Phase-Shift Analysis of Proton Scattering by Nitrogen

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A phase-shift analysis has been made of a set of angular distributions for the elastic scattering of protons by nitrogen in the energy range 1.0 Mev to 3.0 Mev. Two independent S-wave phase shifts and one P-wave phase shift of hard-sphere type have been assumed. Moderately good agreement with the nonresonant scattering below 2.3 Mev and with the scattering at the $\frac{1}{2}+$ resonance at 1.557 Mev has been obtained, indicating that the gross features of the scattering can be represented in this way. Between 2.3 Mev and 3.0 Mev the fits are poor. The results indicate a broad $\frac{1}{2}+$ resonance at 2.32 Mev with a width of 0.55 Mev.

INTRODUCTION

A PHASE-SHIFT analysis of the low-energy elastic scattering of protons by nitrogen is of interest in two ways. In the first place it appears to be necessary for the establishment of spin and parity assignments for some of the observed resonances.[1] In the second place it will provide some information about the average interaction between a nucleon and the N^{14} nucleus through the behavior of the nonresonant scattering. An early report[2] stated that contributions from P waves were small below 2.0 Mev. However, subsequent work showed that the experimental cross sections used there were low by about 10%, and with the revised normalization good fits using S waves only could not be obtained. The next logical step in studying the problem is to introduce P waves. However, due to the spin of nitrogen, the introduction of P waves with full generality involves a large number of arbitrary parameters, substantially more, in fact, than can be obtained from an analysis of angular distributions. It is thus necessary to introduce some simplifying assumption. The present work has been carried out using the assumption that all of the phase shifts describing the P waves are equal. Equality between the two S-wave and between the various P-wave phase shifts will result if the scattering is described by a potential well without spin-orbit coupling. The assumption thus has some resemblance to the "cloudy crystal ball" model[3-5] although it does not include the complex part. Since this part serves to transfer some of the particles from the pure "shape elastic" process to "compound elastic" and reaction processes, its neglect will be justified if the latter are small or absent. For the present case particle reactions are energetically impossible. Compound elastic processes will be represented by transitions between different channel spins which result from off-diagonal elements of the R matrix. It is argued by Teichmann and Wigner[6] that such terms will be small in the nonresonant regions.

Independent phase shifts for the two states formed by S-waves have been used. Although this departs from the requirements of a simple well, any differences found between the phases can be attributed to compound elastic effects and the procedure has consequently somewhat greater generality. An S-wave resonance will be fitted with these parameters, and if the fits are good, they will identify the spin of the state.

The experimental data used are those described in the preceding paper.[7] Fitting of the angular distributions has been done by the least-squares method, the required computations being carried out on the Datatron computer at Chalk River. While the emphasis of the analysis is on the nonresonant scattering, the fits have been carried through most of the resonances. The immediate neighborhood of the resonances at 1.746 Mev and 1.803 Mev have been omitted as these are unquestionably not S-wave ones. Below 2.3 Mev the results are reasonably good, although with some qualifications discussed later in detail. Above this they are poor and suggest that a simple potential well is an inadequate description of the situation.

THEORY

A discussion of charged-particle scattering analysis, together with remarks on the number of parameters involved, has been given by Christy.[8] For the case of nitrogen in the energy range considered, reactions do not occur, and it is found that nine parameters are required to include S and P waves with full generality. Of these, seven are involved with the P waves. If these are known to be absent, the formulation in terms of S waves is simple, and fitting by hand calculation relatively easy. That this simplification is inadequate has been noted so that a single phase shift has been introduced to represent the P waves. Three adjustable parameters are thus used, two S-wave phase shifts for the $1/2+$ and $3/2+$ states and the single phase shift for the P waves.

It is of interest to consider the information content of charged particle scattering. When S and P waves are

[1] Hagedorn, Mozer, Webb, Fowler, and Lauritsen, Phys. Rev. **105**, 219 (1957).
[2] Gove, Ferguson, and Sample, Phys. Rev. **93**, 928(A) (1954).
[3] Melkanoff, Nodvik, Saxon, and Woods, Phys. Rev. **106**, 793 (1957). An extensive bibliography is given here.
[4] J. Bowcock, Proc. Phys. Soc. (London) **A70**, 515 (1957).
[5] G. E. Brown and C. T. de Dominicis, Proc. Phys. Soc. (London) **A70**, 668 (1957).
[6] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).
[7] Ferguson, Clarke, and Gove, preceding paper [Phys. Rev. **115**, 1655 (1959)].
[8] R. F. Christy, Physica **22**, 1009 (1956).

