_{ie}',z_{io}')=z^2+z_{ie}'^2+z_{io}'^2-1-2zz_{ie}'z_{io}',\tag{2.15}$$

$$z_1=z_{ie}'z_{io}'+(z_{ie}'^2-1)^{\frac{1}{2}}(z_{io}'^2-1)^{\frac{1}{2}}.\tag{2.16}$$

The point z_1 is the value of z for which $k=0$. Details are given in the calculation following Eq. (3.19) of the previous paper. Though it is usually most convenient to use these formulas as they stand with the z's given in terms of our usual variables by (2.3) and (2.4), we could write the formulas directly in terms of our usual variables when they would take the form of Eqs. (3.28) and (3.29) of the previous paper.

If one or both the energy denominators in the perturbation terms (2.10) had involved u instead of t, the same formulae would hold, provided we constructed the graph corresponding to Fig. 2 according to our prescription, and always defined z in the manner corresponding to the case just treated. As in the equal-mass case, the spectral function would be A_{12} if one energy denominator of (2.10) involved t, the other u.

Calculation with Subtraction Terms

Let us now deal with the complications which arise when not all the particles are scalar, so that there are subtraction terms in the dispersion relations. As an example, we may consider pion-pion scattering through virtual nucleons. The process would be represented by Fig. 2, in which the external lines are pions and the internal lines nucleons. We again begin with the unitarity equation for A_1, which will be of the type (2.8) but is more complicated owing to the spin of the intermediate state. We shall simply outline the calculation to show how the subtraction terms may be found, leaving out the details of the algebra. The evaluation of the integral for $A_1{}^{(4,\mathrm{i})}$ proceeds in a manner analogous to the calculation without spin and gives a similar result. In this case, however, $A_1{}^{(4,\mathrm{i})}$ remains finite as t tends to infinity, so that it satisfies a dispersion relation of the form

$$A_1{}^{(4,\mathrm{i})}(s,t,u_1)=A_1{}^{(4,\mathrm{i})}(s,t_0,u_1)+\frac{t-t_0}{\pi}\int dt'\frac{A_{13}{}^{(4)}(s,t')}{(t'-t_0)(t'-t)},\tag{2.17}$$

where *both* $A_1{}^{(4,i)}(s,t_0,u_1)$ *and* $A_{13}{}^{(4)}(s,t)$ *are known*, as we have an explicit expression for $A_1{}^{(4,i)}(s,t,u_1)$. By comparing this equation with (2.7d) we observe that $A_{13}{}^{(4)}$ and $f_1{}^{(4)}(s)$ in the representation (2.5) are now determined, the latter just being equal to $A_1{}^{(4,i)}(s,t_0,u_1)$.

If, instead of using the unitarity equation for A_1, we had used the equation for A_3, we would have found that

$$A_3{}^{(4,\mathrm{i})}(s,t,u_1)=A_3{}^{(4,\mathrm{i})}(s_0,t,u_1)+\frac{s-s_0}{\pi}\int ds'\frac{A_{13}{}^{(4)}(s',t)}{(s'-s_0)(s'-s)}.\tag{2.18}$$

The value of $A_{13}{}^{(4)}$ in (2.17) and (2.18) must be the same in a consistent theory. By comparing (2.18) with (2.7f), however, we see that we have in addition now determined $f_3{}^{(4)}(t)$, which is equal to $A_3{}^{(4,\mathrm{i})}(s_0,t,u_1)$. The information provided by the unitarity equation for A_3 is therefore not quite redundant, as $f_3(t)$ cannot be determined without it. In general, therefore, we can use either of two unitarity conditions to determine the spectral functions A_{13}, A_{23}, and A_{12}, but we must use the unitarity condition for the relevant reaction to determine $f_1(s)$, $f_2(u)$ and $f_3(t)$. In the particular case represented by Fig. 2, there is no real intermediate state for the reaction II, and f_2 is zero.

The over-all subtraction term λ in (2.5) cannot be determined from the unitarity equation, but it is an extra constant which has to be inserted into the theory. Such a constant is also present in the conventional theory, where it corresponds to the renormalized value of λ in the $\lambda\phi^4$ term. We have already observed that the only instance in which there is an over-all subtraction term is in scattering of particles of spin zero, and this

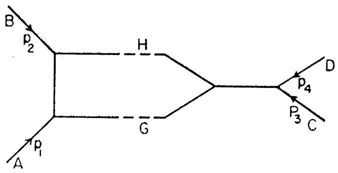

FIG. 3. Construction of reducible diagrams according to our prescription.

corresponds to the result of renormalization theory that a $\lambda\phi^4$ interaction is unrenormalizable in all other instances.

