tentatively ascribe this observation to a 24-hour variation in α-particle intensity, but further experiments are necessary to substantiate this effect.

Both the observation of an independent hourly α-particle intensity variation as well as the change in the ratio of α particle to proton flux between the balloon flights in 1957 and 1958 cannot be explained by the action of a modulation mechanism only, which operates on both primary components. One is led to suspect the possibility of occasional production of primary α particles by the sun. The experimental evidence is scarce and it cannot be justified to draw a more detailed conclusion at the present time, but further experiments should be directed towards answering this question.

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Analytic Properties of Transition Amplitudes in Perturbation Theory*

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The analytic properties of two-particle transition amplitudes as functions of both energy and momentum transfer are examined in perturbation theory. The modified Nambu representation previously proposed by the author for expressing these properties is discussed in a little more detail. It is shown that, as long as the masses do not satisfy certain inequalities connected with the existence of anomalous thresholds, the fourth-order terms, calculated in the usual manner, satisfy the representation. The spectral functions are calculated explicitly for spinless particles. The proof can be extended to the sixth order, but is not worked out here. The modifications necessary when there exist anomalous thresholds are mentioned.

1. INTRODUCTION

IN a previous paper,[1] a representation was proposed for two-particle transition amplitudes when both the energy and the momentum transfer become complex. This representation exhibits analytic properties of the transition amplitude which are generalizations of the analytic properties expressed by the usual dispersion relations, in which one of the variables is kept fixed. The representation is similar in appearance to one proposed earlier by Nambu[2] for Green's functions; however, it differs in detail and its validity is postulated in a much more restrictive form.

Double dispersion representations of this type have not thus far been proved from the general principles of quantum field theory. The usual dispersion relations can be proved by examining the restrictions imposed by causality on the four-point Green's function, provided that the momentum transfer is sufficiently small.[3-5] It is unlikely that a corresponding proof can be carried out in our case, or indeed that the representation follows from these requirements alone. The general principles of field theory contain much more information, since the causality condition enables one to deduce analytic properties of all the Green's functions, which are related to one another by the unitarity conditions. It is therefore very possible that the representation is a consequence of the general principles of field theory, but it

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[2] Y. Nambu, Phys. Rev. **100**, 394 (1955).

[3] Bogoliubov, Medvedev, and Polivanov (unpublished).

[4] Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958).

[5] H. Lehmann, Nuovo cimento **10**, 579 (1958).

seems at present a matter of considerable difficulty to carry out such a proof.

In the absence of a rigorous treatment making use of all the information available from the general principles of field theory, therefore, it should be useful to examine the analytic structure of transition amplitudes in perturbation theory. It is the aim of the present paper to carry out such an investigation and, in particular, to verify that the representation mentioned above is valid in the lower orders. In Sec. 2 the representation is discussed for the general reaction $A+B\rightarrow C+D$, and certain features, such as the form of the representation when subtractions are necessary, are discussed in more detail than in reference 1. In Sec. 3 it is shown that the fourth-order perturbation theory terms satisfy this representation, and an explicit form for these terms is given if the particles are scalar. The masses are restricted to have values for which there are no anomalous thresholds. It is pointed out that the sixth-order terms can also be shown to satisfy the representation, but the method used does not permit extension to general order.

In the final section the analytic form of the fourth-order transition amplitude is examined when the masses are allowed to take on values corresponding to the existence of the anomalous thresholds of Karplus, Sommerfield, and Wichmann.[6] These authors distinguish two types of anomalous thresholds, those that depend on one of the variables only and correspond to similar thresholds in the vertex problem, and those that depend on both the variables and have no analog in the vertex problem. If only the first type of anomalous threshold occurs, it will be shown that the double dispersion representation still holds. The form of the regions in which the spectral functions are nonzero is slightly altered. If, however, the second type occurs, the double dispersion representation breaks down, and there will be singularities in the complex plane.

2. ANALYTIC REPRESENTATION OF TRANSITION AMPLITUDES

Kinematics

We take the general case of the interaction of four particles, A, B, C and D, whose masses are M_1, M_2, M_3, and M_4, though we shall frequently simplify the treatment by assuming that some of the masses are equal to one another. A single Green's function will describe three possible reactions

$$A+B\rightarrow C+D, \quad \text{(I)}$$

$$A+D\rightarrow B+C, \quad \text{(II)}$$

$$A+C\rightarrow B+D. \quad \text{(III)}$$

Let the four-momenta of the four particles be denoted by $p_1\cdots p_4$; they will be taken as positive for incoming particles, negative for outgoing particles. The squares of the energies for the three reactions will be given by

$$s=(p_1+p_2)^2, \quad \text{(reaction I)} \tag{2.1a}$$

$$t=(p_1+p_3)^2, \quad \text{(reaction III)} \tag{2.1b}$$

$$u=(p_1+p_4)^2, \quad \text{(reaction II)} \tag{2.1c}$$

(time-like vectors are taken as positive). They are connected by the relation

$$s+t+u=M_1^2+M_2^2+M_3^2+M_4^2. \tag{2.2}$$

Owing to this relation, the scattering is characterized by only two independent invariants. However, it will prove most convenient to use all three quantities s, t, and u, and to take different pairs of these as independent variables at different times. When an expression such as $A(s,t,u_1)$ is written down, it is implied that s and t are the independent variables, and u_1 is to be defined as a function of s and t by (2.2). Similarly, in expressions such as $A(s,t_1,u)$ or $A(s_1,t,u)$, t_1 and s_1 are to be defined as a function of the other variables by (2.2).

It should be pointed out that the variable t, besides being the square of the energy for the reaction III, is also the square of the momentum transfer between A and C in the reactions I and II (space-like if negative) and u is similarly the square of the momentum transfer between A and D in the reactions I and III.

Ordinary Dispersion Relations

The usual dispersion relation, in which t is kept constant, has the form (for scalar particles)

$$A(s,t,u_1)=\frac{1}{\pi}\int_{M_a^2}^{\infty}ds'\frac{A_1(s',t,u_1)}{s'-s}+\frac{1}{\pi}\int_{-\infty}^{\Sigma M_i^2-M_b^2-t}ds'\frac{A_2(s',t,u_1)}{s'-s}, \tag{2.3}$$

where, as usual, the denominators are given small negative imaginary parts. A_1 and A_2 are the "absorptive parts" associated with the reactions I and II, respectively. M_a and M_b are the lowest masses which can occur in the continuum of intermediate states in the two reactions. If, as is often the case, the lowest intermediate states are discrete, the absorptive parts will have δ-functions at the corresponding values of s; for simplicity, we shall not write them explicitly. The dispersion relations will have this simple form only if A tends to zero as s becomes infinite; otherwise they will have to be modified by subtractions in the usual way.

