Reciprocal Static Metrics and Scalar Fields in the General Theory of Relativity

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In the first part of this paper a variant of the idea of reciprocal static solutions of the gravitational field equations is developed. Thus, given any static solution of Einstein's vacuum equations $R_{kl}=0$, a one-parameter family of pairs of solutions of the field equations with scalar field, *viz.*, $R_{kl}=-\mu V_{;k}V_{;l}$, $g^{kl}V_{;kl}=0$, can be written down by inspection. The special cases of spherical symmetry and axial symmetry are treated as explicit examples. In the former case all the solutions of the field equations are obtained in this way. In the second part of the paper the theory is discussed from a physical point of view, for which purpose the motion of a test particle in the spherically symmetric field is treated in some detail.

1. INTRODUCTION

RECIPROCAL static fundamental tensors have been considered by the author in previous papers,[1] both in the context of Riemannian geometry and of the field theory with asymmetric fundamental tensor. The basic idea is this. Let g_{kl} $(k,l=1,2,\cdots,n\geqslant 4)$ be the fundamental tensor. Suppose that there exists a coordinate x^s such that

$$g_{kl,s}=0,\ \ g_{as}=0,\ \ (a,b=1,2,\cdots,s-1,s+1,\cdots,n) \tag{1}$$

i.e., the fundamental tensor $g_{kl}\equiv(g_{ab},g_{ss})$ is "static with respect to x^s." Then the fundamental tensor $'g_{kl}\equiv('g_{ab},'g_{ss})$ reciprocal to g_{kl} is defined to be

$$'g_{kl}=((g_{ss})^{2/(n-3)}g_{ab},(g_{ss})^{-1}). \tag{2}$$

In particular, if g_{kl} is symmetric and its Ricci tensor R_{kl} is zero, then the Ricci tensor $'R_{kl}$ belonging to $'g_{kl}$ also vanishes. Hence, choosing $n=4$, any "static" solution of the gravitational field equations in empty space at once implies the reciprocal solution $'g_{kl}=((g_{ss})^2g_{ab},(g_{ss})^{-1})$, where s need not necessarily be 4 (x^4 being the time coordinate). Further, if g_{kl} defines a distribution of matter which is static in the ordinary sense ($s=4$) and $'g_{kl}$ the reciprocal distribution, then the total energies of the two distributions are related by

$$'U=-U. \tag{3}$$

In the first part of the present paper a variant of the idea of reciprocal solutions is developed in which the introduction of a scalar field is contemplated, such scalar fields having been introduced for instance by Bergmann and Leipnik[2] and by Yilmaz.[3] It will be supposed that the metrics to be considered are four-dimensional, and static in the ordinary sense, i.e., $n=s=4$, though these restrictions are not essential. Then the main formal result to be presented is that the field equations which arise from the Lagrangian

$$L=R+\mu g^{kl}V_{;k}V_{;l},\quad (\mu=\text{const}) \tag{4}$$

where V is a scalar field, *viz.*,

$$R_{kl}=-\mu V_{;k}V_{;l}, \tag{5}$$

$$g^{kl}V_{;kl}=0, \tag{6}$$

have the one-parameter set of pairs of (static) solutions

$$g_{kl}=((g_{44})^{1-\beta}g_{ab},(g_{44})^{\beta}),$$
$$V=\lambda\ln g_{44},\ \beta=\pm(1-2\mu\lambda^2)^{\frac{1}{2}},\ \lambda=\text{const.} \tag{7}$$

if (g_{ab},g_{44}) is any static solution of the usual *vacuum* equations $R_{kl}=0$. [It should be noticed incidentally that when $V\neq\text{const}$, then Eq. (6), which arises from (4) by variation of V, is already contained in Eqs. (5) by virtue of the contracted Bianchi identities.] A result corresponding to that expressed by Eq. (3) again holds. When λ is set equal to zero one has the special case of the usual reciprocal solutions of the vacuum equations.

In the latter part of the paper spherically symmetric and axially symmetric solutions are written down explicitly, these being generated by the well-known Schwarzschild and Weyl solutions, respectively. The world lines of a test particle, for given g_{kl} and V, correspond to the geodesics of a space whose metric is conformal to g_{kl}. This result is used in the course of a physical discussion of the spherically symmetric solutions.