Before proceeding further we may return to the case where the perturbation expressions (2.10), which are to be inserted into the unitarity equation (2.8) for A_1, contain a term $g^2/(s-M^2)$ or a constant term. If either or both of the terms inserted as A_{ei} and A_{io} into (2.8) are of this form, the integral will be independent of z. This may be seen immediately by expanding A_{ei} and A_{io} in spherical harmonics, or it may be verified by direct calculation. The contributions to A_1 of this type, which are easily evaluated, will thus be functions of s alone, and will represent additions to the function $f_1(s)$ in the single dispersion integrals. They do not affect the spectral functions A_{13} and A_{12}. By looking at the Feynman diagrams which correspond to these terms according to our prescription, such as Fig. 3, we notice that they are the reducible graphs of the usual formalism. In the present method of calculation, however, the absorptive part is found and inserted into a dispersion integral, so that any single or double poles which may occur at $s=M^2$ are omitted. We therefore obtain our results directly in renormalized form.

In cases where some but not all of the dispersion relations have subtraction terms, it may be unnecessary to use both unitarity conditions for the irreducible diagrams. With pion-nucleon scattering, for instance, the appropriate double dispersion relation is given by Eq. (2.17) of the previous paper. The reducible diagrams will contribute to the functions a_1, a_2, a_3, b_1, and b_2, and the unitarity conditions for the three reactions must be used to evaluate them. The irreducible diagrams, however, will only contribute to a_3. It is therefore sufficient to use the unitarity equation for A_3 to determine A_{13} and A_{23}, since a_3 is thereby determined as well. If the unitarity equations for A_1 or A_2 had been used, a_3 would have remained undetermined.

Avoidance of Unproved Hypotheses

It is now necessary to re-examine our calculation of the fourth-order terms in order to show how it may be carried through by using only analyticity properties of the scattering amplitude which have been shown rigorously to be true. We shall begin with the case where all the masses are equal, and shall again neglect subtraction terms for simplicity.

The calculation of $A_1^{(4)}$ and $A_2^{(4)}$, and the demonstration that they could be brought into the form (2.7a) and (2.7b) with known spectral functions, made use of the unitarity condition only. They are therefore justified, provided we restrict ourselves to the physical region in the unprimed variables. These expressions for $A_1^{(4)}$ and $A_2^{(4)}$ can now be substituted into the ordinary dispersion relation (2.6a), which has been proved rigorously if t is space-like and sufficiently small[9,10,4] (less than $2M^2$); on doing so, we arrive at the representation (2.5) (without subtraction terms). In order to justify this substitution, we have to show that $A_1^{(4)}$ and $A_2^{(4)}$ are given by (2.7a) and (2.7b) for all space-like values of t less than $2M^2$, which include points in the unphysical region. However, it has been proved rigorously[4] that these absorptive parts are analytic functions of t for fixed s or u if $-2M^2<t<0$, so that, if they are given by (2.7a) and (2.7b) in the physical part of this region, they are given by the same equations in the entire region.

Our calculation of the fourth-order scattering amplitude from analyticity and unitarity, and its expression in the form (2.5), are thus fully justified if $-2M^2<t<0$. We now make use of another rigorously proved result, that the scattering amplitude is an analytic function of the momentum transfer of one of the reactions, with the energy kept fixed, in the physical region for that reaction.[4] By doing so we can analytically continue our expressions into the entire physical regions for the reactions I and II. Finally, by repeating the proof with the dispersion relation (2.6a) replaced by (2.6b) or (2.6c), we can include the physical region for the reaction III as well.

The above proof is not applicable without modification for arbitrary masses, as the dispersion relations (2.6) have not been rigorously established unless they satisfy certain inequalities. In order to extend the result to the general case it is necessary to invoke more analytic properties of the Green's function. The variables M_1, M_2, M_3, and M_4 in Fig. 2, which correspond to the squares of the external momenta, are not now restricted to have their physical values. For certain real (i.e., noncomplex) ranges of values of these masses the dispersion relations (2.6) are rigorously true, so that, within these ranges, the proof can be carried through as before.