Except for the case of forward elastic scattering, the absorptive parts in (2.3) always contain contributions from the unphysical region. This unphysical region may be divided into two parts. If s approaches sufficiently near the threshold and t is kept fixed, the angle of scattering will become complex. If t is not too large, the value of the absorptive part can be found in this unphysical region from the angular-momentum expansion. In addition, the lowest masses of the intermediate states may be below the sum of the masses of the

[6] Karplus, Somerfield, and Wichmann, Phys. Rev. 114, 376 (1959).

particles participating in the reaction. Between these two mass values we shall always be in the unphysical region. Such as unphysical region occurs, for instance, in nucleon-antinucleon scattering or nucleon-antinucleon annihilation into two pions, when it extends from the square of the mass of the lowest intermediate state ($4\mu^2$), to $4M^2$.

If t is positive (time-like momentum transfer), the greater part of, if not the entire, region of integration in (2.3) lies in the unphysical region, and dispersion relations have not been considered for this case. We shall see, however, that they follow from the general analytic representation for the transition amplitude.

As an absorptive part A_2 in (2.3) is associated with the reaction II, it is more appropriate to use the variable u instead of s; the equation then becomes

$$A=\frac{1}{\pi}\int_{M_a^2}^{\infty} ds' \frac{A_1(s',t,u_1)}{s'-s}+\frac{1}{\pi}\int_{M_b^2}^{\infty} du' \frac{A_2(s_1,t,u')}{u'-u}. \quad (2.4)$$

By interchanging the roles of the particles in the reactions, dispersion relations can also be obtained when u or s is kept constant; they take the form

$$A=\frac{1}{\pi}\int_{M_a^2}^{\infty} ds' \frac{A_1(s',t_1,u)}{s'-s}+\frac{1}{\pi}\int_{M_c^2}^{\infty} dt' \frac{A_3(s_1,t',u)}{t'-t}, \quad (2.5)$$

$$A=\frac{1}{\pi}\int_{M_b^2}^{\infty} du' \frac{A_2(s,t_1,u')}{u'-u}+\frac{1}{\pi}\int_{M_c^2}^{\infty} dt' \frac{A_3(s,t',u_1)}{t'-t}. \quad (2.6)$$

A_3 is the absorptive part associated with the reaction III.

In the case of elastic scattering, two of the reactions, I, II, and III will be identical. If, for instance, A and C are nucleons, B and D pions, then the reactions I and II will be identical. In the dispersion relation (2.4), where the momentum transfer between the two pions (or the two-nucleons) is kept constant, the absorptive part associated with reaction III, the pair-annihilation reaction, will not occur at all. In the dispersion relation (2.5), however, in which the momentum transfer between the incoming pion and the outgoing nucleon is kept constant, the integrand in the "crossing" term will involve A_3, the absorptive part associated with reaction III. Owing to the contribution from the large unphysical region, this dispersion relation has not thus far proved useful in phenomenological analyses.

Double Dispersion Representation

The proposed representation of the transition amplitude, as an analytic function of two complex variables, is obtained by generalizing the analytic properties given by the ordinary dispersion relations in the simplest possible manner. It is assumed that the transition amplitude is analytic in the entire space of the two variables except for cuts along certain planes, the location of the cuts being determined so as to lead to the dispersion relations (2.4)–(2.6). Cauchy's theorem then leads to the following analytic representation of the transition amplitude:

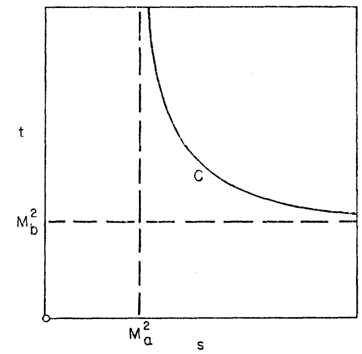

FIG. 1. Region in which the spectral function A_{13} is nonzero.

$$A=\frac{1}{\pi^2}\int ds'dt' \frac{A_{13}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int dt'du' \frac{A_{23}(t',u')}{(t'-t)(u'-u)}+\frac{1}{\pi^2}\int ds'du' \frac{A_{12}(s',u')}{(s'-s)(u'-u)}. \quad (2.7)$$

As in the ordinary dispersion relations, there will in practice be subtraction terms and the representation will be rather more complicated than (2.7). The spectral function A_{13} will be nonzero in a region of the form shown in Fig. 1 (to the right of the curve C). The equation of C has to be determined from the unitarity condition; all that we can say now is that the curve approaches asymptotically the lines $s=M_a^2$ and $t=M_c^2$. A_{12} and A_{23} will be nonzero in corresponding regions. It should be noted that the spectral functions are zero in the physical region for any of the reactions, where one of the invariants—the square of the energy—is positive, while the other two—the squares of the momentum transfers—are negative.

The dispersion relations (2.4)–(2.6) follow from (2.7) if we define the absorptive parts by the equations

$$A_1=\frac{1}{\pi}\int dt' \frac{A_{13}(s,t')}{t'-t}+\frac{1}{\pi}\int du' \frac{A_{12}(s,u')}{u'-u}, \quad (2.8)$$

$$A_2=\frac{1}{\pi}\int dt' \frac{A_{23}(t',u)}{t'-t}+\frac{1}{\pi}\int du' \frac{A_{12}(s',u)}{u'-u}, \quad (2.9)$$

$$A_3=\frac{1}{\pi}\int ds' \frac{A_{13}(s',t)}{s'-s}+\frac{1}{\pi}\int du' \frac{A_{23}(t,u')}{u'-u}. \quad (2.10)$$

This may be verified by a direct substitution of Eqs. (2.8)–(2.10) in any one of the dispersion relations, when (2.7) results. From (2.8)–(2.10) we see that the absorptive parts satisfy dispersion relations in the momentum transfer for their reaction when the energy is kept fixed. The weight functions are A_{13}, A_{23}, and A_{12} which, as we have pointed out, are in the unphysical region. The cuts therefore begin at points on the curve C in Fig. 1 and corresponding curves for A_{13} and A_{12}, i.e., a finite distance above the thresholds M_a^2, M_b^2, and M_c^2. On the other hand, as long as C approaches asymptotically the lines $s=M_a^2$ and $t=M_c^2$, the absorptive parts A_1, A_2, and A_3, as given by (2.8)–(2.10), will begin at the expected thresholds.