included in the most general way, the ratio of the differential cross section to the Rutherford cross section is found to be a linear combination of the seven linearly independent functions of angle $(1-\mu)^2P_0$, $(1-\mu)^2P_1$, $(1-\mu)^2P_2$, M_0, N_0, M_1, and N_1 which are defined below. An identity between the coefficients reduces the number of effectively independent functions to six. Clearly no more than six parameters describing the scattering can be determined from an angular distribution. However, it has been found that the situation is not as favorable as this. Although the six functions are theoretically linearly independent, when the angular range is restricted approximate linear dependencies may arise. Such a situation has been found near 2.3 Mev where it was found that two of the functions could be represented to an accuracy of about 1% by linear combinations of the other four in the range 50° to 150°. Here it is effectively impossible to determine more than four parameters. This degeneracy may be removed by extending the angular range and may disappear at other energies. Its occurrence emphasizes the desirability of measurements over as wide a range of angle as possible.

The theory used is based on the work of Blatt and Biedenharn[9,10] in which the summations over magnetic substates required when polarization is not observed is effected with the aid of Racah coefficients. Under the assumptions described above, the scattering cross section is given by

$$\begin{aligned}d\sigma(\mu)/d\sigma_R(\mu)=&1+2M_0(\sin^2\delta^{1/2+}+2\sin^2\delta^{3/2+})\\&-2N_0(\sin\delta^{1/2+}\cos\delta^{1/2+}+2\sin\delta^{3/2+}\cos\delta^{3/2+})\\&+18M_1\sin^2\phi_1-18N_1\sin\phi_1\cos\phi_1\\&+(1-\mu^2)\eta^{-2}P_0(\mu)(\tfrac{1}{3}\sin^2\delta^{1/2+}+\tfrac{2}{3}\sin^2\delta^{3/2+}\\&+3\sin^2\phi_1)+2(1-\mu^2)\eta^{-2}P_1(\mu)\sin\phi_1[\sin\delta^{1/2+}\\&\times\cos(\phi_1+2\psi_1-\delta^{1/2+})+2\sin\delta^{3/2+}\cos(\phi_1+2\psi_1-\delta^{3/2+})]\\&+6(1-\mu^2)\eta^{-2}P_2(\mu)\sin^2\phi_1. \quad (1)\end{aligned}$$

Here $d\sigma(\mu)/d\sigma_R(\mu)$ is the ratio of the observed differential cross section to the Rutherford cross section, evaluated in the center-of-mass system, μ is the cosine of the angle of scattering, $\delta^{1/2+}$ and $\delta^{3/2+}$ are the phase shifts for the states of spin and parity 1/2+ and 3/2+, respectively, and ϕ_1 is the phase shift for the P waves. $\eta=Ze^2/\hbar v$, Z being the atomic number of the target, e the electronic charge, $\hbar$ Planck's constant, and v the relative velocity of proton and target. $\psi_1=\tan^{-1}\eta$. $P_0(\mu)$, $P_1(\mu)$, and $P_2(\mu)$ are the Legendre polynomials of degree 0, 1, and 2, and M_0, N_0, M_1, and N_1 are functions[11] of angle and energy given by

$$M_0=\frac{(1-\mu)P_0(\mu)}{3\eta}\sin\left[\eta\ln\frac{(1-\mu)}{2}\right],$$

$$N_0=\frac{(1-\mu)P_0(\mu)}{3\eta}\cos\left[\eta\ln\frac{(1-\mu)}{2}\right],$$

$$M_1=\frac{(1-\mu)P_1(\mu)}{3\eta}\sin\left[\eta\ln\frac{(1-\mu)}{2}+2\psi_1\right],$$

$$N_1=\frac{(1-\mu)P_1(\mu)}{3\eta}\cos\left[\eta\ln\frac{(1-\mu)}{2}+2\psi_1\right].$$

The fitted parameters are $\delta^{1/2+}$, $\delta^{3/2+}$, and ϕ_1. The fitting has been done by the least-squares procedure. Since the phase shifts occur nonlinearly, an iterative procedure is required. The standard linearizing method[12] was set up initially, but this procedure was found not to converge on most of the distributions. Convergence was secured by computing a correction to the normal matrix involving the second derivatives of the theoretical cross sections with respect to the unknown phases. It was found that for those cases that required the correction term, the convergence was to a point representing a singularity of the uncorrected matrix. This clearly is the reason for the failure of convergence in these cases, and it is believed that it arises from the fact that an adequate fit is not available. This point will be discussed subsequently.

