Meson-Meson Scattering Term in Pseudoscalar-Pseudoscalar Meson Theory*

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The meson-meson scattering term has been investigated within the static approximation for a nucleon. First a static Hamiltonian is constructed from the renormalizable covariant meson theory in a manner similar to that proposed by one of the present authors. Improvements are that the meson-meson scattering term is included besides the pseudoscalar-pseudoscalar coupling term and that an argument is presented to show that the Foldy transformation is the unique one generating a valid static Hamiltonian, though it was left undetermined before. The resulting static Hamiltonian is then analyzed, for the cases of low-energy S- and P-wave pion-nucleon scattering and threshold photomeson production, in terms of the one-meson approximation of the Chew-Low-Wick formalism, without recourse to perturbation expansion. It is shown in particular that the meson-meson scattering term modifies the Chew-Low effective range plot of the δ_{33}-phase shift, making the renormalized P-wave coupling constant smaller than the conventional plot gives, for a positive coefficient of the meson-meson scattering term in the Hamiltonian. Empirical values of the coupling constant determined through the conventional Chew-Low plot and threshold photomeson extrapolation are shown to be interpretable in terms of the renormalized P-wave coupling constant of 0.08 and the meson-meson scattering term with a coefficient of $\approx +4$ ($\hbar = c = 1$). The present treatment of threshold photoproduction of mesons, however, does not agree with the relativistic dispersion relation. General features of the static model resulting from the *ps-ps* meson theory are summarized in the final section, together with the conclusions obtained. The effects of strange particles and of renormalization have been neglected.

I. INTRODUCTION

A SIMPLE model in which a nucleon is assumed as at rest[1] has been successful in analyzing low-energy phenomena. An entirely covariant approach which makes as few assumptions as possible has also been successfully developed, whose basic equations are known as dispersion relations.[2] These two approaches reflect in many ways characteristics of the renormalizable covariant meson theory. We may, therefore, regard these successes as more or less indirect supports to the current meson theory. It is still quite desirable to have a more direct way of showing its consistency with low-energy data, especially in such a way that two basic coupling constants, the *ps-ps* coupling constant and the meson-meson scattering constant, are uniquely determined. Determination of the latter is particularly interesting since such has never been done and a positive evidence for this term would be a strong support to the renormalizable meson theory. Here one should not confuse the meson-meson scattering term in question with an effective meson-meson interaction[3] which was introduced in connection with high-energy phenomena.

One of the present authors[4] proposed a method of constructing a static Hamiltonian from the relativistic γ_5 Hamiltonian and of carrying out the analysis of the resulting nonlinear Hamiltonian without recourse to perturbation expansion. The essential steps consist of assuming the static approximation for a nucleon after a certain canonical transformation is applied to the original γ_5 Hamiltonian, and then applying to it the Chew-Low-Wick formalism,[1] without using any perturbation expansion, even though the renormalization cannot be carried out rigorously.

In the present paper, the same procedure is applied to the case where the meson-meson scattering term is included, while the pseudoscalar-pseudoscalar (*ps-ps*) coupling term alone was assumed previously.[4] Another improvement is that an argument is presented (Sec. II) to show that the Foldy transformation is the unique one that generates a valid static Hamiltonian, although it was left undetermined before.[4]

The basic question whether this static approximation is good was investigated by the other of the present authors.[5] He estimated the lowest order correction to the static Hamiltonian when the Foldy transformation is applied. The correction was found not to be so great as to invalidate the whole scheme,[4] though it may not be quite negligible.

The previous work[4] has shown that the *ps-ps* coupling term is virtually equivalent, regarding low-energy pion-nucleon scattering, to the static models thus far proposed.[1,6] It is shown in Sec. III that the meson-meson scattering term induces a new term which is quadratic as regards P-wave mesons, besides modifying two coupling constants out of three assumed in the static models.[1,6]

Effects of these modifications are then investigated in Secs. IV and V for the cases of S- and P-wave pion-nucleon scattering, respectively. Interesting points are that the meson-meson scattering term with a positive

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coefficient in the Hamiltonian[7] contributes appreciably to S-wave pair-damping and that it acts to reduce the renormalized P-wave coupling constant from what is determined by the conventional Chew-Low plot of the δ_{33}-phase shift.[1]

The same procedure is applied in Sec. VI to threshold photoproduction of mesons. It is concluded that a smaller coupling constant is prevailing here than the P-wave coupling constant effective in scattering. This point is further criticized in connection with the relativistic dispersion relation on the same subject.[2]

Various parameters relating to the meson-meson scattering term are estimated numerically in Sec. VII, using the Tomonaga intermediate-coupling approximation. These theoretical values are compared with empirical figures. The meson-meson scattering constant is finally estimated as ≈ 4 ($\hbar=c=1$).

In the final section we summarize characteristics of the static model resulting from the renormalizable meson theory and also the main conclusions obtained as regards the meson-meson scattering term.