Representation with Subtractions

Let us return to consider the form the representation (2.7) takes when—as is always the case in practice—there are subtractions. Proceeding as we do for ordinary dispersion relations, we may write the first term of (2.7) in the form

$$\frac{(s-s_0)(t-t_0)}{\pi^2}\int ds'dt'\frac{A_{13}(s',t')}{(s'-s_0)(s'-s)(t'-t_0)(t'-t)}+\frac{s-s_0}{\pi}\int_{M_a^2}^{\infty} ds'\frac{f_1(s')}{(s'-s_0)(s'-s)}+\frac{t-t_0}{\pi}\int_{M_c^2}^{\infty} dt'\frac{f_3(t')}{(t'-t_0)(t'-t)}+c, \tag{2.11}$$

where

$$f_1(s)=\frac{1}{\pi}\int dt'\frac{A_{13}(s,t')}{t'-t_0},$$
$$f_3(t)=\frac{1}{\pi}\int ds'\frac{A_{13}(s',t)}{s'-s_0}, \tag{2.12}$$
$$c=\frac{1}{\pi^2}\int ds'dt'\frac{A_{13}(s',t')}{(s'-s_0)(t'-t_0)}.$$

If, now, the scattering amplitude remains finite at infinity, the first term of (2.7) can still be replaced by (2.11) but f_1, f_2, and c will no longer be given by the integrals (2.12), which may not even converge. On treating the other terms of (2.7) in the same way, we find that A is given by the representation

$$\begin{aligned}A=&\frac{(s-s_0)(t-t_0)}{\pi^2}\int ds'dt'\frac{A_{13}(s',t')}{(s'-s_0)(s'-s)(t'-t_0)(t'-t)}+\frac{(t-t_0)(u-u_0)}{\pi^2}\int dt'du'\frac{A_{23}(t',u')}{(t'-t_0)(t'-t)(u'-u_0)(u'-u)}\\&+\frac{(s-s_0)(u-u_0)}{\pi^2}\int ds'du'\frac{A_{12}(s',u')}{(s'-s_0)(s'-s)(u'-u_0)(u'-u)}+\frac{s-s_0}{\pi}\int_{M_a^2}^{\infty} ds'\frac{f_1(s')}{(s'-s_0)(s'-s)}\\&+\frac{u-u_0}{\pi}\int_{M_b^2}^{\infty} du'\frac{f_2(u')}{(u'-u_0)(u'-u)}+\frac{t-t_0}{\pi}\int_{M_c^2}^{\infty} dt'\frac{f_3(t')}{(t'-t_0)(t'-t)}+\lambda.\end{aligned} \tag{2.13}$$

The first three terms will be referred to as "double dispersion integrals," the fourth, fifth, and sixth as "single dispersion integrals." If any of the reactions can occur through a discrete intermediate state, so that there are poles in the transition amplitude, they will be represented by δ-functions in f_1, f_2, or f_3, below the thresholds M_a^2, M_b^2, and M_c^2. In perturbation theory, it is found that the amplitude for scattering of scalar particles (coupled with fermions) does remain finite at large values of the variables, so that the representation has the form (2.13).

For future reference we note that the dispersion relations (2.8)–(2.10) must now be written

$$A_1=f_1(s)+\frac{t-t_0}{\pi}\int dt'\frac{A_{13}(s,t')}{(t'-t_0)(t'-t)}+\frac{u-u_0}{\pi}\int du'\frac{A_{12}(s,u')}{(u'-u_0)(u'-u)}, \tag{2.8a}$$

$$A_2=f_2(u)+\frac{t-t_0}{\pi}\int dt'\frac{A_{23}(t',u)}{(t'-t_0)(t'-t)}+\frac{s-s_0}{\pi}\int ds'\frac{A_{12}(s',u)}{(s'-s_0)(s'-s)}, \tag{2.9a}$$

$$A_3=f_3(t)+\frac{s-s_0}{\pi}\int ds'\frac{A_{13}(s',u)}{(s'-s_0)(s'-s)}+\frac{u-u_0}{\pi}\int du'\frac{A_{23}(t,u')}{(u'-u_0)(u'-u)}. \tag{2.10a}$$

As the fourth term in (2.13) does not depend on t, it affects only the S waves in the reaction I. The succeeding terms are purely real in the physical region, so that the single dispersion integrals affect the absorptive part of only the S wave. The absorptive part of all the other waves is determined completely by the double dispersion integrals.

The functions f_1, f_2, and f_3 will of course depend on the values of s_0 and t_0 chosen. The subtractions can be performed in such a way that these functions are just the S-wave absorptive for the three reactions. To do this, we note from (3.8) that the S-wave absorptive part for the first reaction is given by

$$A_1^{(0)}(s)=\frac{1}{2}\int_1^1 d(\cos\theta)A_1(s,t,u_1)$$

$$=\frac{1}{\pi[t_{bI}(s)-t_{aI}(s)]}\int_{t_{aI}(s)}^{t_{bI}(s)}dt\int dt'\frac{A_{13}(s,t')}{t'-t}+\frac{1}{\pi[u_{bI}(s)-u_{aI}(s)]}\int_{u_{aI}(s)}^{u_{bI}(s)}du\int du'\frac{A_{12}(s,u)}{u'-u}$$

$$=\frac{1}{\pi[t_{bI}(s)-t_{aI}(s)]}\int dt'A_{13}(s,t')\ln\frac{t'-t_{bI}(s)}{t'-t_{aI}(s)}+\frac{1}{\pi[u_{bI}(s)-u_{aI}(s)]}\int du'A_{12}(s,u')\ln\frac{u'-u_{bI}(s)}{u'-u_{aI}(s)}. \tag{2.14}$$

$t_{aI}(s)$ and $t_{bI}(s)$ are the minimum and maximum values of t for the reaction I, and similarly for $u_{aI}(s)$ and $u_{bI}(s)$. As t or u depends linearly on $\cos\theta$ when s is fixed, the integrations over $\cos\theta$ has been replaced by integrations over these variables. We now see that we can rewrite (2.7) in the form

$$A=\frac{1}{\pi^2}\int ds'dt'A_{13}(s',t')\left(\frac{1}{s'-s}-l_{III}(t',s')\right)\left(\frac{1}{t'-t}-l_{I}(s',t')\right)+\frac{1}{\pi^2}\int dt'du'A_{23}(t',u')$$
$$\times\left(\frac{1}{t'-t}-l_{II}(u',t')\right)\left(\frac{1}{u'-u}-l_{III}(t',u')\right)+\frac{1}{\pi^2}\int ds'du'A_{12}(s',u')\left(\frac{1}{s'-s}-l_{II}(u',s')\right)$$
$$\times\left(\frac{1}{u'-u}-l_{I}(s',u')\right)+\frac{s-s_0}{\pi}\int_{M_a^2}^{\infty}ds'\frac{A_1^{(0)}(s')}{(s'-s_0)(s'-s)}+\frac{u-u_0}{\pi}\int_{M_b^2}^{\infty}du'\frac{A_2^{(0)}(u')}{(u'-u_0)(u'-u)}+\frac{t-t_0}{\pi}$$
$$\times\int_{M_c^2}^{\infty}dt'\frac{A_3^{(0)}(t')}{(t'-t_0)(t'-t)}+\mu, \tag{2.15}$$

where we have written $l_I(s',t')$ for

$$\frac{1}{t_{aI}(s')-t_{bI}(s')}\ln\frac{t'-t_{aI}(s')}{t'-t_{bI}(s')}.$$

The absorptive parts for the S waves then depend only on the single dispersion integrals and those for the higher angular-momentum waves only on the double dispersion integral.

