Quantum-Mechanical Three-Body Problem*

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We treat the quantum-mechanical problem of three spinless particles, with the boundary condition that the logarithmic derivative of the wave function be a prescribed constant at each of the three boundaries $|\mathbf{r}_1-\mathbf{r}_2|=a$, $|\mathbf{r}_2-\mathbf{r}_3|=a$, $|\mathbf{r}_1-\mathbf{r}_3|=a$. This boundary condition is discussed; it is roughly equivalent to an interparticle potential which consists of a hard core plus a strong short-range attractive part. The eigenfunctions and eigenvalues of the system are given by the solutions of an infinite set of coupled homogeneous integral equations. The equations involve partial wave expansions in the interparticle distances and can often be truncated with good approximation by taking only a finite number of partial waves. We discuss the solution of these equations for the ground state of the system, taking relative S-waves only, for which case the infinite set of equations reduces to a single integral equation in one variable.

I. INTRODUCTION

IT goes without saying that we would like to solve the following problem. Three (or more) particles with arbitrary spin, statistics, and interparticle forces interact. What are the eigenfunctions and eigenvalues of the possible bound states of the system? It also goes without saying that the problem in this generality is very difficult and that we must begin by simplifying it. In this paper we present a method for treating a simplified version of this problem, but a version which retains much of the essential difficulty inherent in the fact that it is a three-body problem. The method seems generalizable to more than three bodies.

We begin by making the simplifying assumptions that all the particles are identical and spinless, and by neglecting statistics, in the sense that we make no special requirements on the symmetry or antisymmetry of the wave function. The method appears applicable, but is of course more complicated, if these assumptions are not made. There is a more important simplification that we make. We would like at least to be able to solve the problem for central forces, i.e., when the potential energy, as a function of the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ of the particles, is of the form

$$V_{12}(r_{12})+V_{13}(r_{13})+V_{23}(r_{23}), \tag{1}$$

where the V_{ij} are arbitrary functions and $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$. The simplification in the problem we discuss consists of replacing these potentials by a *boundary condition* on the wave function at each of the three boundaries

$$r_{12}=a, \quad r_{23}=a, \quad r_{13}=a. \tag{2}$$

We use boundary conditions instead of potentials of the form (1) mainly because the boundary conditions are easier to handle mathematically. We do not pretend that they correspond to any real physical problem, although they have been extensively used in nuclear physics.[1] From our point of view they are a mathematical artifice, a stepping stone which enables us to take a fairly long step toward the solution of more complicated, and more realistic[2] three-body problems.

If Ψ is the total wave function of the system, the boundary condition that we use is

$$(1/\Psi)(\partial\Psi/\partial n)=-\gamma, \tag{3}$$

at each of the boundaries in (2). Here γ is some constant, the same for all boundaries and $\partial/\partial n$ is just $\partial/\partial r_{ij}$ for the boundary ij. Although this condition is mainly a mathematical device it does correspond, in an imprecise way, to the following interparticle potential: an infinite repulsive core of radius a, surrounded by a deep attractive well of very short range. We elaborate on this now.

Suppose we have a particle bound to a spherically symmetric potential which is of arbitrary shape except that it has a finite radius r_0: $V=0$ for $r>r_0$. Let the particle be in the ground state with binding energy E. The logarithmic derivative of the internal ($r<r_0$) and external ($r>r_0$) wave functions are then equal to each other at $r=r_0$. Given the potential we could, at least in principle, always calculate the value of this logarithmic derivative. Conversely, if one were given *only* the logarithmic derivative of the wave function at $r=r_0$ and were completely ignorant of the form of the potential for $r<r_0$, one could reconstruct at least the external part of the wave function since the general form of the external solution is known and only the constants in it need be determined. But we would know nothing about the wave function for $r<r_0$, since we could not reconstruct the interior solution from a knowledge *only* of the logarithmic derivative at $r=r_0$.

We can, however, imagine a kind of degenerate potential for which the knowledge of the logarithmic derivative at a certain radius defines the wave function everywhere. Suppose the potential V is defined by $V=+\infty$ for $r<a$, $V=-V_0$ for $a<r<a+d$, $V=0$ for

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[1] H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956) and the reference cited therein.

[2] For a solution of a three-body scattering problem involving potentials of infinitely short range see G. V. Skorniakov and K. A. TerMartirosian, J. Exptl. Theoret. Phys. U.S.S.R. **31**, 775 (1956) [translation: Soviet Phys. JETP 4, 648 (1957)].

$r>a+d$. Then it is well known that there is a multiplicity of wells, i.e., different values of V_0 and d, that will give a binding energy E for the ground state, only the quantity V_0d^2 being determined by the energy. For $r>a+d$, all these wells have the same wave function; they differ only in the region $a<r<a+d$; all the solutions vanish of course for $r<a$. Now if we imagine d to diminish and V_0 to increase in such a way that V_0d^2 is constant, we get in the limit a degenerate well for which there is no interior and for which the exterior solution extends down to $r=a$. In this limit then, and for the ground state, the degenerate potential is equivalent to a boundary condition at $r=a$, for either the one or the other uniquely defines the wave function over all space.

The discussion above deals with a particle in the ground state. What makes boundary conditions never exactly equivalent to potentials, even to degenerate ones, is that the logarithmic derivative of the wave function, evaluated at whatever radius, depends on just which state we treat. Thus, even if we limit ourselves to exterior solutions for nondegenerate potentials, we cannot reproduce these solutions for *all* states of any true potential by an energy-*independent* boundary condition. This kind of boundary condition must, as we have said before, be considered mainly as a mathematical device for uniquely defining a boundary value problem which is, however, very suggestive for the three-body problem involving true potentials.

For later comparison with the three-body problem we write the relation between γ and the energy for a bound state of *two* particles whose center of mass is at rest, with the boundary condition (3) at $r=a$, where r is the interparticle distance. The solution for $r>a$ of the Schrödinger equation is e^{-Kr}/r, where $K^2=m|E|/\hbar^2$, $E=-|E|$, and m is the mass of each particle. The above boundary condition then gives $Ka=\gamma a-1$. There is a bound state only for $\gamma a>1$.

Perhaps we should make clear from the outset what is the nature of the solution we shall develop. The wave function Ψ is a function of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. It is defined either as a solution of a 9-dimensional Schrödinger equation or of the corresponding integral equation. This integral equation as it first appears is not in a form which is practical for numerical computation. Our treatment of the problem consists of deriving an equivalent infinite set of coupled integral equations, which can be truncated with good approximation and which are practical for numerical computation. The "solution" thus consists of replacing an intractable integral equation by a set of tractable ones.

II. THE BASIC IDEA

With the assumptions discussed above, we have a nonseparable boundary value problem involving a partial differential equation in 6 variables (after separating off the center-of-mass motion). It is of a special kind in that it is nonseparable, not because the individual boundaries are nonseparable, but because there is more than one such boundary. We have discussed elsewhere[3] problems which are mathematically similar in three dimensions, e.g., the scattering of a wave from a number of scatterers or the problem of a particle bound to more than one spherical potential well. We found that these problems could be handled if one began by writing general solutions to the wave equation in a special way. This way was as follows: For *each potential or boundary*, one wrote a separate solution of the wave equation in *coordinates appropriate to that potential or boundary*, and took the total wave function to be the sum of such solutions. Then boundary or matching conditions at any given potential, say the ith, were satisfied by the use of formulas for expressing all the other solutions except the ith in the ith coordinate system. This form of the solution was motivated by looking at the integral equation formulation of the problem; it is discussed in reference (3).

We use the same point of view for the three-body problem. Here the boundaries of Eq. (2) are of course "surfaces" in the nine-dimensional space of $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, but this is irrelevant for the application of the general idea. We write the wave function as a sum of solutions of the wave equation, each solution appropriate to one of the boundaries. Then for satisfying boundary conditions at a given one of the three boundaries we use transformation formulas which express the solutions appropriate to the other boundaries in terms of the coordinates of that boundary.

Natural sets of coordinates appropriate to the boundaries (2) are [in addition to the center of mass $\mathbf{R}=\frac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)$]

$$\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2 \quad \text{and} \quad \boldsymbol{\varrho}_3=\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2), \tag{4a}$$

or

$$\mathbf{r}_{13}=\mathbf{r}_1-\mathbf{r}_3 \quad \text{and} \quad \boldsymbol{\varrho}_2=\mathbf{r}_2-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_3), \tag{4b}$$

or

$$\mathbf{r}_{23}=\mathbf{r}_2-\mathbf{r}_3 \quad \text{and} \quad \boldsymbol{\varrho}_1=\mathbf{r}_1-\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3). \tag{4c}$$

These sets of coordinates are of course not independent. The linear relations among them are of the form

$$\begin{aligned} \mathbf{r}_{13}&=+\boldsymbol{\varrho}_1+\tfrac{1}{2}\mathbf{r}_{23}, \quad & \mathbf{r}_{12}&=+\boldsymbol{\varrho}_1-\tfrac{1}{2}\mathbf{r}_{23}, \\ \boldsymbol{\varrho}_2&=+\tfrac{3}{4}\mathbf{r}_{23}-\tfrac{1}{2}\boldsymbol{\varrho}_1, \quad & \boldsymbol{\varrho}_3&=-\tfrac{3}{4}\mathbf{r}_{23}-\tfrac{1}{2}\boldsymbol{\varrho}_1, \end{aligned} \tag{5}$$

and so on. The Schrödinger equation, with the center-of-mass part split off has, of course, the same form in all these three pairs of coordinates. For example, in coordinates $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$ it is, for particles of mass m,

$$(\nabla_{r_{23}}^2+\tfrac{3}{4}\nabla_{\rho_1}^2)\psi=K^2\psi, \tag{6}$$

where for a bound state we have set $E=-|E|$ and $K^2=m|E|/\hbar^2$. Note that there is no factor of two in the definition of K^2.