The significant correlations which exist between the errors in the points of any one distribution due to the uncertainty of normalization have been ignored. To accommodate these would be a substantial complication in the routine and the results would probably not be significantly altered. Two solutions for the phase shifts are ordinarily expected; in the cases of the singular normal matrix noted above, the two solutions degenerate into one with $\delta^{1/2+}=\delta^{3/2+}$.

$\delta^{1/2+}$, $\delta^{3/2+}$, and ϕ_1 are given as functions of E in Figs. 1, 2, and 3. When the two solutions for $\delta^{1/2+}$ and $\delta^{3/2+}$ are distinct, the preferred one is shown as a dot and the second as a cross. The remaining points where the solutions are coincident are shown as dots. Typical angular distributions calculated from these phase shifts are indicated by the solid lines in Figs. 6 and 7 of the preceding paper.[7] No points are shown at the sharp resonances at 1.067 Mev, 1.746 Mev, 1.803 Mev, 2.343 Mev, and 2.458 Mev. At the S-wave resonance at 1.067 Mev the measurements have been distorted by inadequate energy resolution and the phases were not

[9] J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952).

[10] A. J. Ferguson, Atomic Energy of Canada Limited Report No. 157 (unpublished). This extends the charged-particle scattering formulas of Blatt and Biedenharn to the case where a general phase-shift representation is used for the nuclear wave functions. Complicated cases such as those where resonances overlap may be treated in this formulation. The formulas contain some small errors.

[11] These differ slightly from the similar functions given in reference 10.

[12] See, for example, E. T. Whittaker and G. Robinson, *Calculus of Observations* (Blackie and Sons, Ltd., London, 1932), Chap. IX, Sec. 109.

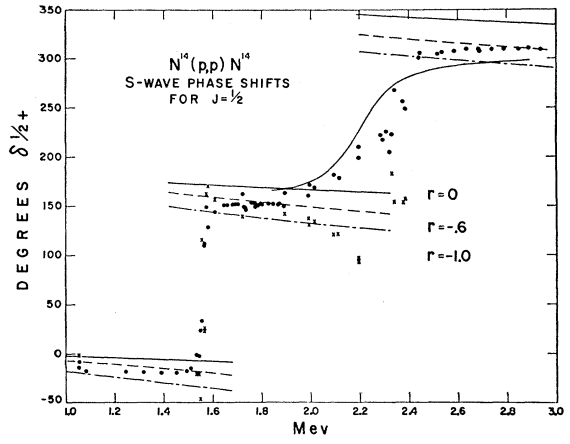

FIG. 1, Energy dependence of $\delta^{1/2+}$. The crosses are alternative solutions. The series of solid and dashed lines labelled $r=0$, -0.6 and -1.0 represent lines of constant logarithmic derivatives. The solid line extending from 1.8 Mev to 2.9 Mev is the phase shift used to fit the broad resonance at 2.3 Mev.

meaningful. The four higher ones are believed not to be S-wave resonances.

The connection between the causality principle and the resonance formulas is discussed by Wigner.[13] This imposes a lower limit on the energy derivative of the phase shift which is governed by the size of the scattering system. Wigner gives for the derivative of this phase shift relative to the wave number, k,

$$\dot{\delta}=\frac{1}{2i}\left[\frac{\dot{I}-\dot{I}'R}{I-I'R}-\frac{\dot{I}^x-\dot{I}'^xR}{I^x-I'^xR}\right]+\frac{\dot{R}}{|I-I'R|^2}$$
$$=\dot{\delta}_0+\frac{\dot{R}}{|I-I'R|^2}. \qquad (2)$$

Here I is the radial factor of the wave function for an incoming particle and R is the reciprocal of the logarithmic derivative. The dot signifies differentiation with respect to k, and the prime signifies differentiation with respect to the radial coordinate. The asterisk indicates the complex conjugate. $\dot{\delta}_0$ is the k derivative of δ for R independent of k, i.e., independent of energy. The essence of the development is that $\dot{R}>0$, which immediately gives $\dot{\delta}>\dot{\delta}_0$. The limitation imposed by causality can thus be conveniently displayed by plotting families of curves representing $R=$const on a graph of δ *versus* energy. The slope of the curve of δ cannot be less than that of any member of the family which it intersects. The application of this principle to charged particle scattering demands only that we use charged particle wave functions[14,15] for computing the family of curves.