Let us now consider the number of subtractions required when the particles A and C are fermions of spin $\frac{1}{2}$. If we write the invariant scattering amplitude in the form

$$-A+\tfrac{1}{2}i\gamma\cdot(p_2-p_4)B, \tag{2.16}$$

we find that the perturbation terms have the following asymptotic behavior:

$t\to$ const,	$s\to\infty$	$A\to$ const, $B\to$ const/s.
$s\to$ const,	t or $u\to\infty$,	Re (A) and Re $(B)\to$ const,
		Im (A) and Im $(B)\to$ const/t (const/u).
s, t, and $u\to\infty$,		A and $B\to$ const/s.

s can be replaced by u in these limits since interchange of these variables simply interchanges the two spin-zero particles. It is then easy to see that the representation must have the form

$$A=\frac{s-s_0}{\pi^2}\int ds'dt'\frac{A_{13}(s',t')}{(s'-s_0)(s'-s)(t'-t)}+\frac{u-u_0}{\pi^2}\int dt'du'\frac{A_{23}(t',u')}{(u'-u_0)(u'-u)(t'-t)}$$
$$+\frac{1}{\pi^2}\int ds'du'\frac{A_{12}(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi}\int_{M_a^2}^{\infty}ds'\frac{a_1(s')}{s'-s}+\frac{1}{\pi}\int_{M_b^2}^{\infty}du'\frac{a_2(u')}{u'-u}+\frac{1}{\pi}\int_{M_c^2}^{\infty}dt'\frac{a_3(t')}{t'-t},$$

$$B=\frac{1}{\pi^2}\int ds'dt'\frac{B_{13}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int dt'du'\frac{B_{23}(t',u')}{(t'-t)(u'-u)}+\frac{1}{\pi^2}\int ds'du'\frac{B_{12}(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi}\int_{M_a^2}^{\infty}ds'\frac{b_1(s')}{s'-s}+\frac{1}{\pi}\int_{M_b^2}^{\infty}du'\frac{b_2(u')}{u'-u}. \tag{2.17}$$

We observe in particular that, in this case, there is no over-all subtraction constant independent of both variables. It is in fact true independently of perturbation theory that the number of over-all subtraction terms in a double dispersion representation is limited (one for the case where all the particles are scalar, none for all other cases). The presence of one over-all subtraction term implies that the S-wave amplitudes for the three reactions behave like a constant at infinity, whereas more than one over-all subtraction term (involving polynomials in the variables) would require at least one angular-momentum state for one of the reactions to increase at infinity. However, the form $-2\pi(e^{i\delta}\sin\delta)w/q\omega_1\omega_2$ of the scattering amplitude for an individual partial wave ($\mathrm{Im}\delta\geqslant 0$) indicates that it is bounded at infinity. On inserting the kinematical factors relating it to the invariant scattering amplitude, we find that the latter must tend to zero at infinity, except for scattering of scalar particles when it may remain finite. We thus arrive at the result just stated.

3. EXPRESSION OF FOURTH-ORDER FEYNMAN GRAPHS AS DOUBLE DISPERSION INTEGRALS

Outline of the Method

We now wish to evaluate the fourth-order Feynman diagrams and to show that they can be expressed in the form of the representation we have been describing. In order to simplify the discussion, all particles will be taken to be scalar and the $\lambda\phi^4$ interaction term will be omitted. Though greatly reducing the amount of algebra to be performed, this assumption does not change any of the essential features of the result.

The reducible graphs do not present any difficulty. They are all functions of one of the variables only, and may be represented by modifying a vertex part or an internal line in a second-order diagram. The modified vertex part or internal line can then be expressed as a dispersion integral in the mass of the internal line. That it is possible to do so is now a well-known fact for the internal line, and has been proved in perturbation theory for the vertex part,[7–9] though in certain cases there may be anomalous thresholds. On inserting these dispersion integrals into the formulas for the scattering diagrams in question, it follows at once that the scattering amplitude satisfies a dispersion relation in the variable on which it depends, again with the possible occurrence of anomalous thresholds.

The reducible graphs thus contribute only to the single dispersion integrals in (2.13). In this case they tend to zero at infinity, so that they can be written without subtractions.

We are left with the irreducible diagrams, which are all topologically identical with Fig. 2. For simplicity we shall suppose all masses to be equal. The amplitude corresponding to this diagram is then given by the formula

$$A^{(4,\mathrm{i})}=\frac{ig^4}{(2\pi)^4}\int\frac{d^4q}{[(p_1-q)^2-M^2][(p_1+p_2-q)^2-M^2][(p_3+q)^2-M^2](q^2-M^2)}. \tag{3.1}$$

The 4 in the superscript is the order of the perturbation theory, and the superscript (i) is to indicate that we are not dealing with the entire fourth-order amplitude.

The integral (3.1) is a function of the two invariants s and t. The method of procedure will be to show that, if t is fixed, real and negative, this integral is an analytic function of s with no singularities in the complex plane. A dispersion relation can therefore be written for $A^{(4,\mathrm{i})}$. The imaginary part which appears in the integrand is next calculated explicitly by examining (3.1). It will then be seen that $A^{(4,\mathrm{i})}$ can be brought into the form of the first term of (2.7).

The easiest way of showing that $A^{(4,\mathrm{i})}$ is an analytic function of s for fixed negative t is to parametrize it in the usual way [see, for instance, Karplus, Sommerfield, and Wichmann,[6] Eq. (9)]. If s is complex, it is not difficult to see that the denominator of the integrand never vanishes over the range of integration of the parameters. This is true whatever the values of the eight masses in Fig. 2, as long as each particle is energetically stable, i.e., its mass must not be greater than the sum of the masses of the other two particles meeting it in one of the vertices. If in addition the condition for the existence of anomalous singularities is not met, the denominator also never vanishes if s is real and below the expected threshold. It then follows immediately that the integral is an analytic function of s with no singularities in the complex plane or below the expected threshold. $A^{(4,\mathrm{i})}$ can thus be written in the form

$$A^{(4,\mathrm{i})}(s,t,u_1)=\frac{1}{\pi}\int_{(2M)^2}^{\infty}ds'\frac{\mathrm{Im}A^{(4,\mathrm{i})}(s',t,u_1)}{s'-s}$$

$$(t \text{ real and } <0). \tag{3.2}$$

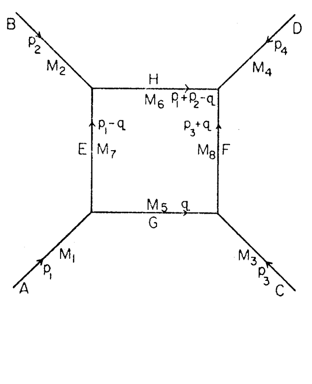

FIG. 2. Fourth-order perturbation graph for the reaction $A+B\rightarrow C+D$.

