# Spherical Gravitational Waves*

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The field equations of the general theory of relativity are solved in the linear approximation for all cases of spherical waves with quadrupole symmetry. Energy is radiated outward by all these waves as determined by the canonical expression for the energy flux. A qualitative check of the validity of this method of calculation is made by the application of the same approximation to cylindrical gravitational radiation, for which an exact solution is known. In this case the exact and the linearized calculations lead to corresponding results.

## I. INTRODUCTION

LIKE all other field theories, the field equations of the general theory of relativity possess solutions which have been interpreted as representing radiation, in this case gravitational radiation. Because of the great complexity of the nonlinear field equations of gravitation, the rigorous solutions obtained so far all have very special symmetries[1–3]; linearized solutions of greater generality are known,[4] but it is not definitely known whether these linearized solutions may be considered as first-order approximations of rigorous solutions having the same symmetry.

In this paper, we shall make no attempt to obtain new rigorous solutions, or even to prove their existence. We shall confine ourselves to the treatment of spherical gravitational waves in the linearized approximation and their transport of energy. The issue of transport of energy in gravitational radiation is beset with a number of complexities, such as the circumstance that in general relativity the concepts of energy density and of energy flux have no invariant local significance. All the proposed energy expressions in general relativity[5–9] obey conservation laws; they all generate infinitesimal coordinate transformations representing time-like displacement.

At best, the total energy, and the total energy flux at infinity, possess invariance properties with respect to certain classes of coordinate transformations, e.g., coordinate transformations that at infinity approach Lorentz transformations. The significance of these restricted classes or groups can be justified only in terms of the preservation of boundary conditions at spatial infinity, such as asymptotic flatness of space-time. Unfortunately, the rigorous solutions in cylindrical coordinates by Einstein and Rosen[1] do not satisfy these boundary conditions on the cylindrical axis. It is at least conceivable that spherical pulses of gravitational radiation are not incompatible with flatness at infinity.[3] Accordingly, we consider that the assignment of a total energy flux to such a spherical wave is not meaningless.

The interest in energy flux by gravitational radiation was originally stimulated by Rosen's discovery[10] that in cylindrical coordinates both energy flux and energy density vanish, a result that was confirmed by Weber and Wheeler.[11] Rosen discovered subsequently that this result is not obtained in quasi-Cartesian coordinates.[12] At any rate, we found that in quasi-Cartesian coordinates all types of spherical quadrupole waves do indeed transport energy (defined in terms of the Poynting vector components of the canonical energy-momentum density pseudotensor), and this result is the principal subject of this paper.

The linearized field equations can be solved to yield spherical waves which arise from (or converge toward) a point mass source. These solutions satisfy the condition of asymptotic flatness at infinity. The validity of the linear approximation can be made plausible by an examination of the relationship between the known exact solution for cylindrical gravitational waves and the corresponding linear-approximation solution. In the linear-approximation solution for spherical waves, as in both the linear-approximation and rigorous[13] solutions for cylindrical waves, energy is observed to travel outward from the source. The existence of this energy flux suggests that also for a rigorous solution of the field equations spherical waves carry energy from the point mass source.

Such radiation presumably exists whenever mass

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points undergo acceleration. Orbiting astronomical systems are examples of such accelerating masses. The existence of energy flux for such systems implies the presence of an outward radiation of energy analogous to electromagnetic radiation. For gravitational systems, the lowest order of mass pole which can give rise to radiation is a mass quadrupole. It will be shown that, just as two physically distinct cases exist for electromagnetic radiation, electric dipoles and magnetic dipoles, so four physically distinct cases exist for spherical gravitational radiation in the linear approximation.

Greek indices will run from 0 to 3, and Latin indices from 1 to 3, except where otherwise noted in Sec. V. Repeated indices indicate that those indices are to be summed over, unless otherwise noted. Partial differentiation is indicated by a comma, i.e., $A_{,\alpha}=\partial A/\partial x^\alpha$. Square brackets around indices indicate antisymmetry, i.e., $F_{[\mu\nu]}=-F_{[\nu\mu]}$. The Minkowski metric is given as $\eta_{\mu\nu}$, where $\eta_{0\kappa}=\delta_{0\kappa}$, $\eta_{kl}=-\delta_{kl}$. In the linearized theory, for which the field quantities and their potentials differ from their flat-space values only by terms which are small in the first order, the subscript 0 will indicate the flat-space value and the subscript 1 will indicate the deviation from this value, e.g.,

$$M^{[\mu\alpha][\nu\beta]}={}_0M^{[\mu\alpha][\nu\beta]}+{}_1M^{[\mu\alpha][\nu\beta]}.$$

This notation will not be used for the metric tensor, for which convention prescribes the linearized expressions

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},$$
$$(-g)^{\frac{1}{2}}g^{\mu\nu}=\eta^{\mu\nu}-\gamma^{\mu\nu}.$$

## II. HERTZ VECTOR FORMULATION OF GRAVITATIONAL THEORY

We shall begin with a review of the Hertz vector method in electromagnetic theory, in order to show how spherical radiation can be studied in this formulation. We shall then apply the Hertz vector formulation to the linear approximation of the general theory of relativity.