The procedure is then to continue analytically in the masses with the aid of the theorem that the Green's function is the boundary value of an analytic function.[5] More precisely, there exists a region, in the six-dimensional complex space of the variables M_1, M_2, M_3, M_4, s and t, for which the Green's function is an analytic function of all six variables, and this region approaches arbitrarily close to any point which can be constructed from real momenta p_1, p_2, p_3, and p_4. It follows that, if $A^{(4)}$ is given by a certain expression for a range of values of the masses, and if the expression is an analytic

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[9] Bogoliubov, Medvedev, and Polivanov, "Problems in the Theory of Dispersion Relations" (unpublished).

[10] Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958).

function of the masses over a wider range, $A^{(4)}$ will be given by this expression over the wider range as well. Now, as long as we keep out of a region for which the mass inequalities which give the anomalous thresholds occur, the calculated expression for $A^{(4)}$ will be an analytic function of its variables unless s and t have their threshold values or unless we are on the curve c of Fig. 1. The expression can accordingly be continued analytically in the masses until they have their physical values, as these singularities are easily avoided.

We have thus proved that the fourth-order perturbation terms can be reproduced from the rigorously proved analytic properties and the unitarity equations alone. As these terms are found to have the double dispersion representation, this provides an alternative proof of the representation in fourth-order perturbation theory.

3. CALCULATION OF HIGHER TERMS FROM ANALYTICITY AND UNITARITY

In the previous section the fourth-order perturbation terms were obtained from the second-order terms using unitarity and analyticity properties. In higher orders there would be further terms on the right of (2.8), corresponding to production processes, and we would require to know the analytic properties of the transition amplitudes for these processes in order to perform the calculation. In this paper we shall neglect production processes, and shall suppose that A_1 is given in all orders by (2.8); the calculation of any term in the transition amplitude from lower-order terms then becomes a straightforward generalization of the fourth-order calculation. We thereby construct a subset of the perturbation series. As before, we do not have to introduce any unproved analytic properties into the calculation, provided that the lower-order terms satisfy dispersion relations, and the higher-order terms are then found to satisfy the double dispersion representation. We accordingly have a proof by induction that all the perturbation terms of the subset have this structure. The approximation (2.8), (which in the case of pion-nucleon scattering is the one-meson approximation of Low[11] and Chew and Low[12]) is being used in attempts to calculate scattering amplitudes from analyticity and unitarity without perturbation theory. The present section therefore proves that, within the framework of this approximation, the individual perturbation terms satisfy the representation (2.5).

We shall also show that the prescriptions given in the last section for determining which of the spectral functions A_{13}, A_{23}, and A_{12} receive contributions from a particular term, and for finding the asymptotes of the regions in which they are nonzero, continue to hold in higher orders.

The calculation is similar to that given in reference 2; since, however, the right-hand side of (2.9) is now known from lower-order perturbation calculations, the result can be obtained explicitly in the form of an integral involving the lower-order absorptive parts. We begin by expressing the transition amplitudes on the right of (2.9) as dispersion relations (2.6c), with the energy s constant, so that the equation for an nth order term adopts the form

$$\begin{aligned}A_1^{(n,i)}\{s,t(z),u_1\} &= \frac{q}{32\pi^2 w}\int d^2\mathbf{n}_i\left\{\frac{1}{\pi}\int du_{ie}'\frac{A_2^{(r)*}(s,t_1,u_{ie}')}{u_{ie}'-u_{ie}(z_{ie})}\right.\\ &\quad\left.+\frac{1}{\pi}\int dt_{ie}'\frac{A_3^{(r)*}(s,t_{ie}',u_1)}{t_{ie}'-t_{ie}(z_{ie})}\right\}\\ &\quad\times\left\{\frac{1}{\pi}\int du_{io}'\frac{A_2^{(n-r)}(s,t_1,u_{io}')}{u_{io}'-u_{io}(z_{io})}\right.\\ &\quad\left.+\frac{1}{\pi}\int dt_{io}'\frac{A_3^{(n-r)}(s,t_{io}',u_1)}{t_{io}'-t_{io}(z_{io})}\right\}. \qquad (3.1)\end{aligned}$$

As in the lowest-order calculations, the two brackets will also contain terms depending on s alone. Again, the contributions to $A_1^{(n)}$ from these terms are easily to work out and affect only the function $f_1(s)$ in dispersion integrals. They correspond as before to the reducible graphs of the usual formalism.