[7] Karplus, Sommerfield, and Wichmann, Phys. Rev. **111**, 1187 (1958).
[8] Y. Nambu, Nuovo cimento **9**, 610 (1958).
[9] R. Oehme, Phys. Rev. **111**, 1430 (1958).

It should be noted that not all dispersion relations can be proved in fourth order with such ease. The dispersion relation for $A^{(4,\mathrm{i})}$ with u instead of t fixed, for instance, does not follow from this type of reasoning. Grisaru[10] has examined a comparable case for nucleon-nucleon scattering. We shall not look at such cases directly, as all the ordinary dispersion relations follow from the double dispersion representation.

Calculation of Im(A)

Returning to consider the imaginary part of (3.1) which must be inserted into (3.2), we proceed along the lines of reference 10. We take (3.1) in the center-of-mass system of the particles, with the momentum transfer in the y direction and the z direction perpendicular to the scattering plane. As we have pointed out, part of the range of integration in (3.2) is in the unphysical region and represents scattering through an angle whose cosine is less than -1. In this region, the x-components of p_1 and p_3, which occur in the first and third factors of (3.1), will be purely imaginary. However, the imaginary components occur only in the form $(p_{1x}-q_x)^2$ and $(p_{3x}-q_x)^2$, and the remainder of the integrand is an even function of q_x. On integrating over q_x from $-\infty$ to ∞, therefore, the imaginary terms arising from this source cancel out.

Each of the four factors of (3.1) really consists of the sum of a principal part and an imaginary δ function. It might therefore naively be supposed that, if these two parts were separated and the resulting expression multiplied out, those terms containing an even number of δ functions would give the imaginary part. The integral over the product of the four principal parts can easily be shown to vanish. The product of the four δ functions also vanishes in our case, as it will be seen below that their arguments can never be zero simultaneously. We are left with all possible terms which are products of two principal parts and two δ functions. By looking more carefully at the integral, we find that this procedure is not correct and must be modified as follows: write each factor in (3.1) in the form

$$\frac{1}{(q_0-p_0)^2-(\mathbf{q}-\mathbf{p})^2-M^2i\epsilon}=\frac{P}{(q_0-p_0)^2-(\mathbf{q}-\mathbf{p})^2-M^2} -\frac{i\pi}{2q_0}\delta\{q_0-p_0+[(\mathbf{q}-\mathbf{p})^2-M^2-i\epsilon]^{\frac{1}{2}}\} -\frac{i\pi}{2q_0}\delta\{q_0-p_0-[(\mathbf{q}-\mathbf{p})^2-M^2-i\epsilon]^{\frac{1}{2}}\}. \quad (3.3)$$

Expand out the product, and examine the terms containing two principal parts and two δ functions. If, in a particular term, the arguments of the two δ functions have infinitesimal imaginary parts of opposite sign, multiply the term by two. If they have imaginary parts of the same sign, do not include the term.

We have therefore to investigate what products of δ functions from two different factors of (3.1) can be nonzero, i.e., can have arguments which are simultaneously zero. The argument of the δ function is zero when the four-momentum of the corresponding internal line in Fig. 2 is on the mass shell. Two adjacent internal lines cannot be on the mass shell simultaneously, as this would imply that one of the vertices had three real lines, which is impossible, even if the p's can have their first component purely imaginary. Further, the two opposite lines E and F cannot be simultaneously on the mass shell and have δ functions which contribute to Im$A^{(4,\mathrm{i})}$, as, assuming this to be the case, we can easily derive a contradiction. The arguments of the δ functions are

$$-p_{10}+q_0\pm[(\mathbf{p}_1+\mathbf{q})^2+M^2-i\epsilon]^{\frac{1}{2}},$$

and

$$p_{30}+q_0\pm[(\mathbf{p}_3+\mathbf{q})^2+M^2-i\epsilon]^{\frac{1}{2}}.$$

If they are to be zero and have imaginary parts of opposite sign, the two fourth components $p_{10}-q_0$ and $p_{30}+q_0$ must have the same sign. As the two lines are both on the mass shell and therefore timelike, and the fourth components of their momenta have the same sign, the sum of their momenta must be time-like. However, the sum of their momenta is equal to t and is therefore, by assumption, space-like. It is therefore impossible for them to be simultaneously on the mass shell and to have δ functions which contribute to Im$A^{(4,\mathrm{i})}$.

We are left with the case where G and H are simultaneously on the mass shell. The arguments of the two δ functions are then

$$q_0-(p_1+p_2)_0\pm(\mathbf{q}^2+M^2-i\epsilon)^{\frac{1}{2}}, \quad (\text{since } \mathbf{p}_1+\mathbf{p}_2=0)$$

and

$$q_0\pm(\mathbf{q}^2+M^2-i\epsilon)^2.$$

If their imaginary parts are to have opposite sign, the energies of the lines G and H, equal to $(p_1+p_2)_0-q_0$ and q_0, must have the same sign, and must therefore be positive, as the incoming pion-nucleon state has positive energy. On combining the δ functions with the principal part of the other two factors, and multiplying by θ functions to restrict the signs of q_0 and $(p_1+p_2)_0-q_0$ to be positive, we obtain for Im$A^{(4,\mathrm{i})}$

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4}{8\pi^2}\int d^4q\frac{1}{\{(p_1-q)^2-M^2\}\{(p_3+q)^2-M^2\}} \times\theta(q_0)\theta\{(p_1+p_2)_0-q_0\}\delta(q^2-M^2) \times\delta\{(p_1+p_2-q)^2-M^2\}, \quad [s>(2M)^2]. \quad (3.4)$$

The arguments of the δ functions vanish when the internal particles G and H are on the mass shell, and we

[10] M. T. Grisaru, Phys. Rev. 111, 1719 (1958).