Let $\phi^\mu$ be an electromagnetic potential which satisfies the Lorentz gauge condition $\phi^\mu{}_{,\mu}=0$. We can then define the Hertz superpotential $Z^{[\mu\nu]}$ so that $\phi^\mu=Z^{[\mu\nu]}{}_{,\nu}$. Similarly, the conservation law of charge $j^\mu{}_{,\mu}=0$ leads to the definition of a "supercharge" $Q^{[\mu\nu]}$ such that $j^\mu=Q^{[\mu\nu]}{}_{,\nu}$. From the field equations $\Box^2\phi^\mu=-j^\mu$ we therefore have $\Box^2Z^{[\mu\nu]}=-Q^{[\mu\nu]}$. In the absence of sources we have $\Box^2Z^{[\mu\nu]}=0$. This equation has spherical wave solutions of the form

$$Z^{[\mu\nu]}=\frac{1}{r}f^{[\mu\nu]}(t-r). \tag{1}$$

The first derivatives of these superpotentials represent possible electromagnetic potentials. The second derivatives represent field strengths of waves having the general character of spherical dipole waves. The six independent superpotential components can be divided into two groups of three:

$$\begin{aligned}&\text{I.}\quad Z^{[01]},\quad Z^{[02]},\quad Z^{[03]};\\&\text{II.}\quad Z^{[23]},\quad Z^{[31]},\quad Z^{[12]}.\end{aligned} \tag{2}$$

Depending upon which ones of these six components are chosen to be nonzero, we can obtain the different types of dipole waves. In investigating the singular sources of the radiation, we deal only with the second spatial derivatives of the superpotentials, since these derivatives contain the terms with the highest negative powers of $r$.

If we take first all $Z^{[\mu\nu]}$ except $Z^{[03]}$ to be zero, we have

$$Z^{[03]}=Z,\quad \phi_0=Z_{,3},\quad \phi_1=\phi_2=0,\quad \phi_3=Z_{,0}.$$

From $F_{[\mu\nu]}=\phi_{\nu,\mu}-\phi_{\mu,\nu}$ we obtain the following electromagnetic field tensor components near the origin:

$$\begin{aligned}&E_1=F_{[01]}=-(1/r)_{,13}f,\quad E_3=F_{[03]}=-(1/r)_{,33}f,\\&E_2=F_{[02]}=-(1/r)_{,23}f,\quad \mathbf{B}=0.\end{aligned} \tag{3}$$

Similarly, if we set $Z^{[12]}=Z'$ and all other components of $Z^{[\mu\nu]}$ equal to zero, we obtain (again near the origin)

$$\begin{aligned}&B_1=-F_{[23]}=(1/r)_{,13}f,\quad B_3=-F_{[12]}=(1/r)_{,33}f,\\&B_2=-F_{[31]}=(1/r)_{,23}f,\quad \mathbf{E}=0.\end{aligned} \tag{4}$$

The fields of (3) and (4) asymptotically satisfy Coulomb's equations. To this extent they may be likened to static solutions of Maxwell's equations. The case (3) represents the radiation of an electric dipole, whereas the expressions (4) represent the radiation of a magnetic dipole.

We shall now turn to the treatment of the linearized equations of gravitation. These equations may be separated and individually given D'Alembert's form with the help of the so-called De Donder (harmonic) coordinate conditions. In the rigorous formulation of the theory, De Donder's coordinate conditions are

$$\mathfrak{g}^{\mu\nu}{}_{,\nu}=0,\quad \mathfrak{g}^{\mu\nu}=(-g)^{\frac{1}{2}}g^{\mu\nu}. \tag{5}$$

As usual we denote the deviations of the $\mathfrak{g}^{\mu\nu}$ from the Minkowski metric by $-\gamma^{\mu\nu}$, the quantities $h_{\mu\nu}$, defined as $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$, being related to the $\gamma^{\mu\nu}$ as follows:

$$\gamma_{\mu\nu}=h_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\operatorname{tr}h. \tag{6}$$