If the orders of integration over $\mathbf{n}_i$ and the primed variables are changed in (3.1), the expression becomes the sum of four integrals such as

$$\begin{aligned}&\frac{q}{32\pi^4 w}\int dt_{ie}'dt_{io}'A_3^{(r)*}(s,t_{ie}',u_1)A_3^{(n-r)}(s,t_{io}',u_1)\\ &\qquad\times\int d^2\mathbf{n}_i\frac{1}{[t_{ie}'-t_{ie}(z_{ie})][t_{io}'-t_{io}(z_{io})]}. \qquad (3.2)\end{aligned}$$

The integral over $\mathbf{n}_i$ has exactly the same form as the corresponding integral in the fourth-order calculation. We may, therefore, repeat the procedure leading to Eq. (2.11) with the modification that the M^2's in the denominators have to be replaced by t_{ie}' and t_{io}'.The function κ is now replaced by

$$\begin{aligned}\gamma(s,t,t_{ie}',t_{io}') &= 4s[(s-4M^2)(t^2+t_{ie}^2+t_{io}'^2\\ &\quad-2tt_{ie}'-2tt_{io}'-2t_{ie}t_{io}')-4tt_{ie}'t_{io}'], \qquad (3.3)\end{aligned}$$

so that the integral takes the form

$$\begin{aligned}&\frac{q}{32\pi^4 w}\int d^2\mathbf{n}_i\frac{1}{[t_{ie}'-t_{ie}(z_{ie})][t_{io}'-t_{io}(z_{io})]}\\ &\qquad=-\frac{1}{8\pi^3}\int dt'\frac{\Gamma(s,t',t_{ie}',t_{io}')}{t'-t}, \qquad (3.4)\end{aligned}$$

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[11] F. E. Low, Phys. Rev. **97**, 1392 (1955).

[12] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

where

$$\Gamma(s,t,t_{ie},t_{io})=1/[\gamma(s,t,t_{ie},t_{io})]^{\frac{1}{2}},\quad \gamma>0,\ t>(t_{ie}^{\frac{1}{2}}+t_{io}^{\frac{1}{2}})^2,\ s>4M^2, \qquad (3.5)$$
$$=0 \quad \text{otherwise.}$$

Equation (3.3) can then be substituted into (3.2), and the orders of integration over t' and the other two variables interchanged. On doing so, we find that the integral has the form of the first term of the dispersion relation (2.7a), with $A_{13}(s,t)$ equal to the expression

$$-\frac{1}{8\pi^2}\int dt_{ie}dt_{io}\Gamma(s,t,t_{ie},t_{io})A_3^{(r)*}(s,t_{ie},u_1)\times A_3^{(n-r)}(s,t_{io},u_1). \qquad (3.6)$$

The primes on the dummy variables t_{ie} and t_{io} have been suppressed.

The other three terms of (3.1) can be found from the symmetry of the problem under two of the interchanges $t\leftrightarrow u$, $t_{ie}\leftrightarrow u_{ie}$, $t_{io}\leftrightarrow u_{io}$, as in the fourth-order calculation. The final result for the evaluation of (3.1) is therefore

$$A_{13}^{(n,i)}(s,t)=-\frac{1}{8\pi^2}\int dt_{ie}dt_{io}\Gamma(s,t,t_{ie},t_{io})\times A_3^{(r)*}(s,t_{ie},u_1)A_3^{(n-r)}(s,t_{io},u_1)$$
$$-\frac{1}{8\pi^2}\int du_{ie}du_{io}\Gamma(s,t,u_{ie},u_{io})\times A_2^{(r)*}(s,t_1,u_{ie})A_2^{(n-r)}(s,t_1,u_{io}), \qquad (3.7a)$$

$$A_{12}^{(n,i)}(s,u)=-\frac{1}{8\pi^2}\int du_{ie}dt_{io}\Gamma(s,u,u_{ie},t_{io})\times A_2^{(r)*}(s,t_1,u_{ie})A_3^{(n-r)}(s,t_{io},u_1)$$
$$-\frac{1}{8\pi^2}\int dt_{ie}du_{io}\Gamma(s,u,t_{ie},u_{io})\times A_3^{(r)*}(s,t_{ie},u_1)A_2^{(n-r)}(s,t_1,u_{io}). \qquad (3.7b)$$