These curves are indicated in Figs. 1, 2, and 3. In Figs. 1 and 2 for the two S-wave phase shifts are given the curves for $r=0$, -0.6 and -1. In Fig. 3 for the

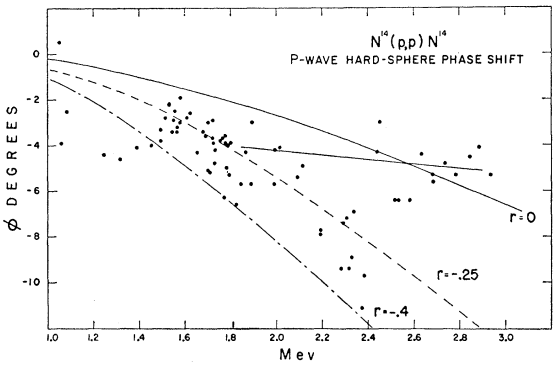

FIG. 3. Energy dependence of ϕ_1. The solid and dashed curves indicate lines of constant logarithmic derivatives as in Fig. 1. The solid line running from 1.8 Mev to 2.9 Mev is the P-wave phase shift used in fitting the 2.32-Mev resonance.

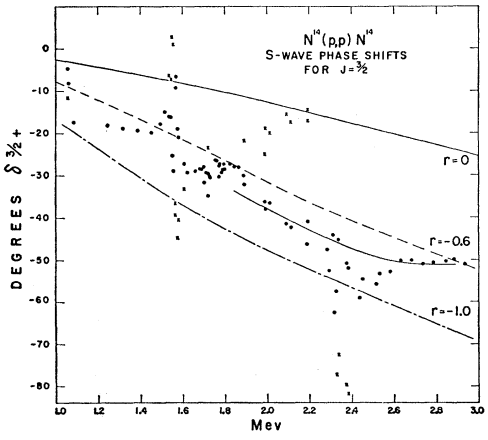

FIG. 2. Energy dependence of $\delta^{3/2+}$. The solid and dashed curves indicate lines of constant logarithmic derivatives as in Fig. 1. The solid line running from 1.8 Mev to 2.9 Mev shows the S-wave hard-sphere phase shifts used in fitting the 2.32-Mev resonance.

P-wave phase shift, the curves are for $r=0$, -0.25 and -0.4. Here $r=R/a$ where a is the nuclear radius and has been taken as $a=1.5A^{1/3}\times10^{-13}$ cm$=3.615\times10^{-13}$ cm. The curves given are adequate to display departures from the condition $\dot{\delta}>\dot{\delta}_0$. $r=0$ corresponds to the vanishing of the wave function at the nuclear surface and this curve is the ordinary hard sphere phase shift. In the Wigner-Eisenbud theory R is given by

$$R=\sum_\lambda \frac{\gamma_\lambda\times\gamma_\lambda}{E_\lambda-E}, \qquad (3)$$

so that if $R=\infty$, then clearly $E=E_\lambda$, the energy of one of the eigenstates of the nucleus. Thus at each resonance the graph of the phase shift crosses that for $R=\infty$, which is not shown, but which lies somewhat above

[13] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[14] Bloch, Hull, Broyles, Bouricius, Freeman, and Breit, Revs. Modern Phys. **23**, 147 (1951).

[15] Sharp, Gove, and Paul, Atomic Energy of Canada Limited, Report TPI-70 (unpublished).

the one for $r=0$. Cross sections are periodic in the phase shift with a period π and the families $R=$ const. in the phase shift graphs repeat with this period.

DISCUSSION OF THE PHASE SHIFTS

Below 2.3 Mev the fits to the angular distributions are moderately good, with the reservations discussed below in more detail. Above this they are consistently poor. It may thus be assumed that the phase shifts represent the gross aspects of the lower energy scattering. Inasmuch as they are not materially different from the background phases used in earlier attempts to fit the 1.746-Mev resonance, the difficulty that this resonance has presented is somewhat surprising. Regarding the region from 2.3 Mev to 3.0 Mev, the D-wave hard-sphere phases are rather small, and it seems likely that the observed discrepancies result from more complicated behavior of the P-waves.