Now we write a general solution to the wave Eq. (6). This equation has elementary separable solutions which

[3] L. Eyges, Ann. Phys. (N. Y.) **2**, 101 (1957); Phys. Rev. **111**, 683 (1958).

are functions of $\mathbf{r}_{23}$ times functions of $\boldsymbol{\varrho}_1$. For satisfying boundary conditions at $r_{23}=a$, it is clearly appropriate to imagine $\nabla_{r_{23}}^2$ expressed in spherical coordinates r_{23}, θ_{23}, φ_{23}, and to work with solutions which are spherical Hankel functions of r_{23} times spherical harmonics in Ω_{23}, (where Ω_{23} stands for θ_{23}, φ_{23}). Since there is no boundary condition on $\boldsymbol{\varrho}_1$ we simply try exponential functions of $\boldsymbol{\varrho}_1$ for that part of Eq. (6) which involves $\nabla_{\rho_1}^2$. Thus we are led to try as an elementary solution[4] of (6)

$$h_l(ikr_{23})Y_{lm}(\Omega_{23})\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\varrho}_1), \tag{7}$$

which *is* a solution if

$$k^2=K^2+\tfrac{3}{4}\kappa^2. \tag{8}$$

The elementary solutions (7) are almost, but not quite, of the correct form. To see what changes must be made in them let us reconsider the two-body problem for a moment. There we found that the radial wave functions for relative motion were $h_l(ikr)$ for $r>a$ and, because of the hard core of the potential, were zero for $r<a$. Formally, we can express this by saying that these wave functions were $\bar{h}(ikr)$ where

$$\begin{aligned}\bar{h}_l(ikr)&=0, \quad r<a\\ &=h_l(ikr), \quad r>a.\end{aligned} \tag{9}$$

The difference between the functions h_l and $\bar{h}_l$ really becomes important only if we take the Fourier transform of the wave functions. This was not necessary to solve the two-body problem. But for the three-body problem, as we shall see, it is essential to the analysis to Fourier-analyze the wave function and it does make a profound difference whether or not we use h_l or $\bar{h}_l$ to build up the elementary solutions. In fact, using the former functions, one is led to inconsistent integral equations. Using $\bar{h}_l$ on the other hand, one is led to a consistent set of equations with reasonable solutions, which is a pragmatic, if not mathematical proof for the necessity of using them. To state it differently, we assume that the boundary value problem we define has a *unique* solution and that sufficient justification for using the $\bar{h}_l$ is that with them we shall be led to a function that satisfies the wave equation and the boundary conditions, namely to *a* solution.

To continue then, we use the functions $\bar{h}_l$ to form the elementary solutions

$$\bar{h}_l(ikr_{23})Y_{lm}(\Omega_{23})\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\varrho}_1). \tag{10}$$

Then we build up a general solution to Eq. (6), for $r_{23}>a$, by multiplying (10) by arbitrary coefficients and summing (and integrating) over the complete set of functions of which (10) is a member. That is, we multiply (10) by an arbitrary function $g_{lm}^{(23)}(\boldsymbol{\kappa})$, sum over l and m, and integrate over $\boldsymbol{\kappa}$. We call the resulting function ψ_{23}.

$$\psi_{23}=\sum_{l,m}\int g_{lm}^{(23)}(\boldsymbol{\kappa})\bar{h}_l(ikr_{23}) \times Y_{lm}(\Omega_{23})\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\varrho}_1)d\boldsymbol{\kappa}, \tag{11a}$$

ψ_{23} is a solution of the wave equation, provided Eq. (8) is satisfied. In accordance with what we have said before, we write similar solutions in $\mathbf{r}_{12}$ and $\boldsymbol{\varrho}_3$ coordinates, which solution we call ψ_{12}, and also in $\mathbf{r}_{13}$ and $\boldsymbol{\varrho}_2$ coordinates, which solution we call ψ_{13}.

$$\psi_{13}=\sum_{l'm'}\int g_{l'm'}^{(13)}(\boldsymbol{\sigma})\bar{h}_{l'}(ikr_{13}) \times Y_{l'm'}(\Omega_{13})\exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_2)d\boldsymbol{\sigma}, \tag{11b}$$

$$\psi_{12}=\sum_{l'm'}\int g_{l'm'}^{(12)}(\boldsymbol{\sigma})\bar{h}_{l'}(ikr_{12}) \times Y_{l'm'}(\Omega_{12})\exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_3)d\boldsymbol{\sigma}. \tag{11c}$$

In these last two expressions we have for later convenience called the integration variable $\boldsymbol{\sigma}$ and the summation variables l' and m'. Of course, $k^2=K^2+\frac{3}{4}\sigma^2$. With these definitions the total wave function ψ is taken to be

$$\Psi=\psi_{12}+\psi_{13}+\psi_{23}, \tag{12}$$

and it is this Ψ which we shall require to satisfy the boundary condition Eq. (3), at *each* of the boundaries (2).

III. ANGULAR MOMENTUM CONSIDERATIONS

For the boundary conditions (3) the Schrödinger equation has solutions which are not only eigenfunctions of the energy but of the total angular momentum (and its z component). It is then clear that the forms of the $g_{lm}^{(ij)}(\boldsymbol{\kappa})$ are not completely arbitrary but are limited in some way special to each angular momentum state. We discuss the form of this limitation in this section.

The usual operators for the components of angular momentum in the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are, e.g.,

$$L_x=-i\hbar\sum_{i=1}^{3}\left(y_i\frac{\partial}{\partial z_i}-z_i\frac{\partial}{\partial y_i}\right),$$

with similar forms for the y and z components. We can express this operator in any of the three sets of coordinates (4a), (4b), and (4c). Let the components of $\mathbf{r}_{23}$ be (x_{23},y_{23},z_{23}) and of $\boldsymbol{\varrho}_1$ be (ξ_1,η_1,ζ_1), with similar notation for the other pairs $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ and $\mathbf{r}_{13}$, $\boldsymbol{\varrho}_2$. The operator for the x component of angular momentum in $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$ coordinates is then

$$\frac{L_x}{-i\hbar}=Y\frac{\partial}{\partial Z}-Z\frac{\partial}{\partial Y}+\eta_1\frac{\partial}{\partial\zeta_1}-\zeta_1\frac{\partial}{\partial\eta_1}+y_{23}\frac{\partial}{\partial z_{23}}-z_{23}\frac{\partial}{\partial y_{23}},$$

with similar expressions for L_y and L_z *and* similar expressions in the other two sets of variables. Since none

[4] We use ikr_{23} instead of kr_{23} as the variable of the Hankel functions in anticipation of the fact that we shall be dealing with wave functions involving decaying exponentials, since we have to do with bound states, and the spherical Hankel functions of imaginary argument are of just that kind.

of the wave functions (11a), (11b), and (11c) depends on $\mathbf{R}$, we see from the form of the commutation relations that the angular momentum in $\mathbf{r}_{23}$, $\boldsymbol{\rho}_1$ coordinates can be looked upon as being that due to the vector composition of angular momentum carried by the "r_{23} particle" and that carried by the "ρ_1 particle." Now the angular momentum properties of the "r_{23} particle" are given by its spherical harmonic expansion, which is explicitly exhibited in (11a). To investigate the angular momentum properties of the "ρ_1 particle," we expand $g_{lm}^{(23)}(\boldsymbol{\kappa})$ in spherical harmonics of Ω_κ, where Ω_κ stands for the angle variables in spherical coordinates in $\boldsymbol{\kappa}$ space.

$$g_{lm}^{(23)}(\boldsymbol{\kappa})=\sum_{l_1=0}^{\infty}\sum_{m_1=-l_1}^{l_1} G_{lml_1m_1}^{(23)}(\kappa)Y_{l_1m_1}(\Omega_\kappa). \quad (13)$$