2. STATIC SOLUTIONS OF THE EQUATIONS WITH SCALAR FIELD

(a) To arrive at the required result it is convenient to deal directly with the variational principle. For this purpose consider the scalar curvature density $\mathfrak{R}$ of the space whose static metric is adopted in the form

$$g_{kl}\equiv(e^{2\sigma}g_{ab},e^{2\gamma}). \tag{8}$$

It is not difficult to show[4] that it is given by

$$\mathfrak{R}=e^{\sigma+\gamma}[\mathfrak{R}^*+2\mathfrak{g}^{ab}(2\sigma_{;ab}+\gamma_{;ab}+\sigma_{;a}\sigma_{;b}+\sigma_{;a}\gamma_{;b}+\gamma_{;a}\gamma_{;b})], \tag{9}$$

where $\mathfrak{R}^*$ is the scalar curvature density of the V_3 whose metric tensor is g_{ab} and the subscripts following

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[1] H. A. Buchdahl, Quart. J. Math. (Oxford) **5**, 116 (1954); Australian J. Phys. **9**, 13 (1956); Nuovo cimento **5**, 1083 (1957).

[2] O. Bergmann and R. Leipnik, Phys. Rev. **107**, 1157 (1957).

[3] H. Yilmaz, Phys. Rev. **111**, 1417 (1958).

[4] H. A. Buchdahl, Quart. J. Math. (Oxford) **5**, 116 (1954).

a semicolon denote covariant differentiation with respect to g_{ab}. Now choose $\sigma=(1-\beta)\nu$, $\gamma=\beta\nu$, where β is a constant, and let $\mathfrak{R}(\beta)$ represent the corresponding scalar curvature density. Then

$$\begin{aligned}\mathfrak{R}(\beta)&=e^{\nu}\{\mathfrak{R}^*+2\mathfrak{g}^{ab}[(2-\beta)\nu_{;ab}+(1-\beta+\beta^2)\nu_{;a}\nu_{;b}]\}\\&\doteq e^{\nu}\{\mathfrak{R}^*+2(\beta^2-1)\mathfrak{g}^{ab}\nu_{;a}\nu_{;b}\},\end{aligned}\tag{10}$$

where the symbol $\doteq$ placed between any two expressions means that they are equal to within an *ordinary* divergence. Divergences which form part of $\mathfrak{R}$ may here be rejected for they will not contribute to the field equations whenever $\mathfrak{R}$ enters additively into a Lagrangian. From (10) it follows that

$$\mathfrak{R}(\beta)+2(1-\beta^2)g^{ab}\nu_{;a}\nu_{;b}\doteq\mathfrak{R}(1),$$

so that

$$\delta\int[\mathfrak{R}(\beta)+2(1-\beta^2)\mathfrak{g}^{ab}\nu_{;a}\nu_{;b}]d^{(4)}x\equiv\delta\int\mathfrak{R}(1)d^{(4)}x \tag{11}$$

for all variations of the g_{ab}, ν which vanish at the boundary. Now if $(g_{ab},e^{2\nu})$ is a (static) solution of the field equations for empty space, then the right-hand member of (11) vanishes. At the same time, keeping in mind the remark in parentheses following Eqs. (7), the field equations (5) and (6) will be satisfied by the functions g_{kl},V $[g_{kl}\equiv('g_{ab},'g_{44})]$ if

$$\delta\int(\mathfrak{R}+\mu\mathfrak{g}^{kl}V_{;k}V_{;l})d^{(4)}x=0 \tag{12}$$

for arbitrary variations which vanish on the boundary. If one now takes $'g_{ab}=e^{2(1-\beta)\nu}g_{ab}$, $'g_{44}=e^{2\beta\nu}$, $V=2\lambda\nu$, where λ is an arbitrary constant and $\beta=\pm(1-2\mu\lambda^2)^{\frac{1}{2}}$, then, according to (11), the condition (12) will indeed be satisfied. The result stated in the Introduction is therefore proved.

(b) The total energy U of a static distribution is given by

$$U=\int(\mathfrak{T}^4{}_4-\mathfrak{T}^a{}_a)dxdydz=-(4\pi)^{-1}\int\mathfrak{R}^4{}_4dxdydz. \tag{13}$$

Using the present notation, one finds[4]

$$\mathfrak{R}^4{}_4=\beta\mathfrak{g}^{ab}(e^{\nu})_{;ab}, \tag{14}$$

so that the values of the energy corresponding to the two possible values of β [according to (7)] are numerically equal but opposite in sign. The result embodied in Eq. (3) is a special case of this.