have seen that they must have positive energies. We may therefore consider G and H to form a *real* intermediate state. As four-momentum is conserved at the vertices of a Feynman diagram, and the initial and final states are taken to be in their center-of-mass systems, this intermediate state is also in its center-of-mass system. The two integrations in (3.4) remaining after the δ functions have been taken into account reduce to integrations over the direction of the particles in the intermediate state, which we shall denote by $\mathbf{n}_i$. There will be a factor from the δ functions equal to $1/(4qw)$, where q is the center-of-mass momentum and w the center-of-mass energy (equal to $s^{\frac{1}{2}}$). The term $(p_1-q)^2$ in the denominator of (3.4) is now just t_{ie}, the square of the momentum transfer between the incoming particle A and the intermediate particle G. $(p_3+q)^2$ is similarly t_{io}, the square of the momentum transfer between the intermediate particle G and the final particle C. Thus (3.4) takes the form

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4q}{32\pi^2w}\int d^2\mathbf{n}_i\frac{1}{(t_{ie}-M^2)(t_{io}-M^2)},\qquad [s>(2M)^2]. \tag{3.5}$$

In order to perform the integrations, t_{ie} and t_{io} must be expressed in terms of the cosines z_{ie} and z_{io} of the corresponding angles by the relations:

$$t_{ie(io)}=2q^2(z_{ie(io)}-1). \tag{3.6}$$

z_{io} is given in terms of z_{ie} and z, the cosine of the angle of scattering from the initial to the final state, by the relation

$$z_{io}=zz_{ie}+(1-z^2)^{\frac{1}{2}}(1-z_{ie}^2)^{\frac{1}{2}}\cos\phi, \tag{3.7}$$

ϕ being the azimuthal angle between the plane of scattering and the direction $\mathbf{n}_i$, with the direction of the incoming particles taken as the pole. On substituting the expressions for t_{ie}, t_{io}, and z_{io} into (3.5), and rewriting the integral over $\mathbf{n}_i$ as an integral over its co-ordinates z_{ie} and ϕ, we obtain

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4q}{32\pi^2w}\int_{-1}^{1}dz_{ie}\int_0^{2\pi}d\phi\frac{1}{[2q^2(z_{ie}-1)-M^2]\{2q^2[zz_{ie}+(1-z^2)(1-z_{ie}^2)\cos\phi-1]-M^2\}}. \tag{3.8}$$

The integral can be evaluated and, after transferring from z to t by (3.6) and expressing q in terms of s by the formula

$$q^2=(s-4M^2)/4, \tag{3.9}$$

we find that

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{1}{16\pi\{\kappa(s,t)\}^{\frac{1}{2}}}\ln\frac{\alpha(s,t)+(q/s^{\frac{1}{2}})[\kappa(s,t)]^{\frac{1}{2}}}{\alpha(s,t)-(q/s^{\frac{1}{2}})[\kappa(s,t)]^{\frac{1}{2}}},\qquad [s>(2M)^2]$$
$$=0,\quad [s<(2M)^2] \tag{3.10}$$

where

$$\kappa(s,t)=4st[st-4M^2(s+t)+12M^4], \tag{3.11}$$

$$\alpha(s,t)=st-2M^2s-4M^2t+6M^4. \tag{3.12}$$

Expression of A as a Double Dispersion Integral

It now follows at once that $\mathrm{Im}A^{(4,\mathrm{i})}$ satisfies a dispersion relation of the form (2.8). The right-hand side of (3.10) is an analytic function of t in the complex t-plane, except for a cut along that portion of the real axis where κ and t are both positive. The discontinuity across the cut is $-1/\{4[\kappa(s,t)]^{\frac{1}{2}}\}$. Hence we may write

$$\mathrm{Im}A^{(4,\mathrm{i})}(s,t,u_1)=\frac{1}{\pi}\int dt'\frac{A_{13}^{(4)}(s,t')}{t'-t}, \tag{3.13}$$

where

$$A_{13}^{(4)}(s,t)=-1/\{8[\kappa(s,t)]^{\frac{1}{2}}\},\qquad \kappa>0,\quad t>0,\quad s>(2M)^2$$
$$=0 \text{ otherwise}. \tag{3.14}$$

Our final step is to substitute (3.13), which has the form of the dispersion relation (2.8), into (3.2), to give

$$A^{(4,\mathrm{i})}(s,t,u_1)=\frac{1}{\pi^2}\int ds'dt'\frac{A_{13}^{(4)}(s',t')}{(s'-s)(t'-t)}. \tag{3.15}$$

We have thus verified that $A^{(4,\mathrm{i})}$ can be written in the form of the representation (2.7), with only the first term contributing. The boundary of the region in which $A_{13}^{(4)}$ is nonzero is given by

$$st-4M^2(s+t)+12M^4=0. \tag{3.16}$$

Equation (3.16) has the form of the curve C of Fig. 1, and approaches asymptotically the lines $s=4M^2$ and $t=4M^2$.

Equation (3.13) has only been established for t real and negative. We can however define an analytic continuation of A for all values of t by this equation. It is then necessary to verify that A is given correctly in the physical region for reaction III, where t is positive, by (3.13). We could do this directly by repeating the same procedure with Fig. 2 turned through 90°, it is then found that $A^{(4,\mathrm{i})}$ is given by (3.15) as long as s is real and negative. This, however, is not really necessary, as the Feynman integral (3.1) and the expression (3.15) are both analytic functions of s and t, so that, if they are equal in a region, they are equal everywhere as long as we are on the same branch. Now, on passing from the region in which t is real and negative to the physical region for the reaction III, the only branch point of (3.15) which has to be crossed is at $t=4M^2$, the threshold for this reaction. The sign of the small imaginary part to be included in the second denominator of (3.15) will depend on the manner in which we go round this point, and it is not difficult to verify that, corresponding to the small negative imaginary parts in the denominators of the Feynman-Dyson integral, we must maintain our convention and insert a small negative imaginary part here. The scattering amplitude is therefore given correctly by (3.13), with the usual convention concerning the imaginary part, in the physical region for the reaction III.

The other fourth-order diagrams are obtained by interchanging some of the external lines in Fig. 2. This has the effect of interchanging a pair of the variables s, t, and u, and these diagrams will therefore correspond to the second or third terms of (2.7). The spectral functions will again be given by (3.14), with the appropriate change of variables. The property of fourth-order perturbation theory that one Feynman diagram corresponds to a single term of (2.7) is not maintained in higher orders.