If we put this into (11a) we exhibit the wave function Ψ_{23} as a sum of products of spherical harmonics, one with Ω_{23} as variable and the other with Ω_κ as variable. By taking the appropriate combinations of these products, according to the vector coupling model, and the known Clebsch-Gordan coefficients, we can make the wave function (11a) an eigenfunction of any L^2 and L_z we choose. Thus we have, using

$$\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}_1)=4\pi\sum_{l_2m_2} i^{l_2}j_{l_2}(\kappa\rho_1)Y_{l_2m_2}(\Omega_{\rho_1})Y_{l_2m_2}{}^*(\Omega_\kappa)$$

and

$$d\boldsymbol{\kappa}=\kappa^2 d\Omega_\kappa d\kappa,$$

$$\psi_{23}=4\pi\sum_{lml_1m_1l_2m_2}\int G_{lml_1m_1}^{(23)}(\kappa)Y_{l_1m_1}(\Omega_\kappa)\bar{h}_l(ikr_{23})$$
$$\times Y_{lm}(\Omega_{23})i^{l_2}j_{l_2}(\kappa\rho_1)Y_{l_2m_2}(\Omega_{\rho_1})Y_{l_2m_2}{}^*(\Omega_\kappa)\kappa^2 d\Omega_\kappa d\kappa.$$

We do the integrals over $d\Omega_\kappa$ to get

$$\psi_{23}=4\pi\sum_{lml_1m_1}\int G_{lml_1m_1}^{(23)}(\kappa)\bar{h}_l(ikr_{23})$$
$$\times Y_{lm}(\Omega_{23})i^{l_1}j_{l_1}(\kappa\rho_1)Y_{l_1m_1}(\Omega_{\rho_1})\kappa^2 d\kappa.$$

Now suppose we want ψ_{23} to be an eigenfunction of a given L and $M=L_z$. We can effect this by writing

$$G_{lml_1m_1}^{(23)}(\kappa)=F_{ll_1}^{(23)}(\kappa)C_{ll_1}(L,M;m,m_1),$$

where the $C_{ll_1}(L,M;m,m_1)$ are the Clebsch-Gordan coefficients, in the notation of Blatt and Weisskopf.[5] For then we have

$$\psi_{23}=4\pi\sum_{l=0}^{\infty}\sum_{l_1=0}^{\infty}\int F_{ll_1}^{(23)}(\kappa)\bar{h}_l(ikr_{23})i^{l_1}j_{l_1}(\kappa\rho_1)$$
$$\times\Big(\sum_{m=-l}^{l}\sum_{m_1=-l_1}^{l_1} C_{ll_1}(L,M;m,m_1)Y_{lm}(\Omega_{23})Y_{l_1m_1}(\Omega_{\rho_1})\Big)\kappa^2 d\kappa.$$

[5] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

This clearly exhibits Ψ_{23} as an eigenfunction of L and M and it is plausible, though unproven that this is the most general way we can solve the problem.

We write similar expressions for ψ_{12} and ψ_{13}, using arbitrary functions $F_{ll_1}^{(12)}(\kappa)$ and $F_{ll_1}^{(13)}(\kappa)$ and the wave function Ψ defined by Eq. (12) will then be an eigenfunction of L and M.

Let us consider the case $L=0$ in a little more detail. For this case the Clebsch-Gordan coefficients vanish unless $l=l_1$, $m=-m_1$; moreover, those that do not vanish are independent of m and in fact are just equal to unity except for a factor $(-)^m$. If we use these facts we get for Ψ_{23}

$$\psi_{23}=4\pi\sum_{l=0}^{\infty}\int i^l\bar{h}_l(ikr_{23})F_{ll}^{(23)}(\kappa)j_l(\kappa\rho_1)$$
$$\times\Big(\sum_{m=-l}^{l}Y_{lm}(\Omega_{23})Y_{lm}{}^*(\Omega_{\rho_1})\Big)\kappa^2 d\kappa \quad (14)$$
$$=\sum_{l=0}^{\infty}\int i^l\bar{h}_l(ikr_{23})F_{ll}^{(23)}(\kappa)j_l(\kappa\rho_1)$$
$$\times(2l+1)P_l(\cos\gamma)\kappa^2 d\kappa,$$

where γ is the angle between $\mathbf{r}_{23}$ and $\boldsymbol{\rho}_1$. It is clear from first principles that this form of the wave function is correct. For $L=0$ the wave function can depend only on the scalars formed from $\mathbf{r}_{23}$ and $\boldsymbol{\rho}_1$, that is, it can depend only on $|\boldsymbol{\rho}_1|$, $|\mathbf{r}_1|$ and $\mathbf{r}_1\cdot\boldsymbol{\rho}_1$. The function (13) is a general function of just that form.

One can make a similar analysis for other L and M, of course, using the appropriate Clebsch-Gordan coefficients. We shall not do this here since this problem in any case is rather academic. For practical problems, one has to deal with the symmetry requirements involved in taking spin and statistics into account. Rather, in this paper we shall take the case $L^2=0$ as leading to integral equations which are more or less representative, and discuss them in some detail. We turn to this now.

IV. THE COUPLED INTEGRAL EQUATIONS: $L^2=0$

The discussion of the last section can be summarized by saying that for $L^2=0$, the $g_{lm}^{(ij)}(\boldsymbol{\kappa})$ that appear in Eqs. (11) must be of the form

$$g_{lm}^{(ij)}(\boldsymbol{\kappa})=F_{ll}^{(ij)}(\kappa)Y_{lm}{}^*(\Omega_\kappa), \quad (15)$$

where $F_{ll}^{(ij)}(\kappa)$ is an arbitrary function of κ. This result came about because the summation in Eq. (13) reduced in this case to a single term. We now wish to determine the functions $F_{ll}^{(ij)}(\kappa)$ so that the wave function satisfies the boundary condition Eq. (3), at each of the three boundaries $r_{12}=a$, $r_{23}=a$, $r_{13}=a$.

In general, one must expect that the forms of the functions $F_{ll}^{(ij)}(\kappa)$ are, as the notation indicates, actually different for different i and j. On the other hand, there may well be states for which all the $F_{ll}^{(ij)}(\kappa)$ are of the same form, i.e., for which the total wave function is completely symmetric among all the particles. It is

clear that this will be true for the ground state, which is a state of maximum symmetry. In this section we shall write the integral equations for states of this symmetry. The same techniques we use can be carried out for other states with some additional complication.

For a symmetrical state of the kind we discuss, the superscripts in the function $F_{ll}^{(ij)}(\kappa)$ are in fact superfluous. We then simplify the notation by writing for this case

$$F_{ll}^{(ij)}(\kappa)=f_l(\kappa). \tag{16}$$

Now it suffices to apply the boundary conditions at only one boundary; it follows from the symmetry of the wave function that the conditions will automatically be satisfied at the other two boundaries as well. Consider first the boundary $r_{23}=a$. To satisfy boundary conditions there, we must express the functions ψ_{12} and ψ_{13} in the coordinates $\mathbf{r}_{23}$ and $\boldsymbol{\varrho}_1$. The general technique for this is to express these functions as Fourier integrals. Then all the space dependence, that is the dependence on $\mathbf{r}_{12}$, $\boldsymbol{\varrho}_3$ and $\mathbf{r}_{13}$, $\boldsymbol{\varrho}_2$ is in an exponent, and transforming these functions to the coordinates $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_2$, corresponding to the boundary $r_{23}=a$, simply means shuffling the variables in this exponent, using the linear relations (5). With this approach we get a set of coupled integral equations relating the $f_l(\kappa)$ to integrals over themselves. These integral equations are homogeneous; hence they have solutions only for special values of K, the eigenvalues, and have corresponding eigenfunctions.

We begin to express $\psi_{12}+\psi_{13}$ in $\mathbf{r}_{23}$, $\boldsymbol{\varrho}_1$ coordinates. To do this we use the integral representation

$$\bar{h}_{l'}(ikr_{12})Y_{l'm'}(\Omega_{12})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int u_{l'm'}(k,\boldsymbol{\lambda})\exp[i\boldsymbol{\lambda}\cdot\mathbf{r}_{12}]d\boldsymbol{\lambda}. \tag{17}$$

On inverting this transform it is found that $u_{l'm'}(k,\boldsymbol{\lambda})$ can be written in the form

$$u_{l'm'}(k,\boldsymbol{\lambda})=(2\pi)^{\frac{3}{2}}v_{l'}(k,\lambda)Y_{l'm'}(\Omega_\lambda).$$

Expressions for the functions $v_{l'}(k,\lambda)$ are derived in the Appendix. We put Eq. (17) and the similar equation involving r_{13} into Eqs. (11b, c) and use Eqs. (15) and (16) to get

$$\psi_{12}+\psi_{13}=\sum_{l'=0}^{\infty}\int\int f_{l'}(\sigma)v_{l'}(k,\lambda)\Big(\sum_{m'=-l'}^{l'}Y_{l'm'}{}^*(\Omega_\kappa)Y_{l'm'}(\Omega_\lambda)\Big)\{\exp[i(\boldsymbol{\lambda}\cdot\mathbf{r}_{12}+\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_3)]+\exp[i(\boldsymbol{\lambda}\cdot\mathbf{r}_{13}+\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_2)]\}d\boldsymbol{\lambda}d\boldsymbol{\sigma}.$$