3. SPECIAL CASES

(a) A case of particular interest is that of the spherically symmetric field. With the customary non-isotropic form of the line element, the Schwarzschild metric

$$\begin{aligned}&ds^2=-q^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2)+qdt^2,\\&q=1-2m/r,\end{aligned}\tag{15}$$

is a solution of the vacuum equations. Accordingly, a solution of the field equations with scalar field (5) and (6) is given by

$$\begin{aligned}&ds^2=-q^{-\beta}dr^2-r^2q^{1-\beta}(d\theta^2+\sin^2\theta d\phi^2)+q^{\beta}dt^2,\\&V=\lambda\ln q.\end{aligned}\tag{16}$$

To reduce the metric to the form in which $g_{22}=-(x^1)^2$, one has to make a change of variables $r\rightarrow\rho$, such that

$$\rho=rq^{(1-\beta)/2}.$$

It is therefore better to use isotropic coordinates at the outset, since if the vacuum solution is written in isotropic form, then the solution with scalar field is also isotropic, *viz.*,

$$\begin{aligned}ds^2=&-(1-m/2r)^{2(1-\beta)}(1+m/2r)^{2(1+\beta)}(dx^2+dy^2+dz^2)\\&+(1-m/2r)^{2\beta}(1+m/2r)^{-2\beta}dt^2,\\V=&\,2\lambda\ln[(1-m/2r)/(1+m/2r)].\end{aligned}\tag{17}$$

Moreover, (16), or (17), is the most general solution of the equations in this case except for a trivial arbitrary constant which may be added to V, since in the static case the fourth member of (5), i.e., $R_{44}=0$, gives

$$[(-\det g^{kl})^{\frac{1}{2}}\nu_{,a}g^{ab}]_{,b}=0, \tag{18}$$

while (6) is an exactly similar equation with V replacing ν. When the field is spherically symmetric, it then follows easily that one must have

$$dV/dr=2\lambda d\nu/dr,\quad \lambda=\text{const}. \tag{19}$$

The solution (17) is further considered in the last section below.

(b) By way of another example, the result of Sec. 2 may be applied to the axially symmetric solutions of Weyl. Thus Eqs. (5) and (6) admit the static solutions

$$\begin{aligned}&ds^2=-e^{2\gamma-2\beta\psi}(d\rho^2+dz^2)-\rho^2e^{-2\beta\psi}d\theta^2+e^{2\beta\psi}dt^2,\\&V=2\lambda\psi,\quad\beta=\pm(1-2\mu\lambda^2)^{\frac{1}{2}},\quad\lambda=\text{const}.\end{aligned}\tag{20}$$

provided $\psi(\rho,z)$ and $\gamma(\rho,z)$ satisfy the equations

$$\begin{aligned}&\rho^{-1}(\rho\psi_{,\rho})_{,\rho}+\psi_{,zz}=0,\quad\gamma_{,\rho}=\rho[(\psi_{,\rho})^2-(\psi_{,z})^2],\\&\gamma_{,z}=2\rho\psi_{,\rho}\psi_{,z}.\end{aligned}\tag{21}$$

4. MOTION OF A TEST PARTICLE

(a) The view will here be taken that the scalar field V (hereafter called "S-field") is a hypothetical "ordinary" long-range field existing in space-time, the situation being entirely analogous to the existence of the electromagnetic field. The S-field will then have sources ("S-charges") and a particle may interact directly with the S-field in consequence of an S-charge which it may possess, i.e., not only indirectly through the fact that the metrical field g_{kl} will depend to some extent on the strengths of the sources of the S-field, the form of the equations for the geodesics thus in turn depending on the S-sources. Moreover, space-time will not necessarily be flat even when the S-field is zero. This attitude is at variance with that taken by Bergmann and Leipnik[2] whose suggested interpretation is, to say the least, difficult to understand, and which seems in any case to be quite unsupportable from the point of view of the formal properties of the equations and their solutions themselves. With the present interpretation in mind, one may then contemplate a test particle whose mass η and S-charge ϵ are both infinitesimal. Its equation of motion in the space-time V_4 in the presence of the scalar field V will be[5]

$$u^l[(\eta-\epsilon V)u^k]_{;l}=-\epsilon V_{;}{}^{k}, \tag{22}$$

where u^k is the four-velocity of the particle. If $\Gamma^r{}_{kl}$ are the Christoffel symbols belonging to g_{kl}, (22) may be written

$$du^k/ds+\Gamma^k{}_{st}u^su^t=-(g^{kl}-u^ku^l)(\eta-\epsilon V)^{-1}\epsilon V_{,l}. \tag{23}$$

These equations obviously have the first integral $u_ku^k=1$, as must be the case.