Then the De Donder conditions in the linearized version take the form

$$\gamma^{\mu\nu}{}_{,\nu}=0. \tag{7}$$

The field equations reduce simply to

$$\Box^2\gamma^{\mu\nu}=-16\pi\kappa\,{}_1T^{\mu\nu}, \tag{8}$$

where $\kappa$ is the gravitational constant.[14]

Just as in the electromagnetic case, we may introduce a "supermetric" in order to complete the separation of the field equations. From the divergence condition (5) we conclude first that there exists a set of

[14] C. Møller, *The Theory of Relativity* (Oxford University Press, Oxford, 1952), p. 314.

quantities $V^{\mu[\nu\alpha]}$, such that

$$\mathfrak{g}^{\mu\nu}=V^{\mu[\nu\alpha]}{}_{,\alpha}. \quad (9)$$

Since $\mathfrak{g}^{\mu\nu}$ is symmetric, we have the following integrability condition:

$$\mathfrak{g}^{\mu\nu}=V^{\mu[\nu\alpha]}{}_{,\alpha}=V^{\nu[\mu\alpha]}{}_{,\alpha}, \quad (10)$$

which may be written as

$$(V^{\mu[\nu\alpha]}-V^{\nu[\mu\alpha]})_{,\alpha}=0. \quad (11)$$

This relationship implies the existence of a quantity $W^{[\mu\nu][\alpha\beta]}$ such that

$$V^{\mu[\nu\alpha]}-V^{\nu[\mu\alpha]}=W^{[\mu\nu][\alpha\beta]}{}_{,\beta}. \quad (12)$$

Interchanging indices, we can also write

$$V^{\alpha[\nu\mu]}-V^{\nu[\alpha\mu]}=W^{[\alpha\nu][\mu\beta]}{}_{,\beta}, \quad (13)$$

$$V^{\alpha[\mu\nu]}-V^{\mu[\alpha\nu]}=W^{[\alpha\mu][\nu\beta]}{}_{,\beta}. \quad (14)$$

When we add Eqs. (12), (13), and (14), we obtain

$$V^{\mu[\nu\alpha]}=U^{\mu\beta[\nu\alpha]}{}_{,\beta}, \quad (15)$$

where

$$U^{\mu\beta[\nu\alpha]}=\tfrac{1}{2}[W^{[\mu\nu][\alpha\beta]}-W^{[\nu\alpha][\mu\beta]}-W^{[\mu\alpha][\nu\beta]}]. \quad (16)$$

The quantities $U^{\mu\beta[\nu\alpha]}$ are potentials of the $V^{\mu[\nu\alpha]}$, arising from the integrability condition (11). We can by-pass the intermediate superpotentials $V^{\mu[\nu\alpha]}$ and combine the $U^{\mu\beta[\nu\alpha]}$ linearly, to obtain a new set of potentials, the "supermetric,"

$$M^{[\mu\alpha][\nu\beta]}=\tfrac{1}{2}[U^{\mu\beta[\nu\alpha]}-U^{\beta\mu[\nu\alpha]}+U^{\nu\alpha[\mu\beta]}-U^{\alpha\nu[\mu\beta]}]. \quad (17)$$

This supermetric is antisymmetric in the index pair $[\mu\alpha]$, as well as in $[\nu\beta]$, and has the further symmetry property

$$M^{[\mu\beta][\nu\alpha]}=M^{[\nu\alpha][\mu\beta]}. \quad (18)$$

By differentiation we verify directly that

$$\mathfrak{g}^{\mu\nu}=M^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}. \quad (19)$$

In the linearized theory, the zeroth-order term of the supermetric will, of course, lead to the Lorentz metric when differentiated twice. Such a supermetric is, for instance,

$${}_0M^{[\mu\alpha][\nu\beta]}=\tfrac{1}{12}(\eta^{\mu\nu}x^\alpha x^\beta+\eta^{\alpha\beta}x^\mu x^\nu-\eta^{\mu\beta}x^\nu x^\alpha-\eta^{\nu\alpha}x^\mu x^\beta). \quad (20)$$

This supermetric formulation of the linearized theory of gravitational radiation is analogous to the Hertz vector formulation of the theory of electromagnetic radiation. With the help of this formalism, we can now establish a far-reaching analogy to the Hertz treatment of an electromagnetic field. In Sec. III these similarities will be employed to study the sources and the energy flux of spherical gravitational radiation.