We use the transformation Eqs. (5) to rearrange the exponents

$$\exp[i(\boldsymbol{\lambda}\cdot\mathbf{r}_{12}+\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_3)]+\exp[i(\boldsymbol{\lambda}\cdot\mathbf{r}_{13}+\boldsymbol{\sigma}\cdot\boldsymbol{\varrho}_2)]=\exp[i\boldsymbol{\varrho}_1\cdot(\boldsymbol{\lambda}-\tfrac{1}{2}\boldsymbol{\sigma})]\{\exp[i\mathbf{r}_{23}\cdot(\tfrac{1}{2}\boldsymbol{\lambda}+\tfrac{3}{4}\boldsymbol{\sigma})]+\exp[-i\mathbf{r}_{23}\cdot(\tfrac{1}{2}\boldsymbol{\lambda}+\tfrac{3}{4}\boldsymbol{\sigma})]\},$$

and we go from the variables $\boldsymbol{\lambda}$, $\boldsymbol{\sigma}$ to new variables $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}'$ by the transformation

$$\boldsymbol{\lambda}-\tfrac{1}{2}\boldsymbol{\sigma}=\boldsymbol{\kappa},\quad \boldsymbol{\sigma}=\boldsymbol{\kappa}'.$$

Then we have

$$\psi_{12}+\psi_{13}=\sum_{l'=0}^{\infty}\int\int f_{l'}(\kappa')v_{l'}\big((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},\,|\boldsymbol{\kappa}+\tfrac{1}{2}\boldsymbol{\kappa}'|\big)$$
$$\times\Big(\sum_{m'=-l'}^{l'}Y_{l'm'}{}^*(\Omega_{\kappa'})Y_{l'm'}(\Omega_{\kappa+\frac{1}{2}\kappa'})\Big)\exp[i\boldsymbol{\varrho}_1\cdot\boldsymbol{\kappa}]\{\exp[i\mathbf{r}_{23}(\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa})]+\exp[-i\mathbf{r}_{23}(\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa})]\}d\boldsymbol{\kappa}'d\boldsymbol{\kappa}.$$

To exhibit the spherical harmonics of Ω_{23} we use

$$\exp[i\mathbf{r}_{23}\cdot(\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa})]+\exp[-i\mathbf{r}_{23}\cdot(\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa})]=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^l[1+(-)^l]j_l(r_{23}|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|)Y_{lm}(\Omega_{23})Y_{lm}{}^*(\Omega_{\kappa'+\frac{1}{2}\kappa}),$$

and

$$\sum_{m'=-l'}^{l'}Y_{l'm'}{}^*(\Omega_{\kappa'})Y_{l'm'}(\Omega_{\frac{1}{2}\kappa'+\kappa})=\frac{2l'+1}{4\pi}P_{l'}(\cos\beta),$$

where

$$\cos\beta=\frac{\boldsymbol{\kappa}'\cdot(\boldsymbol{\kappa}+\frac{1}{2}\boldsymbol{\kappa}')}{|\boldsymbol{\kappa}'|\,|\boldsymbol{\kappa}+\frac{1}{2}\boldsymbol{\kappa}'|}.$$

Then we have

$$\psi_{12}+\psi_{13}=\sum_{l'=0}^{\infty}\int\int f_{l'}(\kappa')v_{l'}\big((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},\,|\boldsymbol{\kappa}+\tfrac{1}{2}\boldsymbol{\kappa}'|\big)(2l'+1)P_{l'}(\cos\beta)\exp[i\boldsymbol{\varrho}_1\cdot\boldsymbol{\kappa}]$$
$$\times\Big\{\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^l[1+(-)^l]j_l(r_{23}|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|)Y_{lm}(\Omega_{23})Y_{lm}{}^*(\Omega_{\kappa'+\frac{1}{2}\kappa})\Big\}d\boldsymbol{\kappa}d\boldsymbol{\kappa}', \tag{18}$$

and the total $\Psi=\psi_{12}+\psi_{13}+\psi_{23}$ is just

$$\Psi=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int \exp[i\boldsymbol{\varrho}_1\cdot\boldsymbol{\kappa}]Y_{lm}(\Omega_{23})\Big\{f_l(\kappa)Y_{lm}^*(\Omega_\kappa)\bar{h}_l(ikr_{23})+i^l[1+(-)^l]$$
$$\times\int[\sum_{l'=0}^{\infty}f_{l'}(\kappa')v_{l'}((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},|\boldsymbol{\kappa}+\tfrac{1}{2}\boldsymbol{\kappa}'|)(2l'+1)P_{l'}(\cos\beta)]Y_{lm}^*(\Omega_{\kappa'+\frac{1}{2}\kappa})j_l(r_{23}|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|)d\boldsymbol{\kappa}'\Big\}d\boldsymbol{\kappa}. \quad (19)$$

This is the desired result. Ψ is explicitly expressed in terms only of $\mathbf{r}_{23}$ and $\boldsymbol{\varrho}_1$.

Now we apply the boundary condition Eq. (3) at $r_{23}=a\cdot(\partial/\partial n)$ is, of course, $\partial/\partial r_{23}$ which we form by differentiating under the integral sign. We insert the expression (19) for Ψ into this equation. Then we equate the coefficients of the $Y_{lm}(\Omega_{23})$ on each side of the resulting equation; and, since this boundary condition holds for all $\boldsymbol{\varrho}_1$ we equate the integrands of the $d\boldsymbol{\kappa}$ integration. This gives the set of equations which holds for all l and m.

$$-\gamma\Big\{f_l(\kappa)Y_{lm}^*(\Omega_\kappa)h_l(ika)+i^l[1+(-)^l]$$
$$\times\int[\sum_{l'=0}^{\infty}f_{l'}(\kappa')v_{l'}((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},|\boldsymbol{\kappa}+\tfrac{1}{2}\boldsymbol{\kappa}'|)(2l'+1)P_{l'}(\cos\beta)]Y_{lm}^*(\Omega_{\kappa'+\frac{1}{2}\kappa})j_l(a|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|)d\boldsymbol{\kappa}'\Big\}$$
$$=ikf_l(\kappa)Y_{lm}^*(\Omega_\kappa)h_l'(ika)+i^l[1+(-)^l]$$
$$\times\int[\sum_{l'=0}^{\infty}f_{l'}(\kappa')v_{l'}((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},|\boldsymbol{\kappa}+\tfrac{1}{2}\boldsymbol{\kappa}'|)(2l'+1)P_{l'}(\cos\beta)]Y_{lm}^*(\Omega_{\kappa'+\frac{1}{2}\kappa})|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|j_l'(a|\boldsymbol{\kappa}'+\tfrac{1}{2}\boldsymbol{\kappa}|)d\boldsymbol{\kappa}'.$$

The prime on the functions j_l and h_l denotes differentiation with respect to their argument, and we have replaced $\bar{h}_l$ by h_l.

This set can be reduced somewhat further by noting that the integrals in them are functions only of $|\boldsymbol{\kappa}|$, $|\boldsymbol{\kappa}'|$, $\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}'$. Thus we can in the usual way imagine $\boldsymbol{\kappa}$ is along the z axis. This simply amounts to a rotation of the vector $\boldsymbol{\kappa}'$ and since eventually we integrate over the whole solid angle of $\boldsymbol{\kappa}'$ this makes no difference in the final result. With $\boldsymbol{\kappa}$ along the z-axis, $Y_{lm}^*(\Omega_\kappa)$ becomes $[(2l+1)/4\pi]^{\frac{1}{2}}P_l(0)=[(2l+1)/4\pi]^{\frac{1}{2}}$ for all l. Moreover, $\kappa_z'=\kappa'\cos\theta'$ and $Y_{lm}^*(\Omega_{\kappa'+\frac{1}{2}\kappa})$ becomes just $[(2l+1)/4\pi]^{\frac{1}{2}}P_l(\Omega_{\kappa'+\frac{1}{2}\kappa})=[(2l+1)/4\pi]^{\frac{1}{2}}P_l(\cos\alpha)$ where α is the angle between $\boldsymbol{\kappa}'+\frac{1}{2}\boldsymbol{\kappa}$ and the z-axis. That is

$$\cos\alpha=\frac{\boldsymbol{\kappa}\cdot(\boldsymbol{\kappa}'+\frac{1}{2}\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|\,|\boldsymbol{\kappa}'+\frac{1}{2}\boldsymbol{\kappa}|}=\frac{\kappa'\cos\theta'+\frac{1}{2}\kappa}{(\frac{1}{4}\kappa^2+\kappa'^2+\kappa\kappa'\cos\theta')^{\frac{1}{2}}}. \quad (21)$$

Similarly we have

$$\cos\beta=\frac{\kappa\cos\theta'+\frac{1}{2}\kappa'}{(\frac{1}{4}\kappa'^2+\kappa^2+\kappa\kappa'\cos\theta')^{\frac{1}{2}}}. \quad (22)$$