The calculation just performed can also be carried out in the general mass case, and the result will again be obtained by integrating the fourth-order result over the absorptive parts in (3.1). As in Sec. 2, it is now more convenient to work in terms of the variable z, the cosine of the angle of scattering, than in terms of the momentum transfers t and u. Since t and u depend linearly on z, the dispersion integrals in (3.1) can be replaced by dispersion integrals in z'.[13] The procedure is then exactly the same as in the equal-mass case. The right-hand side of the equation corresponding to (3.4) will be a dispersion integral in z, and the weight function is now exactly the same as in the fourth-order term, Eq. (2.14) (except that the kinematical factor is now $q_i/8w$). The equations corresponding to (3.7) will therefore be

$$A_{13}^{(n,i)}\{s,t(z)\}=-\frac{1}{8\pi^2}\frac{q_i}{w}\int dz_{ie}dz_{io}K(z,z_{ie},z_{io})\times\{A_3^{(r)*}(s,t_{ie}(z_{ie}),u_1]A_3^{(n-r)}[s,t_{io}(z_{io}),u_1]+A_2^{(r)*}[s,t_1,u_{ie}(z_{ie})]A_2^{(n-r)}[s,t_1,u_{io}(z_{io})]\}, \qquad (3.8a)$$

$$A_{12}^{(n,i)}\{s,u(z)\}=-\frac{1}{8\pi^2}\frac{q_i}{w}\int dz_{ie}dz_{io}K(z,z_{ie},z_{io})\times\{A_2^{(r)*}[s,t_1,u_{ie}(z_{ie})]A_3^{(n-r)}[s,t_{ie}(z_{io}),u_1]+A_3^{(r)*}[s,t_{ie}(z_{ie}),u_1]A_2^{(n-r)}[s,t_1,u_{io}(z_{io})]\}, \qquad (3.8b)$$

where

$$K(z,z_{ie},z_{io})=1/[k(z,z_{ie},z_{io})]^{\frac{1}{2}},\quad z>z_1,\ s>(M_5+M_6)^2 \qquad (3.9)$$
$$=0 \quad \text{otherwise.}$$

The general mass case could also be handled in terms of the momentum transfers. The kernel Γ in (3.7) is now replaced by an expression of the form (3.28) of the previous paper, with the momentum transfers t_{ie} and t_{io} replacing the masses M_7^2 and M_8^2. For the integrals over u_{ie} or u_{io}, the variables x must be defined with Fig. 3 of the previous paper replaced by the appropriate graph, drawn according to our usual prescription by taking the relevant second-order terms in (5.1) as the perturbation terms.

Owing to the close correspondence between the fourth- and higher-order calculations, it is easy to generalize the formulas quoted in Sec. 2 for the region in which the spectral functions are nonzero. We can draw a graph such as Fig. 4 for one of the higher-order terms. The shaded portion, instead of representing terms $1/(t_{ie}-M_7^2)$ and $1/(t_{io}-M_8^2)$, will now represent dispersion integrals in t beginning at some lowest mass M_{7L} and M_{8L}, which will be the sum of the masses of the particles being exchanged. We have just seen that the transition amplitude corresponding to this diagram

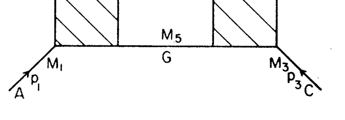

FIG. 4. Diagram for a higher-order term.

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[13] When a change of variables is made from a momentum transfer t or u to a cosine z of an angle, we always take the angle between the particles corresponding to the momentum transfer t or u. With this convention, the variables z' in the dispersion integrals, like the variables t' and u', are always positive.

is obtained by integrating the fourth-order transition amplitude over the masses corresponding to the momentum transferred across the shaded portion. Since the region in which the spectral function is nonzero decreases as the intermediate masses increase, it will be obtained by putting $M_7=M_{7L}$, $M_8=M_{8L}$ in the fourth-order formulas (2.16). We observe that it approaches the asymptote $t=(M_{7L}+M_{8L})^2$ as s becomes infinite, which is to be expected, as $M_{7L}+M_{8L}$ is the mass of the lowest intermediate state of the reaction III represented by Fig. 4.

As in the fourth-order case, we should expect to be able to calculate the contribution to A_{13} by using the unitarity equation for A_3. From Fig. 4, however, we notice that the intermediate state of the reaction III is now not a two-particle state, so that we cannot check this explicitly without going beyond the approximation (2.8). The fact that the contribution to A_3 begins at the correct threshold indicates the consistency of the unitarity requirements.