An unexpected result is that for all but one of the fits in the regions of energy away from resonances, $\delta^{1/2+}$ and $\delta^{3/2+}$ are identical, or equivalently differ by integral multiples of π. This would arise from hard-sphere scattering, or from scattering by a simple potential well. However two aspects of these results indicate that in general they are not physically significant. These are the only solutions obtained in regions where the fit is conspicuously bad, e.g., above 2.5 Mev, and where the formula is clearly inadequate to represent the scattering. Secondly, the two parameters are statistically completely correlated. This would ordinarily imply that the normal matrix was singular and would prevent any solution being found. In the present case the second derivative corrections mentioned above have made this solution possible. It is easily shown that it is always present as a formal solution to the equations, although in many cases it represents a saddle point and not a true minimum of the fit. In this regard it is interesting to note that without the second derivative corrections, the normal matrix is positive definite and does not admit saddle points.

The nature of these results can be illuminated by introducing the transformation

$$A=\sin^2\delta^{1/2+}+2\sin^2\delta^{3/2+},$$

$$B=\sin\delta^{1/2+}\cos\delta^{1/2+}+2\sin\delta^{3/2+}\cos\delta^{3/2+}.$$

$\delta^{1/2+}$ and $\delta^{3/2+}$ are readily eliminated from Eq. (1) by this transformation. In terms of A and B, the cross section is much less restricted than in terms of the δ's if only real values are allowed. Real $\delta^{1/2+}$ and $\delta^{3/2+}$ limits A and B to an annular region in the A,B plane, the outer edge of which is the locus for $\delta^{1/2+}=\delta^{3/2+}$. Equation (1) is not valid for complex phase shifts, so that values of A and B outside this region are not admissible. A least-squares fit with the parameters A, B, and ϕ_1 converges without the introduction of second derivative corrections. A number of such fits to typical distributions have shown that where a nondegenerate solution was previously obtained, the identical result was found, while when only the degenerate solution was obtained, a better fit was found which lay outside the region of real δ's. In the latter cases the degenerate solution corresponded to a nearby point on the edge of the real region. It is thus to be concluded that essentially all of the nonresonant distributions correspond to points outside the region of real phase shifts. This implies a deficiency in the assumptions underlying Eq. (1). However, the fact that many of the degenerate fits obtained are quite good ones shows that in these cases the deficiency is a small one.

Between 1.2 Mev and 1.4 Mev systematic differences between the experimental points and the best theoretical curve are apparent. Degenerate phase shifts are also obtained in this region. At this low energy the presence of D-waves or higher ones is extremely unlikely so that these anomalies must reflect more complicated behavior of the P-waves. The most probable interpretation is that channel spin mixing, i.e., transitions from one channel spin to the other due to off-diagonal elements of the S-matrix, is occurring in the 1/2- and 3/2-states. While this has not been studied specifically for the nonresonant case, studies of P-wave resonances with a strong channel spin mixing have produced angular distributions with shapes characteristic of those obtained in this region. Coupling of the spin of the incident proton to its orbital motion or to the angular momenta of the target nucleus will produce such effects.

Nondegenerate solutions for the S-wave phases have appeared at the 1/2+ resonance at 1.557 Mev and in the region from 1.9 Mev to 2.4 Mev. At 1.557 Mev, $\delta^{1/2+}$ follows the course expected for this resonance. However, the energy dependences of both $\delta^{1/2+}$ and $\delta^{3/2+}$ are distorted here. For $\delta^{3/2+}$ a dispersion-like anomaly is present where a smooth variation is expected. The energy dependence of $\delta^{1/2+}$ implies a width of 13 kev whereas a fit made directly to the cross sections using the one-level formula gives 34 kev.[1] It may be remarked that a phase-shift analysis like the present one will not give good values for widths unless all of the parameters show reasonable behavior. These anomalies are attributable to inadequacies of the formula such as appear in the nonresonant regions. From 2.0 Mev to 2.4 Mev the phases give a strong indication of a broad 1/2+ resonance which will be discussed fully in the following section.