With these definitions, and the further definitions

$$s=(\tfrac{1}{4}\kappa^2+\kappa'^2+\kappa\kappa'\cos\theta')^{\frac{1}{2}}, \quad (23)$$
$$t=(\tfrac{1}{4}\kappa'^2+\kappa^2+\kappa\kappa'\cos\theta')^{\frac{1}{2}}, \quad (24)$$

the equations are

$$-\gamma\Big\{f_l(\kappa)h_l(ika)+2\pi i^l[1+(-)^l]$$
$$\times\int_0^{\infty}\int_0^{\pi}[\sum_{l'=0}^{\infty}f_{l'}(\kappa')v_{l'}((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},t)(2l'+1)P_{l'}(\cos\beta)]P_l(\cos\alpha)j_l(as)\sin\theta'\kappa'^2d\theta'd\kappa'\Big\} \quad (25)$$
$$=ikf_l(\kappa)h_l'(ika)+2\pi i^l[1+(-)^l]$$
$$\times\int_0^{\infty}\int_0^{\pi}[\sum_{l'=0}^{\infty}f_{l'}(\kappa')v_{l'}((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}},t)(2l'+1)P_{l'}(\cos\beta)]P_l(\cos\alpha)sj_l'(as)\sin\theta'\kappa'^2d\theta'd\kappa'.$$

Setting $l=0, 1, 2\cdots$ successively gives the infinite set of coupled integral equations in which each $f_l(\kappa)$ is related to integrals over all other $f_{l'}(\kappa)$. Since the set is homogeneous, or if one likes, since it is equivalent to the original Schrödinger equation, it can have solutions only for certain eigenvalues and corresponding eigenfunctions. That is, for each eigenvalue K, the form of all the functions $f_l(\kappa)$ is determinate from these equations.

Although this set of equations is an infinite one, it is natural to try the approximation of truncating it and keeping only a finite number of partial waves, i.e., of functions $f_l(\kappa)$. The dimensionless parameters that determine the nature of the solutions of these equations and the degree of approximation required are of course Ka and γa since these are the only such parameters that enter the equations. These are in fact not independent, since the eigenvalue Ka is determined from γa. The magnitude of Ka affects the nature of the solutions of Eqs. (25) in that it sets the scale for some of the functions that appear in the integral equations as for example, the function $h_l(i(K^2+\frac{3}{4}\kappa^2)^{\frac{1}{2}}a)$ whose behavior is quite different according as Ka is much greater or much less than κa.

Now we ask why, for given Ka, we can get away with keeping only a finite number of partial waves. Speaking generally, this comes about because of the asymptotic properties of spherical Bessel functions, namely, that those of high order are small when their arguments are small: $j_l(x)\approx 0$ for $x\ll l$. This is, of course, the same property that limits the number of partial waves in ordinary scattering theory. Physically, we imagine that any two particles have a relative wavelength and that if this relative wavelength is of the same order as the radius a, we need only a few relative partial waves. For a small number of particles we expect this relative wavelength to be roughly the deBroglie wavelength corresponding to the total energy of the system. Mathematically, we can see from Eq. (25), by a rather circular argument, how a limited number of partial waves might suffice. In the integrands of these equations are the functions $f_l(\kappa)$. Now it seems likely that each of these functions is of finite range, i.e., is essentially zero beyond a certain κ. If this is in fact the case this means that there is, speaking roughly, a maximum value of the argument $(\frac{1}{4}\kappa^2+\kappa'^2+\kappa\kappa'\cos\theta')^{\frac{1}{2}}$ of the Bessel functions which comes into play; for κ greater than this argument, the j_l vanish, which is consistent with the fact that the functions $f_l(\kappa)$ have finite range. Even if one has an idea of the number of partial waves required for a given Ka, in applying these equations it must be remembered that Ka is not given in advance, but comes out of the solution. In practice, then, one must guess at Ka, choose an appropriate number of partial waves, and then having *found* Ka, verify that it was indeed permissible to neglect higher order ones. These points are discussed in somewhat more detail in Sec. Vc.

V. THE TRUNCATED EQUATIONS

a. The Equation for S-Waves Only

In this section we discuss in some detail the solution of the infinite set Eq. (25), when we truncate them and retain only $f_0(\kappa)$ as being different from zero. The equations then reduce to the single homogeneous integral equation

$$-\gamma\left\{h_0(ia(K^2+\tfrac{3}{4}\kappa^2)^{\frac{1}{2}})f_0(\kappa)+4\pi\int_0^\infty \kappa'^2 d\kappa' \int_0^\pi f_0(\kappa')j_0(as)v_0((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}}, t)\sin\theta' d\theta'\right\}$$
$$=i(K^2+\tfrac{3}{4}\kappa^2)^{\frac{1}{2}}h_0'(ia(K^2+\tfrac{3}{4}\kappa^2)^{\frac{1}{2}})f_0(\kappa)-4\pi\int_0^\infty \kappa'^2 d\kappa' \int_0^\pi f_0(\kappa')sj_1(as)v_0((K^2+\tfrac{3}{4}\kappa'^2)^{\frac{1}{2}}, t)\sin\theta' d\theta'. \quad (26)$$

We have used $j_0'=-j_1$ here.

We reduce this equation in the following way. We make the substitution $y=\sin\theta'$, introduce dimensionless parameters K_0, γ_0

$$Ka=K_0, \quad \gamma a=\gamma_0,$$

and the definitions

$$\bar{s}=K_0(\tfrac{1}{4}\xi^2+\xi'^2+\xi\xi' y)^{\frac{1}{2}}, \quad (27)$$

$$\bar{t}=K_0(\tfrac{1}{4}\xi'^2+\xi^2+\xi\xi' y)^{\frac{1}{2}}. \quad (28)$$

We use the definitions of the Hankel functions, the expression for $v_0(k,\lambda)$ from the Appendix and introduce a new function

$$f_0^*(\xi)=f_0(\xi)\exp[-K_0(1+\tfrac{3}{4}\xi^2)^{\frac{1}{2}}]/(1+\tfrac{3}{4}\xi^2)^{\frac{1}{2}}. \quad (29)$$

Then after some rearrangement the integral equation for $f_0^*(\xi)$ is, written out in full,

$$f_0^*(\xi)\left[(1+\tfrac{3}{4}\xi^2)^{\frac{1}{2}}-\frac{\gamma_0-1}{K_0}\right]$$
$$=\frac{2}{\pi}\int_0^\infty f_0^*(\xi')\xi'^2 d\xi' \int_{-1}^1 \frac{\gamma_0 j_0(K_0(\xi'^2+\frac{1}{4}\xi^2+\xi\xi' y)^{\frac{1}{2}})-K_0(\xi'^2+\frac{1}{4}\xi^2+\xi\xi' y)^{\frac{1}{2}}j_1(K_0(\xi'^2+\frac{1}{4}\xi^2+\xi\xi' y)^{\frac{1}{2}})}{\xi^2+\xi'^2+\xi\xi' y+1}$$
$$\times\left[\cos(K_0(\xi^2+\tfrac{1}{4}\xi'^2+\xi\xi' y)^{\frac{1}{2}})+(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}\frac{\sin(K_0(\xi^2+\frac{1}{4}\xi'^2+\xi\xi' y)^{\frac{1}{2}})}{(\xi^2+\frac{1}{4}\xi'^2+\xi\xi' y)^{\frac{1}{2}}}\right]dy. \quad (30)$$

When an eigenvalue K_0 and corresponding eigenfunction $f_0^*(\xi)$ of this integral equation is found then the (unnormalized) wave function of the system corresponding to this K_0 and $f_0^*(\xi)$ is

$$\Psi = F(r_{12},\rho_3)+F(r_{13},\rho_2)+F(r_{23},\rho_1),$$

where

$$F(r,\rho)=\int_0^\infty f_0^*(\xi)\frac{\exp[-(r/a-1)K_0(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}]\sin(K_0\rho\xi/a)}{r\rho}\xi d\xi.$$

Before discussing the solution of Eq. (30) it is interesting to note that it is the right-hand side of Eq. (30) that reflects the fact that we have a three body problem. This side comes from the transformation of $\psi_{12}+\psi_{13}$ to 23 coordinates. The left-hand side comes from ψ_{23} itself. If the right-hand side were zero, then the left-hand side would describe the system of particles 2 and 3 and if the center of mass of this system were at rest $f_0^*(\xi)$ would simply be a delta function of ξ, i.e., the left-hand side would be zero except for $\xi=0$. Thus, for Eq. (30) to be satisfied for this case, for all ξ, including $\xi=0$, we would have to have $1+K_0=\gamma_0$, which is of course just the relation we found previously for the two-body problem.

b. Qualitative Discussion of the *S*-Wave Equation

In this section we shall present a qualitative discussion of the integral Eq. (30). First, we shall try to show why it can have solutions only for certain values of K_0, the eigenvalues. The point is clear in principle, of course, since the integral equation is equivalent to a Schrödinger equation which we know has solutions only for certain eigenvalues. Nonetheless, it is illuminating to see from a direct analysis of the integral equation how this comes about. We shall also try to find the qualitative form of the eigenfunctions and make a rough estimate of the eigenvalues, as a guide to the more refined treatment in the next section.