(b) Consider now a Riemann space $^*V_4$ whose metric tensor is $^*g_{kl}$, and which is in conformal correspondence with V_4, i.e., there exists a scalar function χ such that

$$^*g_{kl}=e^{2\chi}g_{kl}. \tag{24}$$

The geodesics of $^*V_4$, using the arc length $^*s$ as parameter, have the equations

$$\frac{d^2x^k}{d^*s^2}+{}^*\Gamma^k{}_{st}\frac{dx^s}{d^*s}\frac{dx^t}{d^*s}=0. \tag{25}$$

Now

$$^*\Gamma^k{}_{st}=\Gamma^k{}_{st}+(\delta^k{}_s\chi_{,t}+\delta^k{}_t\chi_{,s}-g_{st}g^{kr}\chi_{,r}), \tag{26}$$

while

$$d^*s=e^{\chi}ds. \tag{27}$$

Using (26) and (27) in (25), the latter take just the form (23) if one chooses

$$\chi=\ln(\eta-\epsilon V). \tag{28}$$

Hence the world lines of the particle whose motion is described by the equations (22) correspond[6] to the geodesics of the space whose metric is

$$ds^2=(\eta-\epsilon V)^2g_{kl}dx^kdx^l. \tag{29}$$

5. THE SPHERICALLY SYMMETRIC FIELD

(a) Bergmann and Leipnik[2] have solved the equations (5) and (6) above in the spherically symmetric field, but their formal work is disfigured partly by the occurrence of certain errors, as recently pointed out by Treder,[7] and partly by an avoidable complexity which is largely the result of an unfortunate choice of coordinate system and the failure to observe at the outset the result embodied in Eq. (19). Equation (17) is in fact equivalent to all the solutions they obtain. It is convenient to write $m=M/\beta$ when $\beta>0$, with an analogous substitution when $\beta<0$ [see also Sec. 5(c)]. Then (17) becomes

$$ds^2=-f(dx^2+dy^2+dz^2)+gdt^2,$$

where

$$\begin{aligned}f&=1+2M/r+O(r^{-2}),\\ g&=1-2M/r+2M^2/r^2+O(r^{-3}).\end{aligned} \tag{30}$$

[It may be noted in passing that the solution of Yilmaz[3] follows from (17) on taking $\mu=-2$ and letting λ tend to infinity.] Now to study the physical meaning of a field theory of this kind, one should examine the problem of the equations of motion by one of the usual approximation procedures, after first introducing continuous distributions of field-producing and field-responding matter into the Lagrangian. On a somewhat more superficial level, one may content oneself with the assumptions and results of Sec. 4 above. Then let it be first supposed that the field be explored by examining the motion of a test particle carrying no S-charge ($\epsilon=0$), so that its motion is given by the geodesics of the space whose metric is (30). Now in (30) β occurs in f and g only through the terms $O(r^{-2})$ and $O(r^{-3})$, respectively, and the terms explicitly written out are identical with the corresponding terms of the Schwarzschild metric. It follows that the test particle will carry out the familiar motion, including the correct precession of the pericenter, this result being subject to two qualifications: (i) that the orbit is to be calculated only to the usual accuracy, i.e., by an iterative procedure in which only the post-Newtonian terms are retained, that is, all terms which have factors $c^{-2n}(n>1)$ are rejected; (ii) that β be not so small that this process of iteration becomes meaningless, which implies that for all points on the orbit one must require

$$M/r\ll 4\beta^2. \tag{31}$$

Subject to these qualifications the fields corresponding to different choices of the value of β will be observationally indistinguishable as far as their exploration with uncharged test particles is concerned.

(b) When the S-charge of the test particle is not zero, set $\epsilon/\eta=\zeta$, where η is of course assumed to be not zero. It is also convenient to construct a formal definition of the strength σ of the source of the S-field. By analogy with the electrostatic field, take

$$\sigma=(4\pi)^{-1}\int \mathfrak{g}^{st}V_{;st}dxdydz=-2\lambda m, \tag{32}$$

in view of (17). Incidentally, therefore,

$$\lambda=-\sigma(4M^2+2\mu\sigma^2)^{-\frac{1}{2}}. \tag{33}$$

Then in virtue of the results of Sec. 4 the trajectories

[5] E.g., J. Plebański and J. Sawicki, Acta Phys. Polon. **14**, 455 (1955).

[6] Since writing the manuscript it has been pointed out to me that the result (29) occurs essentially already in a slightly different context in a recent paper by C. Jankiewicz, Bull. acad. polon. sci. **6**, 765 (1958).