BROAD RESONANCE AT 2.3 MEV

Between 1.9 Mev and 2.4 Mev the two solutions for $\delta^{1/2+}$ and $\delta^{3/2+}$ are distinct. The phase shifts from one solution follows a course which would be expected for a 1/2+ resonance, while those from the other give an energy dependence for $\delta^{1/2+}$ that is incompatible with the causality condition so that it can be discarded. For the acceptable solution $\delta^{3/2+}$ and ϕ_1 show a reasonably smooth behavior except at points where they are

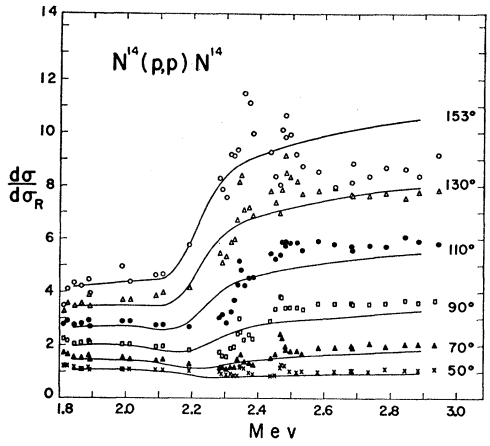

FIG. 4. Experimental and calculated yield curves at the 2.32-Mev resonance at six different angles.

perturbed by the narrow resonances at 2.344 Mev and 2.468 Mev.

In order to interpret the scattering near 2.3 Mev as a 1/2+ resonance, it is necessary to show that the scattering cross sections are reasonably well approximated by smoothly varying potential phase shifts plus a resonance phase shift for the 1/2+ state. In smoothing the background phases we must expect the fits to be impaired since this operation demands departure from the best least-squares values. The result will be satisfactory if the impairment is slight. If the smoothed phases represent the average in a small energy interval, then the resulting cross sections will be the average cross sections over that interval.

$\delta^{1/2+}$ is given by the formula[16]

$$\tan\delta^{1/2+}=\frac{\frac{1}{2}\Gamma}{E_0-E}+R_0 \qquad (4)$$

where the first term on the right-hand side gives the resonance contributions and R_0 gives the nonresonant background. We assume that R_0 is obtained from $\delta^{3/2+}$, i.e.,

$$\tan\delta^{3/2+}=R_0. \qquad (5)$$

[16] This equation applies properly to the nuclear phase shift, $\delta^{J\pi}-\phi_0$, in our notation, where ϕ_0 is the hard-sphere phase shift. The correction of this inaccuracy will change the curves only slightly, and as a formal fit is not being attempted it has been omitted.

The width Γ is given by

$$\Gamma=2P\gamma^2,$$

where P is the penetrability and γ the reduced width of the level.

$\delta^{1/2+}$ has been calculated from Eqs. (4) and (5), using for $\delta^{3/2+}$ and ϕ_1 the solid lines in Figs. 2 and 3 which run from 1.8 Mev to 2.9 Mev. The resonant energy and width are $E_0=2.32$ Mev and $\Gamma=0.55$ Mev. The energy dependence of Γ through the penetrability has been included, and the above value is the width at E_0. This choice of resonant energy and width has been made to force a fairly good fit to angular distributions at 1.99 Mev and 2.44 Mev. The solid line of Fig. 1 running from 1.8 Mev to 2.8 Mev shows $\delta^{1/2+}$ as calculated in this way.

$d\sigma/d\sigma_R$ computed from these phase shifts for six different angles are shown in Fig. 4, for the energy range 1.8 Mev to 2.9 Mev. The solid lines represent the computed results and the various circles, triangles, etc., the experimental values. The cross section is fairly well represented for the largest angles up to 2.4 Mev. For angles less than 120°, the fit becomes poor at about 2.2 Mev. Above 2.4 Mev neither the resonance fit nor the unrestricted one is adequate. Better agreement at the smaller angles can be obtained by increasing E_0 and Γ, but this makes the fit at higher energies even worse.

The $N^{14}(p,\gamma)O^{15}$ reaction shows a resonance-like behavior in this region that has been interpreted by Duncan and Perry[17] in terms of S-wave resonance at 2.6 Mev with a width of 1.25 Mev. The resonant energy is somewhat higher and the width substantially greater than found from the scattering experiments. This disparity may be due to the presence of an appreciable nonresonant contribution to the (p,γ) cross section whose effect would be to increase the apparent width of the resonance if not explicitly included.

ACKNOWLEDGMENTS

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[17] D. B. Duncan and J. E. Perry, Phys. Rev. 82, 809 (1957).