Before we proceed, it is useful to have fixed in our minds the range of values of the parameters that enter the equation. First we consider K_0, the eigenvalue we seek. As we have pointed out before, the magnitude of K_0, among other things, determines the number of partial waves one must retain in the analysis. K_0 cannot turn out to be too large if the truncated Eq. (30) is to make sense, that is if it is really justifiable to go from the coupled set of Eqs. (25) to the single Eq. (30). Just how large is a moot question, but we are probably safe in retaining only S waves when K_0 is, say, smaller than about $\frac{1}{2}$. We shall think of K_0 as being in this range. This means that the values of γ_0 we must consider are necessarily close to unity. For we have found for the two-body problem the relation $K_0=\gamma_0-1$ showing K_0 increasing with γ_0. Now, for $\gamma_0=1.5$ we already have a K_0 of 0.5. And it must certainly be true that the ground-state binding energy is greater for the three-body than for the two-body problem; thus to avoid dealing with a K_0 greater than 0.5 in the three-body problem we must take γ_0 considerably less than 1.5. We shall in fact discuss Eq. (30) for γ_0 between 1 and 1.2, although it should be noted that there are bound states of the three-body problem for γ_0 less than unity. These are qualitatively similar to those for γ_0 greater than unity.

The technique of the present qualitative discussion is to show that Eq. (30) is roughly equivalent to a degenerate equation with a separable kernel. That is to say, we replace the integral equation with a crude approximation to itself which approximate equation is, however, exactly soluble. In doing this we shall in general begin by simply stating the approximations we use and will discuss and, insofar as is possible, justify them only later.

First consider the function

$$j_0(K_0(\xi'^2+\tfrac{1}{4}\xi^2+\xi\xi'y)^{\frac{1}{2}})$$

which appears in the integrand of Eq. (30). This can be expanded by the addition formula

$$j_0(K_0(\xi'^2+\tfrac{1}{4}\xi^2+\xi\xi'y)^{\frac{1}{2}})\equiv j_0(\bar{s})$$
$$=\sum_{l=0}^{\infty}(2l+1)(-)^l j_l(K_0\xi')j_l(K_0\xi/2)P_l(y). \quad (31)$$

Now for K_0 small and neither ξ nor ξ' too large we can take only the first term in this formula, i.e., we use

$$j_0(\bar{s})\approx j_0(K_0\xi')j_0(K_0\xi/2). \quad (32)$$

In like manner we could expand the other similar functions $\bar{s}j_1(\bar{s})$, $\cos\bar{t}$, $\sin\bar{t}/\bar{t}$. We shall, however, approximate these even more crudely, i.e., still assuming that $K_0\xi$ and $K_0\xi'$ never get very large we replace these functions by their values at the origin, i.e., for $\xi=\xi'=0$. Thus we set $\bar{s}j_1(\bar{s})$ equal to zero and set $\cos\bar{t}$ and $\sin\bar{t}/\bar{t}$ equal to unity. The integral equation then becomes

$$f_0^*(\xi)=\frac{2\gamma_0 j_0(\frac{1}{2}K_0\xi)}{\pi\{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}}\int_0^\infty \xi'^2 f_0^*(\xi')j_0(K_0\xi')[1+K_0(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}]d\xi'\int_{-1}^{1}\frac{dy}{\xi^2+\xi'^2+\xi\xi'y+1}. \quad (33)$$

Now we consider the y integral in this last equation,

$$\int_{-1}^{1}\frac{dy}{\xi^2+\xi'^2+\xi\xi'y+1}=\int_{-1}^{1}\frac{dy}{(\xi^2+\xi'^2+1)[1+\xi\xi'y/(\xi^2+\xi'^2+1)]}. \quad (34)$$

We observe that $\xi\xi' y/(\xi^2+\xi'^2+1)$ is always less than unity, and we can expand the denominator in powers of this quantity. We can then conclude that the integral (34) is represented to sufficient accuracy (within 10%) by the function

$$2/(\xi^2+\xi'^2+1). \tag{35}$$

With this approximation we reduce the integral Eq. (33) to

$$f_0^*(\xi)=\frac{4\gamma_0 j_0(K_0\xi/2)}{\pi\{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}}\int_0^\infty \frac{\xi'^2 f_0^*(\xi')j_0(K_0\xi')[1+K_0(1+\frac{3}{4}\xi'^2)^{\frac{1}{2}}]d\xi'}{\xi^2+\xi'^2+1}. \tag{36}$$

Now the equation is almost separable. It can be made exactly so by one last and most drastic approximation. We consider the possibility of replacing the kernel $1/(\xi^2+\xi'^2+1)$ by the separable kernel $1/(1+\xi^2)(1+\xi'^2)$. This approximation is excellent when either ξ or ξ' or both are small. It is worst when $\xi\approx\xi'$ and both are large. Even so, for ξ and ξ' not too large it is tolerable. For example, for $\xi=\xi'=1$ the exact kernel has the value $\frac{1}{3}$ and the approximate kernel $\frac{1}{4}$. On the other hand the approximation rapidly becomes poor for ξ and ξ' large. Thus for $\xi=\xi'=2$ the exact kernel is $\frac{1}{9}$ and the approximate one 1/25. Nonetheless, if we make this approximation, crude as it may be, we have the great advantage of being led to an equation whose solution is transparent.

$$f_0^*(\xi)=\frac{4\gamma_0 j_0(\frac{1}{2}K_0\xi)}{\pi\{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}(1+\xi^2)}\int_0^\infty \frac{\xi'^2 f_0^*(\xi')j_0(K_0\xi')[1+K_0(1+\frac{3}{4}\xi'^2)^{\frac{1}{2}}]d\xi'}{1+\xi'^2}. \tag{37}$$

The essential form of the function $f_0^*(\xi)$ stands in front of the integral sign; all that remains to determine it completely is to find the eigenvalue K_0. To do this we multiply the equation by

$$\xi^2 j_0(K_0\xi)[1+K_0(1+\tfrac{3}{4}\xi^2)^{\frac{1}{2}}]/(1+\xi^2)$$

and integrate with respect to ξ. We get

$$1=\frac{4\gamma_0}{\pi}\int_0^\infty \frac{\xi^2 j_0(K_0\xi)j_0(\frac{1}{2}K_0\xi)[1+K_0(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}]}{[(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0](1+\xi^2)^2}d\xi. \tag{38}$$

For a given γ_0 the eigenvalues are determined from this equation. It is illuminating to see qualitatively why at least one eigenvalue must exist. The right-hand side of Eq. (38) must equal unity for the eigenvalue K_0. More generally, let us consider the magnitude of the integral on the right-hand side as a function of K_0, which we imagine for the moment to be a variable parameter. From our previous discussion of the two-body problem K_0 must be greater than γ_0-1 (for γ_0 greater than one). If for a moment we imagine K_0 close to its minimum value of γ_0-1, we see that the denominator of the integrand almost vanishes for small ξ. Such a small value of K_0 maximizes the integral; if in fact one calculates the integral for this case one finds that the right-hand side of Eq. (38) is greater than unity. On the other hand, for large K_0 it is easy to see that the integral vanishes, essentially because of the oscillatory character of $j_0(K_0\xi)$ and $j_0(K_0\xi/2)$. There is some intermediate K_0 then for which the right-hand side is unity. Numerical calculation gives the results: for $\gamma_0=1.1$, $K_0=0.22$; for $\gamma_0=1.2$, $K_0=0.46$.