1. In empty space, the supermetric's components individually satisfy D'Alembert's equation. In the presence of matter, the energy tensor itself, because of its own conservation law, may be obtained from a tensor $P^{[\mu\alpha][\nu\beta]}$, according to the equation

$$T^{\mu\nu}=P^{[\mu\alpha][\nu\beta]}{}_{,\alpha\beta}. \quad (21)$$

In that case the supermetric obeys the set of equations

$$\Box^2\,{}_1M^{[\mu\alpha][\nu\beta]}=-16\pi\kappa\,{}_1P^{[\mu\alpha][\nu\beta]}. \quad (22)$$

2. If ${}_1P^{[\mu\alpha][\nu\beta]}$ vanishes (except possibly for a singular contribution at the origin of the spatial coordinate system), we shall consider solutions of D'Alembert's equation having the form

$${}_1M^{[\mu\alpha][\nu\beta]}=(1/r)f^{[\mu\alpha][\nu\beta]}(t-r). \quad (23)$$

This type of solution leads to spherical waves having the symmetry properties of quadrupole waves.

3. The supermetric has 21 components, just as the Hertz superpotential in the electromagnetic case has six components. We can obtain 21 linearly independent solutions of the field equations by setting one component at a time to be nonzero. These solutions fall at first sight into six distinct symmetry classes. Within each class, solutions may be carried over into each other by simple rotation of the coordinate system. We shall show in Sec. III that two of these classes can be carried over into others by means of infinitesimal coordinate transformations.

The procedure outlined here leads to quadrupole radiation. By differentiating the resulting metric with respect to arbitrary combinations of the spatial coordinates, we may obtain higher multipole symmetries at will, but no monopole or dipole radiation. These types of radiation are excluded, even in the linearized theory, by the conservation laws of general relativity. The conservation of mass prevents the occurrence of monopole radiation, just as the conservation of charge does in electrodynamics; the conservation of linear and angular momentum likewise prevents the occurrence of spherical dipole waves.[15] Hence our procedure is capable of furnishing us with all the spherical waves compatible with the field equations and the harmonic coordinate condition.

## III. SPHERICAL GRAVITATIONAL WAVES IN THE LINEAR APPROXIMATION

We shall now consider in detail the six different symmetry classes into which the 21 independent nontrivial components of the linearized supermetric can be arranged. The spatial indices $r$, $s$, and $t$ below shall be all different from each other. The summation convention is not to operate. The six classes are these:

I. ${}_1M^{[0r][0r]}={}_1M^{\mathrm{I}}$,

II. ${}_1M^{[0s][0t]}={}_1M^{\mathrm{II}}$,

III. ${}_1M^{[0r][st]}={}_1M^{\mathrm{III}}$,

IV. ${}_1M^{[0s][rs]}={}_1M^{\mathrm{IV}}$,

V. ${}_1M^{[st][st]}\quad{}_1M^{\mathrm{V}}$,

VI. ${}_1M^{[tr][rs]}={}_1M^{\mathrm{VI}}$.

[15] R. Sachs and P. G. Bergmann, Phys. Rev. 112, 674 (1958).

Each class contains three independent components, except for class IV, which contains six. The physical significance of this classification will be discussed below.

Infinitesimal coordinate transformations, analogous to the gauge transformation of electrodynamics, will be found to reduce the number of independent classes to four. This reduction results from an examination of the curvature tensor, which in this approximation is Lorentz-covariant as well as invariant under an infinitesimal coordinate transformation. The components of the curvature tensor for classes V and VI are (except for a change of sign) identical with the components of the curvature tensor for classes I and II, respectively. If we take

$$f^{\mathrm{V}}(t-r)=-f^{\mathrm{I}}(t-r),\quad f^{\mathrm{VI}}(t-r)=-f^{\mathrm{II}}(t-r),\quad (24)$$

then the metrics for these classes are related by the following infinitesimal coordinate transformations:

$$\gamma_{\mu\nu}{}^{\mathrm{V}}-\gamma_{\mu\nu}{}^{\mathrm{I}}=\xi_{\mu,\nu}+\xi_{\nu,\mu}-\eta_{\mu\nu}\xi^{\rho}{}_{,\rho},\quad (25)$$

where

$$\xi_0=-\tfrac{1}{2}\,{}_1M^{\mathrm{I}}{}_{,0},\quad \xi_1=\tfrac{1}{2}\,{}_1M^{\mathrm{I}}{}_{,1},\quad \xi_2=\tfrac{1}{2}\,{}_1M^{\mathrm{I}}{}_{,2},\quad \xi_3=-\tfrac{1}{2}\,{}_1M^{\mathrm{I}}{}_{,3},\quad (26)$$

and

$$\gamma_{\mu\nu}{}^{\mathrm{VI}}-\gamma_{\mu\nu}{}^{\mathrm{II}}=\xi_{\mu,\nu}+\xi_{\nu,\mu}-\eta_{\mu\nu}\xi^{\rho}{}_{,\rho},\quad (27)$$

where

$$\xi_0=0,\quad \xi_1=-{}_1M^{\mathrm{II}}{}_{,2},\quad \xi_3=-{}_1M^{\mathrm{II}}{}_{,1},\quad \xi_3=0.\quad (28)$$