The General Mass Case

We shall conclude this section by generalizing the procedure to the case where all eight masses are different. Instead of (3.5), we now have

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4q_i}{32\pi^2w}\int d^2\mathbf{n}_i\frac{1}{(t_{ie}-M_7^2)(t_{io}-M_8^2)},\qquad [s>(M_5+M_6)^2] \tag{3.17}$$

q_i denoting the center-of-mass momentum of the intermediate state. The equation connecting the momentum transfer and the cosine of the angle is, in the general case,

$$t_{\alpha\beta}=2q_\alpha q_\beta z_{\alpha\beta}-q_\alpha^2-q_\beta^2+[(M_\alpha^2+q_\alpha^2)^{\frac12}-(M_\beta^2+q_\beta^2)^{\frac12}]^2, \tag{3.18}$$

where q_α and q_β are the center-of-mass momenta, and M_α and M_β the masses, of the particles between which the momentum transfer is to be found. It is more convenient to work in terms of cosines of angles than of momentum. As t_{ie} and t_{io} are linearly related to z_{ie} and z_{io}, with constants of proportionality q_eq_i and q_oq_i (q_e and q_o being the center-of-mass momenta of the initial and final states), Eq. (3.17) may be written

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4}{32\pi^2q_eq_oq_iw}\times\int d^2\mathbf{n}_i\frac{1}{(z_{ie}-z_{ie}')(z_{io}-z_{io}')}, \tag{3.19}$$

where z_{ie}' and z_{io}' are the cosines of the (complex) angles corresponding to momentum transfers $t_{ie}=M_7^2$ and $t_{io}=M_8^2$. According to (3.8), they will be given by

$$z_{ie}'=\{q_e^2+q_i^2+M_7^2-[(M_1^2+q_e^2)^{\frac12}-(M_5^2+q_i^2)^{\frac12}]^2\}/(2q_iq_e), \tag{3.20a}$$

$$z_{io}'=\{q_o^2+q_i^2+M_8^2-[(M_3^2+q_o^2)^{\frac12}-(M_5^2+q_i^2)^{\frac12}]^2\}/(2q_iq_o), \tag{3.20b}$$

and there will be a corresponding equation connecting z with t:

$$z=\{q_e^2+q_o^2+t-[(M_1^2+q_e^2)^{\frac12}-(M_7^2+q_o^2)^{\frac12}]^2\}/(2q_eq_o). \tag{3.20c}$$

The formula connecting q with s is

$$q^2=[s^2-2s(M_a^2+M_b^2)+(M_a^2-M_b^2)^2]/(4s), \tag{3.20d}$$

where M_a and M_b are the masses of the particles in the state in question (M_1 and M_2 for the initial state, M_5 and M_6 for the intermediate state, and M_3 and M_4 for the final state).

On expressing z_{io} by (3.7) and performing the integration over z_{ie} and ϕ, we obtain the formula

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4}{16\pi q_eq_oq_iW}\frac{1}{\{k(z,z_{ie}',z_{io}')\}^{\frac12}}\times\ln\frac{z-z_{ie}'z_{io}'+[k(z,z_{ie}',z_{io}')]^{\frac12}}{z-z_{ie}'z_{io}'-[k(z,z_{ie}',z_{io}')]^{\frac12}}, \tag{3.21}$$

where

$$k(z,z_{ie}',z_{io}')=z^2+z_{ie}'^2+z_{io}'^2-1-2zz_{ie}'z_{io}'. \tag{3.22}$$

The right-hand side of (3.21) is an analytic function of z, except for a cut running from the value z_1, where

$$z_1=z_{ie}'z_{io}'+(z_{ie}'^2-1)^{\frac12}(z_{io}'^2-1)^{\frac12}, \tag{3.23}$$

to infinity. The discontinuity across the cut is $-1/\{4[k(z,z_{ie}',z_{io}')]^{\frac12}\}$, so that (3.21) may be written

$$\mathrm{Im}A^{(4,\mathrm{i})}=\frac{g^4}{8\pi q_eq_oq_iw}\times\int_{z_1}^{\infty}dz'\frac{1}{\{k(z',z_{ie}',z_{io}')\}^{\frac12}}\frac{1}{z'-z}, \tag{3.24}$$

where the lower limit z_1 is given by (3.23).

Since t is linearly related to z by (3.20c), Eq. (3.24) may be rewritten as a dispersion integral in t instead of z. As in the simple case, we can then substitute this dispersion relation into (3.2) and obtain (3.15), with $A_{13}^{(4)}$ now given by the equation

$$\begin{aligned}A_{13}^{(4)}&=-\frac{1}{8q_eq_oq_iW\{k(z,z_{ie}',z_{io}')\}^{\frac12}},\quad (z>z_1)\\&=0.\quad (z<z_1).\end{aligned} \tag{3.25}$$

z_1 is again the value of z in (3.23). The curve C of Fig. 1, which bounds the region in which $A_{13}^{(4)}$ is nonzero, is thus given by (3.23). On inserting the values of z_{ie}', z_{io}' and z from (3.20) into (3.25) and (3.23), and expressing q_e, q_o, q_i and W in terms of s and the masses,

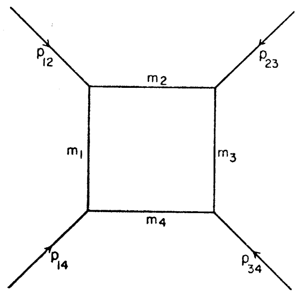

FIG. 3. Definition of the momenta used in Eq. (3.26).

we can obtain $A_{13}^{(4)}$, and the curve C which bounds the region in which it is nonzero, in terms of s and t.

For most cases of practical interest, in which not all the masses are different, it is probably more convenient to work out (3.20) with the masses involved, and then to substitute into (3.25) and (3.23), rather than to use these formulas expressed in terms of s and t and general masses $M_1\cdots M_8$. However, for the general case, the result of the substitution assumes a neat form in the notation of reference 6, so we give it here. The masses and momenta are now re-defined according to Fig. 3, together with the equations

$$\begin{aligned} p_{13}&=p_{12}+p_{23},\\ p_{24}&=p_{23}+p_{34}. \end{aligned} \tag{3.26}$$

We further define the variables[11]

$$\begin{aligned} &x_{ii}=-1,\\ &x_{ij}=(p_{ij}^2-m_i^2-m_j^2)/(2m_im_j), \quad (i,j=1\cdots4). \end{aligned} \tag{3.27}$$

Apart from proportionality factors and constant terms, the variables x_{12}, x_{23}, x_{34}, and x_{14} correspond to the external masses, x_{24} corresponds to s and x_{13} to t. Equation (3.25) then reduces to the formula[12]

$$A_{13}^{(4)}=-\frac{1}{8m_1m_2m_3m_4(\mathrm{Det}x)^{\frac{1}{2}}} \quad \text{or} \quad 0, \tag{3.28}$$

and the region in which it is nonzero is that portion of the $x_{13}-x_{24}$ plane which lies to the right and above the curve:

$$\mathrm{Det}\,x=0, \quad x_{13}>1, \quad x_{24}>1. \tag{3.29}$$

If the condition for anomalous singularities is fulfilled, the inequalities in (3.29) have to be modified. As long as this is not the case, the equation, when expressed in terms of s and t, has the form of the curve C in Fig. 1.