With these values of K_0 at hand let us reconsider the approximations we have made in deriving the approximate integral equation (38). For the sake of being specific we take the case $\gamma_0=1.1$. Then the function $f_0^*(\xi)$, normalized to unity at the origin is just

$$f_0^*(\xi)=\frac{0.55}{[(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-0.45](1+\xi^2)}.$$

This function has the value 0.316 at $\xi=1$, 0.071 at $\xi=2$, and has dropped to 0.026 at $\xi=3$. Now, we have already discussed the approximations involved in truncating the expansion in Eq. (35). The next approximation was that of Eq. (32), the expansion of $j_0(\bar{s})$ retaining only the $l=0$ term. To get an idea of its validity we evaluate it for those parameters of interest for which it converges least well, say $\xi=\xi'=3$ and $K_0=0.40$, to choose an arbitrary value. Then the series (31) is $0.731-0.199P_1+0.0101P_2-0.00024P_3$ which is not too bad for this worst case, since the convergence becomes rapidly better for either ξ or ξ' smaller than 3. Less justified was the third approximation: the dropping of the term $\bar{s}j_1(\bar{s})$ and the setting of $\cos\bar{s}$ and $\sin\bar{s}/\bar{s}$ equal to unity. Our reason for so doing was heuristic; if we had expanded them in the manner of Eq. (31) we would not have been led to a separable equation. However, we see that these last mentioned approximations make no qualitative difference in the integral equation. That is, these func-

tions are essentially functions which oscillate faster and faster the larger K_0 is. As such they supplement the roles of $j_0(K_0\xi)$ and $j_0(K_0\xi/2)$ as convergence factors, without changing anything essential. It is difficult to assess quantitively the last approximation, the replacement of the kernel $1/(\xi^2+\xi'^2+1)$ by $1/[(1+\xi^2)(1+\xi'^2)]$. Roughly speaking, it is good up to $\xi=\xi'=1$ and quite bad by the time it gets to $\xi=\xi'=2$. Now the value of $f_0^*(\xi)$ has dropped appreciably by $\xi=1$, but it is still not negligible at $\xi=2$, so the error in the kernel may be quite important. All told it is hard to assess the approximations, except to repeat that they seem almost certain to be qualitatively correct and to give the right general behavior of the eigenfunction and the correct order of magnitude of K_0.

c. Quantitative Discussion of the *S*-Wave Equation

Having established the general features of the solution of Eq. (30), we turn to a more quantitative, although still approximate, discussion. We start in a way similar to the last section, by expanding the functions $j_0(\bar{s})$, $\bar{s}j_1(\bar{s})$, $\cos\bar{t}$, and $\sin\bar{t}/\bar{t}$ in Legendre polynomials in y and keeping only the first term in the expansion. These expansions are easily got by differentiating the expansion Eq. (31) for j_0. Thus we have, for example,

$$\cos\bar{t}=\left[\frac{d}{da}\frac{\sin(a\bar{t})}{\bar{t}}\right]_{a=1}=\left[\frac{d}{da}aj_0(a\bar{t})\right]_{a=1}$$
$$=\frac{d}{da}\left(a\sum_{l=0}^{\infty}(2l+1)(-)^l j_l(a\xi)j_l(a\xi'/2)P_l(y)\right)\Bigg|_{a=1}.$$

This gives, keeping only the $l=0$ term,

$$\cos\bar{t}\approx j_0(K_0\xi)j_0(K_0\xi'/2)-\tfrac{1}{2}K_0\xi' j_0(K_0\xi)j_1(K_0\xi'/2)$$
$$-K_0\xi j_0(K_0\xi'/2)j_1(K_0\xi).$$

In a similar way we can expand $\bar{s}j_1(\bar{s})$ and get

$$\bar{s}j_1(\bar{s})\approx\tfrac{1}{2}K_0\xi j_0(K_0\xi')j_1(K_0\xi/2)$$
$$+K_0\xi' j_0(K_0\xi/2)j_1(K_0\xi').$$

Putting these approximate expansions into Eq. (30), we can do the y integral; this integral is:

$$\int_{-1}^{1}\frac{d\xi'}{\xi^2+\xi'^2+\xi\xi'y+1}\approx\frac{2}{\xi^2+\xi'^2+1}. \tag{39}$$

Here we have used the same approximation as in the qualitative discussion of the last section, and as we have seen there the average error this introduces into the kernel is of the order of a few percent. The integral equation (30) is now reduced to

$$f_0^*(\xi)=\frac{4}{\pi\{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}}\int_0^{\infty}\frac{f_0^*(\xi')A(\xi,\xi')B(\xi,\xi')\xi'^2d\xi'}{\xi^2+\xi'^2+1}, \tag{40}$$

where

$$A(\xi,\xi')=j_0(K_0\xi)j_0(K_0\xi'/2)\left\{\frac{\cos(K_0\xi'/2)}{j_0(K_0\xi'/2)}+\frac{\cos K_0\xi}{j_0(K_0\xi)}-1+K_0(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}\right\},$$

$$B(\xi,\xi')=j_0(K_0\xi')j_0(K_0\xi/2)\left\{(\gamma_0-2)+\frac{\cos(K_0\xi/2)}{j_0(K_0\xi/2)}+\frac{\cos(K_0\xi')}{j_0(K_0\xi')}\right\}.$$

From Eq. (40) we see that $f_0^*(\xi)$ is given by the function standing in front of the integral times that function of ξ represented by the integral itself. We discuss this latter. We observe first that for the values of ξ and ξ' which we are likely to be interested in—and which are, from the results of the previous section between, say, 0 and 3 for each variable—the functions $A(\xi,\xi')$ and $B(\xi,\xi')$ are slowly varying compared to the function $1/(\xi^2+\xi'^2+1)$. The main ξ dependence of the integral in Eq. (40) derives from this factor. This suggests that a reasonable approximation to the true solution is a function of the form

$$f_0^*(\xi)\approx\frac{1}{\{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}(1+\epsilon\xi^2)}, \tag{41}$$

where ϵ is a constant still to be determined. If we try this and put it into Eq. (40) we get the following equation to be satisfied for all ξ.

$$1=\frac{4}{\pi}(1+\epsilon\xi^2)\int_0^{\infty}\frac{A(\xi,\xi')B(\xi,\xi')\xi'^2d\xi'}{\{(1+\frac{3}{4}\xi'^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}(1+\epsilon\xi'^2)(\xi^2+\xi'^2+1)}. \tag{42}$$

Since the functions $A(\xi,\xi')$ and $B(\xi,\xi')$ are slowly varying compared to the other factors in the integrand, we shall in the argument that follows imagine them as evaluated at some average value $\xi=\bar{\xi}$. With this approximation, and a little rearranging of the denominator Eq. (42) becomes

$$1=\frac{4}{\pi}(1+\epsilon\xi^2)\int_0^{\infty}\frac{A(\bar{\xi},\xi')B(\bar{\xi},\xi')\xi'^2d\xi'}{\{(1+\frac{3}{4}\xi'^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}[1+\xi^2/(\xi'^2+1)](1+\xi'^2)(1+\epsilon\xi'^2)}. \tag{43}$$

Now the left-hand side of this equation is a constant; the right-hand side ostensibly depends on ξ. How is this to be reconciled? To answer this question let us anticipate a bit, and take ϵ of the order of $\frac{1}{2}$. Then we observe that the factor

$$\xi'^2/\{[(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0](1+\xi'^2)(1+\epsilon\xi'^2)\}$$

is one which is peaked fairly sharply around $\xi'=1$. If it were very sharply peaked to the extent, say, of being a delta function, at some $\xi'=\xi_0'$, we could do the integral and we see that by choosing $\epsilon=1/(\xi_0^2+1)$ the integral equation would be satisfied exactly, i.e., the dependence of the right-hand side on ϵ would cancel out. To the extent that the integrand has a finite width, the proposed solution (41) is necessarily approximate and the right-hand side is not strictly a constant independent of ξ, but we shall see that it is so to a good approximation.

Although it is clear that, since the integrand peaks at ξ' about unity, ϵ is about $\frac{1}{2}$, we would like to determine ϵ more closely and of course determine the eigenvalue K_0 as well. To determine these two constants we must have two equations; we can get these by demanding that the integral Eq. (40) holds exactly at two points and if the representation of Eq. (41) is reasonable it will hold approximately at intermediate points as well.

The requirement that the integral equation be satisfied at $\xi=0$ leads to the equation

$$1=-\frac{4}{\pi}\int_0^\infty \frac{A(\xi,\xi')B(\xi,\xi')\xi'^2 d\xi'}{\{(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}(1+\xi'^2)(1+\epsilon\xi'^2)}. \tag{44}$$

This is an equation very similar to the eigenvalue Eq. (38) we found previously and a similar qualitative discussion applies, remembering that $\epsilon\approx\frac{1}{2}$. It is convenient to choose $x=2$ as the other point at which we demand the integral equation be satisfied. At this point the function f_0^* has dropped appreciably from its value at the origin, without being excessively small. The condition that the integral equation be satisfied exactly at $x=2$ gives the equation

$$1=-\frac{4}{\pi}(1+4\epsilon)\int_0^\infty \frac{A(\xi,\xi')B(\xi,\xi')\xi'^2 d\xi'}{\{(1+\tfrac{3}{4}\xi'^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0\}(1+\epsilon\xi'^2)(5+\xi'^2)}. \tag{45}$$

From Eq. (44) we get a set of possible allowed values of K_0 for a given ϵ, i.e., a curve in the K_0, ϵ plane. From Eq. (45) we get a different curve; the intersection of these curves gives then unique values of K_0 and ϵ, and we must then verify that these values lead to a function which satisfies the integral equation at least approximately at other points than the two chosen above.

The results of calculations along these lines are given in Table I. Because of the various approximations, in particular that represented by Eq. (39) we do not consider the eigenvalues K_0 to be accurate to, say, better than 10 percent.