Furthermore, three of the six independent components in class IV can be related to the other three by suitable infinitesimal coordinate transformations. The metric $\gamma_{\mu\nu}{}^{\mathrm{IV}}$ obtained from the supermetric ${}_1M^{[0s][rs]}$ is related to the metric $\bar{\gamma}_{\mu\nu}{}^{\mathrm{IV}}$ obtained from ${}_1M^{[0t][rt]}$ by

$$\bar{\gamma}_{\mu\nu}{}^{\mathrm{IV}}-\gamma_{\mu\nu}{}^{\mathrm{IV}}=\xi_{\mu,\nu}+\xi_{\nu,\mu}-\eta_{\mu\nu}\xi^{\rho}{}_{,\rho},\quad (29)$$

where

$${}_1M^{[0s][rs]}=-{}_1M^{[0t][rt]},\quad \xi_0=-{}_1M^{[0s][rs]},\quad \xi_r={}_1M^{[0s][rs]},\quad (30)$$

and the other components of $\xi_\mu$ are zero.

In carrying our calculations, numerical values will be assigned to $r$, $s$, and $t$ in such a manner that the $z$ direction is "preferred" (for the index pair [$0r$] the $r$ direction is preferred, while for the index pair [$rs$] the $t$ direction is preferred). In classes II and IV, where each of the two antisymmetric pairs of indices refers to a different direction, values are assigned to the indices in such a way that the preferred direction for the first pair is the $x$ direction, and the preferred direction for the second pair is the $y$ direction. Thus the over-all preferred direction is the $z$ direction. The following assignment of indices is therefore made:

I. $r=3$,
II. $s=1,\quad t=2$,
III. $r=3,\quad s=1,\quad t=2$,
IV. $r=3,\quad s=1$.

The components of the metric tensor and of the curvature tensor will be calculated for two physically interesting regions: large distances from the source, where only terms of the lowest order in $1/r$ are significant, and small distances from the source, where only terms of the highest order in $1/r$ are significant. The metric in the former region enables us to calculate the components of the energy-momentum density pseudotensor at large distances from the origin, and thus to obtain the energy flux radiated outward through a sphere of large radius centered at the origin. In the latter region we can study the distribution of sources that gives rise to each type of radiation.

At large distances, the linear corrections to the Lorentz metric have the form $[f''(t-r)/r]F(\phi,\theta)$, where $\phi$ and $\theta$ are the polar and azimuthal angles, respectively. For example, we obtain for class I:

$$\gamma^{00}=(f''/r)\cos^2\theta,\quad \gamma^{03}=(f''/r)\cos\theta,\quad \gamma^{33}=(f''/r).\quad (31)$$

If the field equations $R_{\mu\nu}=0$ are satisfied, the independent components of the curvature tensor for class I are, again at large distances,

$$\begin{aligned}
R_{0101}&=\tfrac{1}{4}(f''''/r)\sin^2\theta(\sin^2\phi-\cos^2\theta\cos^2\phi),\\
R_{0202}&=\tfrac{1}{4}(f''''/r)\sin^2\theta(\cos^2\phi-\cos^2\theta\sin^2\phi),\\
R_{0102}&=-\tfrac{1}{4}(f''''/r)\sin^2\theta\sin\phi\cos\phi(1+\cos^2\theta),\\
R_{0103}&=\tfrac{1}{4}(f''''/r)\sin^3\theta\cos\theta\cos\phi,\\
R_{0203}&=\tfrac{1}{4}(f''''/r)\sin^3\theta\cos\theta\sin\phi,\\
R_{0212}&=-\tfrac{1}{4}(f''''/r)\sin^3\theta\cos\phi,\\
R_{0112}&=\tfrac{1}{4}(f''''/r)\sin^3\theta\sin\phi,\\
R_{0113}&=\tfrac{1}{4}(f''''/r)\sin^2\theta\cos\theta(\sin^2\phi-\cos^2\phi),\\
R_{0123}&=-\tfrac{1}{2}(f''''/r)\sin^2\theta\cos\theta\sin\phi\cos\phi,\\
R_{0312}&=0.
\end{aligned}\quad (32)$$

Similar expressions are obtained for the other symmetry types. The energy fluxes for all the different symmetry classes will be discussed in Sec. IV.