Since the terms in A_{13} which we obtain by applying the approximation (2.8) to A_1 and A_3 are not the same, we cannot consistently apply (2.8) to all three reactions. We might apply the approximation to the reactions I and II alone, as we have seen that the unitarity equation for any one of the reactions is redundant except for determining subtraction terms, in which case it could be applied to the S-waves of the reaction III. The spectral functions A_{13} and A_{23} are then uniquely obtained, but we shall still have two different approximations for A_{12}. The difficulty, of course, is due to the fact that a graph which represents the reaction I going through a two-particle intermediate state represents the reaction II going through a multiparticle intermediate state and vice-versa. We would therefore have to include in the series for A_{12} all terms obtained from the unitarity conditions for both the reactions I and II. By doing so we include some of the contributions of the multiparticle intermediate states to the unitarity equations for A_1 and A_2. If desired, the unitarity condition for the reaction III could also be used, and the resulting contributions to A_{13} and A_{23} added to those already obtained.

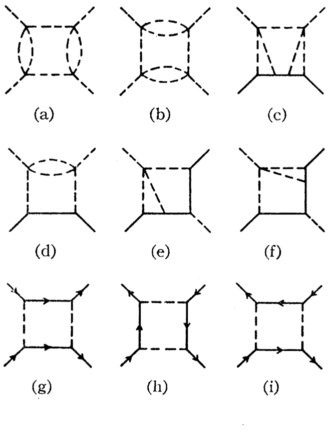

FIG. 5. Diagrams with the lowest intermediate states for various processes.

The demonstration given in the last section, that the perturbation terms can be constructed by using only proved analyticity properties of the transition amplitude, can be extended to general order. We need not repeat the reasoning, but should refer to the difficulty just mentioned, which did not occur in the fourth-order case. The procedure, as before, is to calculate A_1 and A_2 by unitarity and to insert them into the dispersion relation (2.6a). The expression then obtained is

$$\begin{aligned}A^{(n,\mathrm{i})}=&\frac{1}{\pi^2}\int ds'dt'\frac{A_{13}{}^{(n,\mathrm{i})}(s',t')}{(s'-s)(t'-t)}\\&+\frac{1}{\pi}\int ds'du'\frac{A_{12}{}^{(n,\mathrm{i})}(s',u')}{(s'-s)\{u'-u(s',t)\}}\\&+\frac{1}{\pi^2}\int dt'du'\frac{A_{23}{}^{(n,\mathrm{i})}(t',u')}{(t'-t)(u'-u)}\\&+\frac{1}{\pi^2}\int ds'du'\frac{A_{12}{}^{(n,\mathrm{ii})}(s',u')}{\{s'-s(t,u')\}(u'-u)}.\end{aligned}\tag{3.10}$$

The form $u(s',t)$ indicates that u is to be expressed as a function of s' and t by (2.2), and similarly for $s(t,u')$. $A_{12}{}^{(n,\mathrm{i})}$ and $A_{12}{}^{(n,\mathrm{ii})}$ are the contributions to $A_{12}{}^{(n)}$ obtained from the unitarity condition for the reactions I and II.

The equation (3.10), which has thus far only been obtained rigorously if t is sufficiently small, has now to be extended into the physical regions for the first two reactions. The second and fourth terms, however, cannot be continued beyond the (algebraically) smallest value of t for which $A_{12}(s,u)$ is nonzero. In the fourth-order case $A_{12}{}^{(4,\mathrm{i})}$ was equal to $A_{12}{}^{(4,\mathrm{ii})}$, and the sum of the two terms could be continued into the entire physical region. Now we have to add to $A_1{}^{(n)}$ and $A_2{}^{(n)}$ the terms

$$\frac{1}{\pi}\int du'\frac{A_{12}{}^{(n,\mathrm{ii})}(s,u')}{u'-u}\quad\text{and}\quad\frac{1}{\pi}\int ds'\frac{A_{12}{}^{(n,\mathrm{i})}(s',u)}{s'-s}.$$

If these values of $A_1{}^{(n)}$ and $A_2{}^{(n)}$ are inserted into (2.6a) and the result added to (3.10), the two energy denominators $u'-u(s',t)$ and $s'-s(t,u')$ become simply $u'-u$ and $s'-s$, so that the complete expression has the required properties. It should be emphasized that these additions to A_1 and A_2, which must come from higher intermediate states, were imposed by the requirement that the scattering amplitude have its rigorously proved analytic properties, and not by postulating the double dispersion representation.

In the sixth and higher orders, one Feynman graph will correspond to the sum of several terms calculated by out formalism. If, for instance, Fig. 4 consists of

$n/2$ lines being exchanged between the particles, there will be $(n/2)-1$ terms depending on how many lines are assigned to each half. There will in addition be terms corresponding to such a diagram which are not included in our unitarity approximation.