The results of Table I are quoted for a minimum value of γ_0 equal to unity, because our main purpose is to compare the three body binding energy with the two-body and the two-body system is not bound for γ_0 less than unity. As we have mentioned before, there are in fact solutions of the integral Eq. (30) for γ_0 less than one (although not for γ_0 too small). From Table I we see that the ratio of three-body binding energy to two-body binding energy is infinite for $\gamma_0=1$, nine for $\gamma_0=1.1$, and four for $\gamma_0=1.2$. This seems reasonable.

TABLE I. As a function of γ_0: the eigenvalue parameter $K_0=m|\mathrm{E}|/\hbar^2$ and the parameter ϵ which appears in the function $f_0^*(\xi)\approx 1/\{[(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}-(\gamma_0-1)/K_0](1+\epsilon\xi^2)\}$.

γ_0	K_0	ϵ
1.00	0.27	0.27
1.10	0.30	0.50
1.20	0.40	0.60

There is one final point we would like to discuss, namely, to show qualitatively why the higher order partial waves we neglected in truncating the general integral Eqs. (25) are in fact negligible for small K_0. To discuss this, we observe that if the amplitudes of these waves are in fact small we can estimate them from the integral Eqs. (25). From these equations we see that the factor $[1+(-)^l]$ precludes relative P waves. Thus the next amplitude after $f_0(\kappa)$ which we have to discuss is the D-wave one, $f_2(\kappa)$. We get an approximate equation relating $f_2(\kappa)$ to $f_0(\kappa)$ from Eqs. (25) by setting $l=2$, $l'=0$, i.e., by neglecting all partial waves except S and D. This gives

$$f_2(\kappa)=\frac{4\pi\int_0^\infty\int_0^\pi f_0(\kappa')v_0(k,t)P_2(\cos\alpha)[\gamma j_2(as)+sj_2'(as)]\kappa'^2\sin\theta' d\theta' d\kappa'}{ikh_2'(ika)+\gamma h_2(ika)}.$$

As before, we introduce dimensionless parameters $Ka=K_0$, $\gamma a=\gamma_0$, new variables $\xi=\kappa a/K_0$, $\xi'=\kappa' a/K_0$, and define

a new function

$$f_2^*(\xi)=\frac{f_2(\xi)\exp[-K_0(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}]}{(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}}.$$

Then we get

$$f_2^*(\xi)=\frac{\frac{2}{\pi}\int_0^\infty\int_{-1}^1 f_0^*(\xi')\dfrac{\{\cos K_0\bar{t}+K_0(1+\frac{3}{4}\xi'^2)^{\frac{1}{2}}\sin K_0\bar{t}/K_0\bar{t}\}}{\xi^2+\xi'^2+\xi\xi'y+1}P_2(\cos\alpha)[\gamma_0 j_2(K_0\bar{s})+K_0\bar{s}j_2'(K_0\bar{s})]\xi'^2d\xi'dy}{[1+(4/p)+(9/p^2)+(9/p^3)]-\gamma_0[(1/p)+(3/p^2)+(3/p^3)]},$$

where $p=K_0(1+\frac{3}{4}\xi^2)^{\frac{1}{2}}$. Now we can see the basic reason that, for small K_0, $f_2^*(\xi)$ is small. It is that we have to integrate in this equation $f_0^*(\xi)$, a function sharply peaked for small ξ, with $j_2'(K_0\bar{s})$ and $j_2(K_0\bar{s})$, which are functions which vanish for their argument small. Thus for K_0 small the overlap integral is small and f_2^* is small compared with f_0^*.

APPENDIX: INTEGRAL REPRESENTATION FOR $\bar{h}_l(ikr)Y_{lm}(\Omega)$

We want to represent the function $\bar{h}_l(ikr)Y_{lm}(\Omega)$, defined by Eq. (9) of the text, as a Fourier integral. We write

$$\bar{h}_l(ikr)Y_{lm}(\Omega)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int u_{lm}(k,\boldsymbol{\lambda})\exp[i\boldsymbol{\lambda}\cdot\mathbf{r}]d\boldsymbol{\lambda}.$$

Inverting this we get

$$u_{lm}(k,\boldsymbol{\lambda})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\bar{h}_l(ikr)Y_{lm}(\Omega)\exp[-i\boldsymbol{\lambda}\cdot\mathbf{r}]d\mathbf{r}$$

$$=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\bar{h}_l(ikr)Y_{lm}(\Omega)(4\pi\textstyle\sum_{l'm'}(-i)^{l'}j_{l'}(\lambda r)Y_{l'm'}(\Omega_\lambda)Y_{l'm'}{}^*(\Omega))r^2drd\Omega,$$

or

$$u_{lm}(k,\boldsymbol{\lambda})=(2/\pi)^{\frac{1}{2}}(-i)^lY_{lm}(\Omega_\lambda)\int_0^\infty\bar{h}_l(ikr)j_l(\lambda r)r^2dr.$$

We define a function $v_l(k,\lambda)$ by

$$u_{lm}(k,\boldsymbol{\lambda})=(2\pi)^{\frac{3}{2}}v_l(k,\lambda)Y_{lm}(\Omega_\lambda).$$

whence

$$v_l(k,\lambda)=\frac{(-i)^l}{2\pi^2}\int_0^\infty\bar{h}_l(ikr)j_l(\lambda r)r^2dr$$

$$=\frac{(-i)^l}{2\pi^2}\int_a^\infty h_l(ikr)j_l(\lambda r)r^2dr. \tag{A-1}$$

The integral in Eq. (A-1) is an example of a well-known general integral[6]

$$\int f_l(\alpha x)g_l(\beta x)x^2dx=\frac{x^2}{\alpha^2-\beta^2}[\beta f_l(\alpha x)g_{l-1}(\beta x)-\alpha f_{l-1}(\alpha x)g_l(\beta x)],$$

where f_l and g_l can be spherical Bessel, Neumann, or Hankel functions. This gives

$$v_l(k,\lambda)=\frac{(-i)^la^2}{2\pi^2(\lambda^2+k^2)}[\lambda h_l(ika)j_{l-1}(\lambda a)-ikh_{l-1}(ika)j_l(\lambda a)].$$

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[6] See, for example, P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1951), p. 1574.

The case $l=0$ is, written out explicitly,

$$v_0(k,\lambda)=-\frac{e^{-ka}}{2\pi^2 k\lambda}\frac{(\lambda\cos\lambda a+k\sin\lambda a)}{(k^2+\lambda^2)}.$$

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Elastic Scattering of Protons by Nitrogen

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The cross sections for the elastic scattering of protons by nitrogen have been measured in 105 angular distributions ranging in angle from 53° to 155° and in energy from 1.05 Mev to 2.93 Mev. Resonances have been observed at 1065±5 kev, 1557±6 kev, 1743±7 kev, 1803±7 kev, 2344±10 kev, and 2468±10 kev.

INTRODUCTION

THE elastic scattering of protons by nitrogen in the energy range covered by electrostatic accelerators shows, in addition to a number of well-established resonances, a large background of potential scattering. It has been reported from 0.6 Mev to 4.1 Mev in several recent papers.[1–4] Spin and parity assignments have been found for the resonances at 1.065 Mev and 1.557 Mev. The lack of knowledge of the background scattering, on which the emphasis of the present work rests, has been the principal obstacle to establishing the assignments for the others. An early report[5] that the P-wave potential phase shifts were small below 2.0 Mev has been found incorrect. This conclusion arose from an incorrect normalization for the cross sections, which were about 10% too low. The cross sections of Tautfest and Rubin[6] are approximately the same as those of reference 5. Hagedorn *et al.*[2] have compared these results with the more recent data and show that they are consistently low. The results of the present work are available in tabular form in an unpublished report.[7] The present data, which are in agreement with the recent data[1,2] cannot be satisfactorily analyzed with S-waves only and thus imply the presence of P-waves and possibly higher ones. A phase shift analysis of these results which includes P-waves is described in the following paper.[8]

APPARATUS

The proton beam for the work was supplied by the Chalk River electrostatic accelerator. The upper limit to the proton energy available was 3 Mev. The lower limit was about 1 Mev, which was the minimum energy where a resolved peak in the spectrum from the scintillation proton counter could be obtained. The energy of the proton beam was measured by deflecting it through 90° in a uniform magnetic field which, in turn, was measured and controlled by a proton gyromagnetic resonance detector.

The scattering chamber is shown in Figs. 1 and 2.

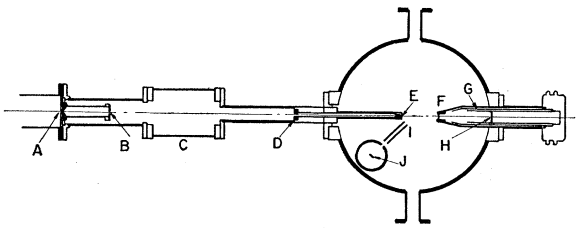

FIG. 1. Gas scattering chamber. A, B, and D are collimating apertures. E and F are thin nickel windows for beam entry and exit. F, G, and H are the beam catcher assembly. I is the counter collimator and J a small scintillator for particle counting mounted on the rotating cover.

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