Near the sources, the only physically significant components of $\gamma^{\mu\nu}$ are given below for the four symmetry classes:

$$\begin{aligned}
\text{I.}\quad & \gamma^{00}=2\frac{f}{r^3}P_2{}^0(\cos\theta),\\
\text{II.}\quad & \gamma^{00}=\frac{1}{2}\frac{f}{r^3}P_2{}^2(\cos\theta)\sin2\phi,\\
\text{III.}\quad & \gamma^{01}=-\frac{f}{r^3}P_2{}^1(\cos\theta)\sin\phi,\\
& \gamma^{02}=\frac{f}{r^3}P_2{}^1(\cos\theta)\cos\phi,\\
\text{IV.}\quad & \gamma^{01}=\frac{f}{r^3}P_2{}^1(\cos\theta)\cos\phi,\\
& \gamma^{03}=-\frac{f}{r^3}[P_2{}^0(\cos\theta)-\tfrac{1}{2}P_2{}^2(\cos\theta)\cos2\phi],
\end{aligned}\quad (33)$$

where $P_2^0$, $P_2^1$, and $P_2^2$ are associated Legendre polynomials.

These twelve solutions (one for each possible preferred direction in each class) to the field equations are seen to be linearly independent. The four symmetry classes represent four physically distinct types of linearized spherical gravitational quadrupole waves. The corresponding superpotentials form a complete set; any gravitational quadrupole potential in the linearized theory may be expressed as a linear combination of the above twelve.

The four symmetry classes can be interpreted physically as different distributions of mass and mass flux in the pulsating point quadrupole at the origin of spatial coordinates. This mass distribution is given for each symmetry class by the field equations at small distances from the sources:

$$-16\pi\kappa\,{}_1T^{\mu\nu}=\Box^2\gamma^{\mu\nu}=\left(\Box^2\frac{1}{r}f^{[\mu\alpha][\nu\beta]}(t-r)\right)_{,\alpha\beta}=\frac{\partial^2}{\partial x^\alpha\partial x^\beta}F^{[\mu\alpha][\nu\beta]}(t)\delta^3(\mathbf{r}),\quad(34)$$

since

$$\Box^2\frac{1}{r}f(t-r)=F(t)\delta^3(\mathbf{r}).\quad(35)$$

Rosen and Shamir,[16] in discussing a radiating system of the type which we call class I, set

$$F^{[03][03]}(t)=p_0e^{-i\omega t},\quad(36)$$

and all other components of $F^{[\mu\alpha][\nu\beta]}(t)$ equal to zero. The source model for this supermetric consists of two massive particles situated on the $z$ axis and connected by a massless spring, and emits radiation characteristic of an axial quadrupole. The choice of (36) leads to an outgoing sinusoidal gravitational wave solution.

Equations (33) enable us to construct source models for the four types of linearized spherical gravitational waves.[15] The source of class I is a linear mass quadrupole oriented in the $z$ direction, while the source of class II is a plane mass quadrupole in the $x$-$y$ plane. Insofar as the analogy with electromagnetic dipole radiation is appropriate, these two classes of radiation may be regarded as "electric-type" in character. The sources of classes III and IV are distributions of mass flux; the former may be thought of as two circular mass-currents whose directed normals point in opposite directions along the $z$ axis, and which are joined by a massless spring along this axis; and the latter as two circular mass-currents whose normals are also oriented along the $z$ axis but which are joined by a massless spring in the direction of the $x$ axis. Classes III and IV may be called "magnetic-type" sources.

## IV. ENERGY FLUX

We shall take the Poynting vector of gravitational radiation to be the $t_0^k$ components of the canonical energy-momentum density pseudotensor in quasi-Cartesian coordinates[17]:

$$16\pi\kappa(-g)^{\frac{1}{2}}t_\mu{}^\nu=\Gamma_{\alpha\beta}{}^\nu\mathfrak{g}^{\alpha\beta}{}_{,\mu}-\mathfrak{g}^{\nu\alpha}{}_{,\mu}(\ln\sqrt{-g})_{,\alpha}+\delta_\mu{}^\nu\mathfrak{g}^{\lambda\sigma}[\Gamma_{\lambda\beta}{}^\alpha\Gamma_{\sigma\alpha}{}^\beta-\Gamma_{\lambda\sigma}{}^\alpha(\ln\sqrt{-g})_{,\alpha}].\quad(37)$$

In the linear approximation, we can write

$$16\pi\kappa t_0{}^k=-\Gamma_{\alpha\beta}{}^k\gamma^{\alpha\beta}{}_{,0}+\gamma^{\alpha k}{}_{,0}(\ln\sqrt{-g})_{,\alpha}.\quad(38)$$

If the three-volume $V$ is a coordinate sphere centered at the origin, and if $S$ is the surface of the sphere, then the total outward flux of energy across $S$ is

$$\frac{dP_0}{dt}=\int_S t_0{}^kn_kdS,\quad(39)$$

where $n_k=x_k/r$ is the $k$ component of the unit vector normal to $S$.