It has been verified in sixth order that, if the right-hand side of (2.8) is supplemented by a term for three-particle intermediate states (which is the most we can get in this order), the result still satisfies the double dispersion representation. Thus any perturbation terms which violate it must occur only in eighth or higher orders, and not at all if the unitarity approximation (2.8) is used.

4. REGIONS IN WHICH THE SPECTRAL FUNCTIONS ARE NONZERO IN PRACTICAL CASES

In this section we shall apply to some simple cases the formulas given in the last two sections for the region in which the spectral functions are non-zero. We have seen that, for graphs of the form of Fig. 4, the bounding curves are obtained by inserting the lowest masses of the multiparticle intermediate states into (2.16). The bounding curves for the whole perturbation series will be obtained by choosing those graphs with the lowest possible intermediate states. It may be that terms not included in our unitarity approximation give spectral functions which extend out of the regions so calculated, but, as the intermediate states in such terms are necessarily not the lowest, this is unlikely. The asymptotes certainly lie further away from the physical region than the asymptotes of the curves we calculate.

Pion-Pion Scattering

The pion-pion amplitude is symmetric under interchange of any pair of the variables s, t, and u (for neutral pions). All three spectral functions are therefore equal to one another, and each is a symmetric function of its two arguments. Let us consider for definiteness $A_{13}(s,t)$. The graphs with the lowest intermediate states are Figs. 5(a) and (b). In Fig. 5(a), $M_{7L}=M_{8L}=2\mu$, and the remaining masses are equal to μ. The cosines z_{ie} and z_{io} of the angles between the intermediate and the initial or final states corresponding to a momentum transfer 2μ are

$$z_{ie}'=z_{io}'=1+(2\mu)^2/(2q^2),$$

while the cosine z of the angle between the initial and the final state corresponding to a momentum transfer t is

$$z=1+t/(2q^2).$$

In this case, q is simply expressed in terms of s by the relation

$$q^2=\tfrac{1}{4}s-\mu^2.$$

On putting these formulas into (2.16), we find that the bounding curve is given by

$$st-4\mu^2(4s+t)=0$$
$$(\text{Asymptotes: } s=4\mu^2 \text{ and } t=16\mu^2). \tag{4.1a}$$

The bounding curve corresponding to Fig. 5(b) is obtained by interchanging s and t, and has the equation

$$st-4\mu^2(s+4t)=0$$
$$(\text{Asymptotes: } s=16\mu^2 \text{ and } t=4\mu^2). \tag{4.1b}$$

The area in which A_{13} is nonzero thus has the form of the shaded area in Fig. 6.

Pion-Nucleon Scattering

We take p_1 and p_3 to represent the external pion lines, p_2 and p_4 the external nucleon lines. Thus the reactions I and II are pion-nucleon scattering, while the reaction III is the pair-annihilation reaction.

The amplitude is a symmetric function of s and u (for neutral pions and scalar nucleons), so that A_{13} and A_{23} are equal, and A_{12} is a symmetric function of s and u. The graphs giving contributions to A_{13} are those for which both reactions I and III can take place through real intermediate states. Figure 5(c) and (d) give the diagrams with the lowest intermediate states. The relations between the z's, q and s and t will be a little more complicated than in the last case, and are obtained by putting the appropriate mass values in (2.3a) and (2.4a). On substituting in (2.16), we find, for the bounding curves from Figs. 5(c) and (d), respectively,

$$(t-16\mu^2)[s-(M+\mu)^2][s-(M-\mu)^2]-64\mu^4s=0$$
$$(\text{Asymptotes: } s=(M+\mu)^2 \text{ and } t=16\mu^2). \tag{4.2a}$$

$$(t-4\mu^2)[s-(M+2\mu)^2][s-(M-2\mu)^2]-16\mu^4(s+3M^2-3\mu^2)=0$$
$$(\text{Asymptotes: } s=(M+2\mu)^2 \text{ and } t=4\mu^2). \tag{4.2b}$$

The graphs contributing to A_{12} will represent the reactions I and II taking place through real intermediate states. Those with the lowest intermediate states are Figs. 5(e) and (f), and, on calculating as before, the

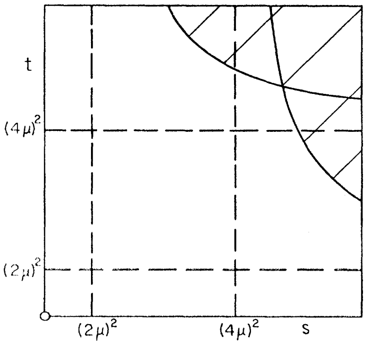

FIG. 6. Region in which A_{13} is nonzero for pion-pion scattering.