[7] H. Treder, Phys. Rev. **112**, 2127 (1959).

of the test particle correspond to the geodesics of the space whose metric is

$$ds^2 = -{}^*f(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + {}^*g dt^2,$$

where

$$\begin{aligned} {}^*f &= 1 + 2(1+w)M/r + O(r^{-2}), \\ {}^*g &= 1 - 2(1-w)M/r + (2 - 4w + w^2)M^2/r^2 + O(r^{-3}), \end{aligned} \tag{34}$$

with

$$w = \sigma\zeta/M. \tag{35}$$

Only the case in which the particle does not recede to infinity need be considered, so that the condition

$$w < 1 \tag{36}$$

is to be imposed. (Physically: if σ and ϵ have the same sign and $\sigma\epsilon$ is large enough the repulsion of the charges will exceed the gravitational attraction.) Then the equation of an orbit lying in the plane $\theta = \pi/2$ may be obtained in the usual way, and one obtains the following results. In first approximation the orbit is of course an ellipse, corresponding to the classical Kepler motion under the joint action of two inverse-square fields. In second approximation, if $\tilde{\omega}$ is the precession of the pericenter per revolution for given w, and $\tilde{\omega}_0$ is the precession when $w=0$, then

$$\tilde{\omega}/\tilde{\omega}_0 = 1 - 2w/3 - w^2/6. \tag{37}$$

The precession can therefore have its "usual" value, *viz.*, when $w = -4$. There is, however, nothing in the theory to say why w should have always just this or any other particular value.

(c) Normally the condition $T^4{}_4 > 0$ is imposed upon any physical field, which implies $\mu > 0$, because of (5), and hence β is restricted to the range of values $-1 \leqslant \beta \leqslant +1$. On the other hand, the total energy of the field is given by Eqs. (13) and (14). As one of the field equations (5) is just $R^4{}_4 = 0$, the foregoing solution corresponds of course to the presence of a δ-function source at the origin and it is not difficult to confirm by explicit calculation that the value of the integral on the right of (13) is

$$U = \beta m \ (= M), \tag{38}$$

in harmony with (30). Hence one must take $\beta \geqslant 0$ or $\beta \leqslant 0$ according as $m > 0$ or $m < 0$, respectively. But these two families of solutions are then in effect not distinct, as inspection of (17) shows at once (except insofar as the sign of λ is reversed). Hence in (17)

$$0 \leqslant \beta \leqslant 1, \quad m \geqslant 0. \tag{39}$$

[It might be noted that Bergmann and Leipnik refer to solutions which resemble the Schwarzschild solution "except for the disappearance of the famous finite singularity." This remark is difficult to understand; one can always make the finite singularity of the Schwarzschild solution disappear by simply choosing the wrong (i.e., unphysical) sign for the constant of integration m.]

The solution (17) is remarkably different from the corresponding solution describing an electrically charged particle. Now it is not difficult to see that whereas in linear approximation an electric charge e contributes a term $4\pi e^2/r^2$ to g_{44}, the charge σ of the S-field at the origin does not contribute to g_{44} at all. This peculiar difference arises essentially from a difference in sign of the stresses of the two fields: when $T^4{}_4$ has the same value for both fields then the stresses of the S-field are equal to, but opposite in sign from, those of the electrostatic field (space-time being temporarily taken as flat in this context).

One may perhaps consider the situation in the following way. From Eqs. (32) and (33),

$$M^2 = m^2 - \tfrac{1}{2}\mu\sigma^2. \tag{40}$$

If one now contemplates the family of solutions of given total energy, i.e., of given gravitation-producing mass M, then the analog of the usual Schwarzschild singularity, r_0 $(=\tfrac{1}{2}m)$, increases with increasing source strength of the S-field, r_0 tending to infinity with $|\sigma|$. From this point of view the limiting solution (for which $\beta \to 0$) loses any physical meaning. On the other hand (although the difficulty of clearly distinguishing between the two different types of sources which take part in generating the gravitational field must be conceded), the considerations above, and especially Eq. (40), suggest that one may consistently regard σ and m as in some sense characterizing the strength of the "S-source" and the "non-S-source", respectively. Then, keeping the latter, i.e., m, constant and varying σ, the limiting solution (for which $\beta \to 0$) corresponds to a state of affairs in which both sources are finite, but the stresses of the S-field contribute an amount of negative gravitational potential energy just sufficient to make the total energy zero.

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