If the values of the metric tensor for the four symmetry classes are substituted into (38) and (39), we obtain the following values for $t_0^k$ and for $dP_0/dt$, respectively, for large $r$:

$$\begin{aligned}
&\text{I.}\quad 16\pi\kappa t_0{}^k=\frac{(f''')^2}{r^2}\frac{x^k}{4r}\left(1-\frac{z^2}{r^2}\right)^2, && \frac{dP_0}{dt}=\frac{1}{30\kappa}(f''')^2,\\
&\text{II.}\quad 16\pi\kappa t_0{}^k=\frac{(f''')^2}{r^2}\frac{x^k}{r}\left(1-\frac{x^2}{r^2}\right)\left(1-\frac{y^2}{r^2}\right), && \frac{dP_0}{dt}=\frac{1}{10\kappa}(f''')^2,\\
&\text{III.}\quad 16\pi\kappa t_0{}^k=\frac{(f''')^2}{r^2}\frac{x^k}{r}\left(1-\frac{z^2}{r^2}\right)^2, && \frac{dP_0}{dt}=\frac{2}{15\kappa}(f''')^2,\\
&\text{IV.}\quad 16\pi\kappa t_0{}^k=\frac{(f''')^2}{r^2}\frac{x^k}{r}\left(1-\frac{x^2}{r^2}\right)\left(1-\frac{y^2}{r^2}\right), && \frac{dP_0}{dt}=\frac{1}{10\kappa}(f''')^2.
\end{aligned}\quad(40)$$

The linear approximation to the field equations of the general theory of relativity therefore yields spherical gravitational waves, all classes of which carry a finite energy flux.

Since the components of the Poynting vector are

[16] N. Rosen and H. Shamir, Revs. Modern Phys. **29**, 429 (1957). Cf. also: W. B. Bonnor, Roy. Soc. London Phil. Trans. **251**, 233 (1959).

[17] Reference 14, p. 341.

quadratic in the metric tensor, a source which is represented by a linear combination of two or more of the twelve above-mentioned solutions would give rise to an energy flux which would contain cross-terms involving the metrics of both classes.

## V. CYLINDRICAL GRAVITATIONAL RADIATION

Einstein and Rosen[1] have shown that the field equations of the general theory of relativity have rigorous solutions representing cylindrical gravitational waves. These waves are propagated outwards from a singularity located on the cylinder axis; this singularity can be interpreted as being a material source.

These waves do not seem to carry energy and momentum if the canonical energy-momentum density pseudotensor is calculated in cylindrical coordinates.[10,11] However, it has been pointed out[12] that this method of calculation is invalid, since the coordinate system employed does not go over into a Minkowskian system at large distances. Accordingly, the cylindrically symmetric solution of the field equations of the general theory of relativity will be treated in quasi-rectangular coordinates.

A suitable choice of coordinates[1] puts the cylindrically symmetric metric in the form:

$$\begin{aligned} g_{00}&=e^{2(\gamma-\psi)},\quad g_{0k}=g_{03}=g_{k3}=0,\\ g_{kl}&=-e^{-2\psi}[(\delta_{kl}-n_kn_l)+n_kn_le^{2\gamma}],\\ g_{33}&=-e^{2\psi},\end{aligned}\tag{41}$$

where Latin indices run from 1 to 2, $n_i=x_i/\rho$, $n_in^i=1$, and $\gamma$ and $\psi$ are functions of $\rho=(x^2+y^2)^{\frac{1}{2}}$ and $t$ only. The variables $\gamma$ and $\psi$ have the same meaning as in Einstein's and Rosen's papers.[1,10] In the quasi-Cartesian coordinate system which we use, the components of the canonical energy-momentum density pseudotensor differ from theirs. For these components we have

$$16\pi\kappa(-g)^{\frac{1}{2}}t_0{}^k=-2n^ke^{2\gamma}\gamma_{,t}/r.\tag{42}$$

The time rate of outward flow of the linear energy density $\Lambda_0$ in a volume $V$ whose cylindrical surface is $S$ is

$$\frac{d\Lambda_0}{dt}=\int_S(-g)^{\frac{1}{2}}t_0{}^kn_kdS=-\frac{1}{4\kappa}\gamma_{,t}e^{2\gamma}.\tag{43}$$