bounding curves are found to be

$$[u-(M+2\mu)^2][u-(M-2\mu)^2][s-(M+\mu)^2] \times[s-(M-\mu)^2]-16suM^2\mu^2=0$$

$$[\text{Asymptotes: } s=(M+\mu)^2 \text{ and } u=(M+2\mu)^2]. \quad (4.3\text{a})$$

$$[u-(M+\mu)^2][u-(M-\mu)^2][s-(M+2\mu)^2] \times[s-(M-2\mu)^2]-16suM^2\mu^2=0$$

$$[\text{Asymptotes: } s=(M+2\mu)^2 \text{ and } u=(M+\mu)^2]. \quad (4.3\text{b})$$

The regions in which the spectral functions are nonzero are again similar in form to the shaded area of Fig. 6.

Nucleon-Nucleon Scattering

The reaction I is nucleon-nucleon scattering, whereas the reactions II and III are both nucleon-antinucleon scattering. The scattering amplitude is symmetric in t and u. A_{12} is thus equal to A_{13}, and A_{23} is a symmetric function of its arguments.

The graphs contributing to A_{13} will again represent the reactions I and III going through real intermediate states. In this case, the lowest intermediate states occur in a single graph, Fig. 5(g). The bounding curve is given by the equation

$$(t-4\mu^2)(s-4M^2)-4\mu^4=0$$

$$(\text{Asymptotes: } s=4M^2 \text{ and } t=4\mu^2). \quad (4.4)$$

The graphs contributing to A_{23} will represent the reactions II and III going through real intermediate states. The lowest intermediate states occur in Figs. 5(h) and (i), and the bounding curves are

$$(t-4\mu^2)(u-4M^2)-4\mu^4=0$$

$$(\text{Asymptotes: } t=4\mu^2 \text{ and } u=4M^2). \quad (4.5\text{a})$$

$$(t-4M^2)(u-4\mu^2)-4\mu^4=0$$

$$(\text{Asymptotes: } t=4M^2 \text{ and } u=4\mu^2). \quad (4.5\text{b})$$

5. CONCLUDING REMARKS

The development in the foregoing section shows that an appreciable class of perturbation terms satisfies the double dispersion representation. It may be that some of the perturbation terms not considered in this paper have additional singularities and do not satisfy the representation. Even if this were the case, it would not affect the calculations that have been proposed on the basis of the representation, as such calculations make use of the unitarity approximation (2.8), and the terms of the perturbation series generated by this approximation have been shown to satisfy the representation. If there are other singularities in the perturbation terms not treated, they should be able to be classified and incorporated in any calculations which may be proposed to improve on the approximation (2.8). In fact, the nature of the singularities would probably become evident in the course of the calculation.

A more serious limitation on the validity of the double dispersion representation is that it has thus far not been proved except in perturbation theory. As we have remarked, it is unlikely that it can be proved from causality along without introducing unitarity. In fact, for the general mass case, counterexamples have been found which satisfy all the causality requirements and yet do not satisfy even forward dispersion relations. On the other hand, the unitarity condition gives equations connecting an infinite number of Green's functions, and it would appear to be very difficult to obtain rigorous results with its aid. The natural thing to do would be to approximate the unitarity condition by an equation such as (2.8), and to use the same approximation as is used in the application of the theory to calculations. In other words, we probably do not need all the analyticity properties of (2.5) in order to perform the calculations from unitarity, but it may be sufficient to use only the rigorously established properties—as is the case in perturbation theory. The difficulty with this approach is that the unitarity conditions for the three reactions overdetermine the scattering amplitude, and contradict one another if approximations are made. However, it may be possible to formulate our approximations in a way which overcomes this—for example, by taking into account only singularities which begin sufficiently near the physical region.

It is therefore probably not a formidable problem to calculate the scattering amplitude using only unitarity properties, in a certain approximation, as well as proved analyticity properties, and to show that the result satisfies (2.5). Whether this representation holds independently of any approximations is a question which would be very hard to answer, and we would not like to make speculations about this point here. Though of interest, this is not really relevant in connection with the application of the representation as it does not seem possible to apply it except in conjunction with a unitarity approximation. Even if one attempted to find what happened in the limit of including more and more processes in the unitarity condition, the question of interest would be whether the approximations converged, and not whether the limit had any further singularities in the complex plane.

ACKNOWLEDGMENT

The author would like to acknowledge many helpful discussions with, and useful suggestions of, Professor G. F. Chew.