4. FOURTH-ORDER TERM WITH ANOMALOUS THRESHOLDS

We have assumed thus far that the masses of the particles were such that no anomalous thresholds of the type discussed by Karplus, Sommerfield, and Wichmann[6] occur. In that case the curve C of Fig. 1 is always bounded by its asymptotes as shown, and the representation (2.7) has its simple form. We now wish to discuss what happens when there are anomalous thresholds. No detailed discussion will be given, but we shall merely indicate in what way the representation must be modified.

We again use the variables (3.27). x_{12}, x_{23}, x_{24}, and x_{14} correspond to the four external masses, x_{24} to s, and x_{13} to t. The normal thresholds for the reactions I and III are $x_{24}=1$, $x_{13}=1$.

If the region in which A_{13} is nonzero is to extend out of the region bounded by its asymptotes, there must be some point of horizontal or vertical tangency on the curve bounding it. The equation of this curve is given by (3.29), and it is shown in reference 6 that the equations for the horizontal or vertical tangents are (if x_{24} is plotted horizontally, x_{13} vertically):

$$\text{Horizontal tangents} \quad K_2=0, \quad \text{or} \quad K_4=0, \tag{4.1a}$$

$$\text{Vertical tangents} \quad K_1=0, \quad \text{or} \quad K_3=0, \tag{4.1b}$$

where

$$K_r=\mathrm{Det}x_{ij}; \quad i,j=1\cdots4; \quad i,j\neq r. \tag{4.2}$$

(4.1a) is the equation for the possible anomalous thresholds which Karplus *et al.*[6] find for the reaction III, (4.1b) the equation for the possible anomalous thresholds in the reaction I. We note that $K_r=0$ is the equation for the anomalous threshold of the vertex part obtained by contracting the line m_r (Fig. 3) out of the diagram.

If (4.1a) is solved for x_{13}, it is found that the solutions, which we shall denote by L_2 and L_4, are always between 1 and -1. If one or more of the masses is varied so that the condition for an anomalous threshold of the reaction III is approached, the solution of one of Eqs. (4.1a) approaches 1; let us suppose it is L_2. It reaches 1 when

$$\theta_{14}+\theta_{34}=\pi, \tag{4.3}$$

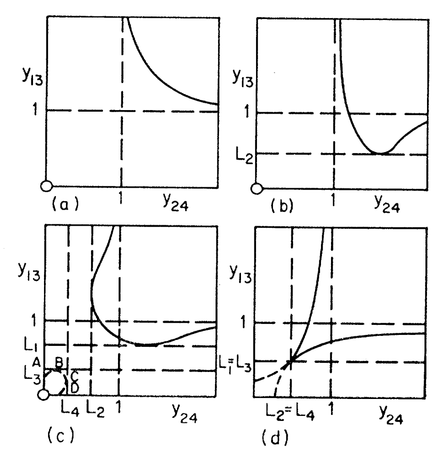

FIG. 4. Behavior of the spectral function when there are anomalous thresholds.

[11] In order to have no reversal in sign between our new variables and s and t, we have defined the x's to be opposite in sign to the y's of reference 8.

[12] This can most easily be shown by a method due to T. W. B. Kibble (to be published.).

where

$$\theta_{ij}=\cos^{-1}(-x_{ij}). \tag{4.4}$$

It then decreases again from 1, and now represents an anomalous threshold. Meanwhile, L_4 increases towards 1, and the two solutions will cross at the point where

$$\theta_{12}+\theta_{23}+\theta_{34}+\theta_{14}=2\pi. \tag{4.5}$$

If (4.1b) is solved for x_{24}, the solutions will behave in the same manner. As (4.5) is symmetric in the indices, the solutions of the two equations (4.1a), and those of the two equations (4.1b), will cross simultaneously. Beyond this point, we have the threshold behavior of case (iii) of reference 6.

Let us now study how the spectral functions behave in these cases. We have seen that, in the normal case, the curve bounding the region in which A_{13} is nonzero has the form of Fig. 4(a). As long as a horizontal or vertical tangent does not meet one of the asymptotes, it must continue to have this form.

Figure 4(b) indicates the behavior when L_2 has reached, and moved away from, the line $x_{13}=1$. The region in which A_{13} is nonzero is now not bounded by the asymptotes, but the scattering amplitude can still be written in the form (3.15). This may be verified by analytic continuation in the masses from the normal case. A_3, given by (2.10), is now nonzero if x_{13} is between L_2 and 1. In other words, we have a typical anomalous threshold.

When L_2 and L_1 are both anomalous thresholds, the situation is as indicated in Fig. 4(c). We have shown in addition the lines $x_{13}=L_4$, $x_{24}=L_3$. The curve drawn below and to the left of these lines is also given by (3.29), but it does not yet play a part in the analytical structure of the scattering amplitude.

At the point given by (4.5), we have the situation indicated in Fig. 4(d). All the points of tangency now coincide—if this were not the case, the equation (3.29), if solved for x_{24} when $x_{13}=L_2$, would have two double roots, which is impossible as it only involves x_{24} quadratically. The scattering amplitude is still given by (3.15), and the curves below and to the left of the lines $x_{13}=L_4$, $x_{24}=L_3$ play no part.

When the sum of the angles in (4.5) becomes greater than 2π, we again have the situation in Fig. 4(c), except that L_2 and L_4, and L_1 and L_3, are interchanged. However, if we assume that (3.15) is true before we reach this condition, and continue it analytically past the point where the situation indicated in Fig. 4(d) holds, the analytic continuation will no longer be of this form (3.15). The scattering amplitude now has a nonzero imaginary part in the area to the right and above $ABCD$. That part of the curve given by (3.29) between B and C plays the role, therefore, of an anomalous threshold which depends on all the variables and has no counterpart in the vertex function. This has previously been shown by Karplus, Sommerfield, and Wichmann[8] by examining the parametrized Feynman integral. For values of x_{24} between L_1 and L_3, the absorptive part A_1 will have a similar analytic form, as a function of x_{13}, to the normal case, [Eq. (3.10), modified for nonequality of the masses] and its singularities will again be given by (3.29). Now, however, the solution of this equation will yield complex values of x_{13} (for real x_{24}), so that A_1 cannot be written as a dispersion integral in x_{13}.

It is thus evident that, once the sum of the angles θ_{12}, θ_{23}, θ_{34} and θ_{14} is greater than 2π, the representation (2.7) will no longer hold. One can still construct the fourth-order term by analytic continuation in the external masses from the normal case.

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