Rosen has solved the field equations for both pulse waves and periodic waves.[10] However, the periodic wave solution is unphysical, since the boundary conditions at infinity cannot be satisfied. If we take the pulse wave solution

$$\begin{aligned}\psi(r,t)&=\frac{1}{2\pi}\int_0^{t-r}\frac{f(\beta)d\beta}{[(t-\beta)^2-r^2]^{\frac{1}{2}}},\\ \gamma_{,t}&=2r\psi_{,r}\psi_{,t},\\ \gamma_{,r}&=r(\psi_{,r}{}^2+\psi_{,t}{}^2),\end{aligned}\tag{44}$$

where $f(\beta)$ is an arbitrary bounded function which vanishes for negative values of $\beta$, and represents the strength at time $\beta$ of the wave source at $r=0$, the following expression is obtained for the time rate of outward flow of linear energy density:

$$\frac{d\Lambda_0}{dt}=\frac{1}{16\pi^2\kappa}\left[\frac{f(0)}{(t^2-r^2)^{\frac{1}{2}}}+\int_0^{t-r}\frac{f'(\beta)d\beta}{[(t-\beta)^2-r^2]^{\frac{1}{2}}}\right]\times\left[\frac{tf(0)}{(t^2-r^2)^{\frac{3}{2}}}+\int_0^{t-r}\frac{(t-\beta)f'(\beta)d\beta}{[(t-\beta)^2-r^2]^{\frac{3}{2}}}\right]e^{2\gamma}.\tag{45}$$

A finite flux of linear energy density thus exists across a cylindrical surface whose axis is the line $r=0$.

The linear approximation to the cylindrical gravitational field equations may be obtained by the expansion of the components of the metric (41) into power series and retaining only terms linear in $\gamma$ and $\psi$. Equation (43) then becomes

$$d\Lambda_0/dt=-(1/4\kappa)\gamma_{,t}.\tag{46}$$

The linear approximation thus also gives a finite flux of linear energy density across a cylindrical surface whose axis is the line $r=0$. As $\psi_{,r}$ and $\psi_{,t}$ always have opposite signs, $d\Lambda_0/dt$ is always positive.

## VI. DISCUSSION

According to the calculations presented in this paper, spherical gravitational quadrupole waves are of four distinct symmetry types, just as there are two distinct types of radiating dipoles in electrodynamics. All of them carry energy in the direction of propagation. The validity of these conclusions is contingent upon the assumptions made: that spherical waves in the linear approximation represent first-order approximations to solutions of the complete Einstein equations, and that the flux of energy in quasi-Cartesian coordinates, expressed by means of the canonical Poynting vector at spatial infinity, represents an intrinsic property of such a solution, independent of the choice of coordinate system.

Though we have presented in the last section some evidence which appears to speak in favor of these assumptions, its importance should perhaps not be overestimated. The linearized field equations permit infinite wave trains, including sinusoidal waves, which are, however, certainly incompatible with the boundary conditions that the metric be asymptotically flat at spatial infinity. This is because the integral over the energy density of the radiation field extended over large values of $r$ diverges in this case and thus presents us with a self-contradictory task in the second approximation. Though no such obvious internal contradiction besets a wave pulse solution, other difficulties might arise which preclude the existence of rigorous solutions.

Our physical interpretation of the components of the canonical energy-momentum density pseudotensor $t_0{}^k$

as those of the energy flux (Poynting vector) is, of course, based on the conservation law satisfied by $t_\mu{}^\nu$. To the extent that the three-dimensional integral over $t_0{}^0$ may be considered the total mass of the physical system, this interpretation is justified. Studies with test particles of the type carried out by Bondi with cylindrical waves[18] may throw additional light on the physical meaning of energy density and energy flux in general relativity. Because of the involved transformation properties of $\int t_0{}^0 dV$, our hope that it represents an invariant feature of a given solution is inextricably tied up with the boundary conditions at infinity, and with the existence of a very restricted group of coordinate transformations which preserve these boundary conditions. Although it is easy to demonstrate that infinite spherical wave trains are incompatible with such boundary conditions, the converse assumption, that finite wave pulses exist and are susceptible to approximate treatment, is presently at best a reasonable conjecture.

[18] H. Bondi, Revs. Modern Phys. **29**, 423 (1957).

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