II. FOLDY TRANSFORMATION

The relativistic γ_5 Hamiltonian is given in the Schrödinger representation by

$$H=\int\bar{\psi}(i\mathbf{p}\cdot\boldsymbol{\gamma}+m)\psi d\mathbf{x}+\tfrac{1}{2}\int[\boldsymbol{\pi}^2+\boldsymbol{\phi}\cdot(\mu^2-\Delta)\boldsymbol{\phi}]d\mathbf{x}$$
$$+if\int\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi d\mathbf{x}+d\int(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2d\mathbf{x}-\delta m\int\bar{\psi}\psi d\mathbf{x}$$
$$-\delta\mu^2\int(\boldsymbol{\phi}\cdot\boldsymbol{\phi})d\mathbf{x}-\delta d\int(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2d\mathbf{x},\quad(1)$$

where the *ps-ps* coupling constant f and the meson-meson scattering constant d are two adjustable parameters. We impose upon the canonical transformation which generates a valid static Hamiltonian, in just the same way as before,[4] the following conditions: it is (i) a charge scalar, (ii) a scalar against space rotation and inversion, (iii) not always invariant under the full Lorentz transformation, (iv) an odd function of $\boldsymbol{\phi}$, and (v) not a function of the derivatives of $\boldsymbol{\phi}$ and its conjugate $\boldsymbol{\pi}$. Another motivation for the final requirement is that the transformation should commute with the meson-meson scattering term. Thus the transformation assumes an expression

$$\exp\left[(if/2m)\int\bar{\psi}\{iv(\chi)+\gamma_4w(\chi)\}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi d\mathbf{x}\right],\quad(2)$$

where $v(\chi)$ and $w(\chi)$ are two arbitrary real and even functions of $\chi=(f/2m)\sqrt{(\boldsymbol{\phi}^2)}$.

[7] This positive coefficient is also consistent with the classical correspondence principle. This point was pointed out to the author by Professor G. Wentzel.

Although (2) is more general than was assumed by Berger, Foldy, and Osborn,[8] we now impose on (2) the same requirements as theirs, namely that this transforms $if\int\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi d\mathbf{x}+\int\bar{\psi}m\psi d\mathbf{x}$ in (1) into a totally even term as regards Dirac matrices. This requirement is equivalent to assuming an expansion of (1) in inverse powers of the nucleon mass m. This explains our motivation for this requirement, since the static approximation is valid only when the nucleon rest energy is the most dominant among all energies involved. Following the same method as Berger *et al.*,[8] we can show that this requirement determines $v(\chi)$ and $w(\chi)$ uniquely as

$$v(\chi)=0,\quad w(\chi)=(\tan^{-1}2\chi)/2\chi,\quad(3)$$

which gives exactly the Foldy transformation. We therefore adopt the Foldy transformation as the unique transformation that permits a valid static approximation to (1). It is added that it was for this transformation that Kanazawa[5] investigated the appropriateness of our procedure.

III. MATHEMATICAL TREATMENT OF THE MESON-MESON SCATTERING TERM

Since the Foldy transformation (2) and (3) commutes with the meson-meson scattering term in (1), the static Hamiltonian which is obtained by approaching the static limit for a nucleon after the Foldy transformation is applied to (1) is the simple sum of those obtained in the previous work[4] and the meson-meson scattering term (d-term):

$$H'=(f/2m)[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\tau}\cdot\boldsymbol{\phi}]f(\chi)$$
$$+(f/2m)^3(\boldsymbol{\tau}\cdot\boldsymbol{\phi})[(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\tau}\cdot\boldsymbol{\phi}](\boldsymbol{\tau}\cdot\boldsymbol{\phi})g(\chi)+2mh(\chi)$$
$$+(f/2m)^2[\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}k(\chi)+k(\chi)\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}]/2$$
$$+d\int(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2d\mathbf{x},\quad(4)$$

where we used the same notations as before,[4] $f(\chi)$, $g(\chi)$, $h(\chi)$ and $k(\chi)$ being defined by (4) and (7) or (9) of reference 4. It is remarked that no perturbation expansion was used to get (4), while the renormalization could not be done and therefore all the infinite terms were simply dropped after the transformation was applied to (1). It is understood correspondingly that the d-term in (4) is well-ordered so that it does not include any self-energy processes.

In the one-meson approximation of the Chew-Low-Wick formalism,[1] we need to evaluate physical nucleon expectation values of commutators of the d-term with meson creation and annihilation operators, $a_{\mathbf{k}\alpha}^*$ and $a_{\mathbf{k}\alpha}$. According to the previous work,[4] it is most convenient to define

$$A_{\mathbf{k}\alpha}=(a_{\mathbf{k}\alpha}^*+a_{\mathbf{k}\alpha})/\sqrt{2},\quad B_{\mathbf{k}\alpha}=i(a_{\mathbf{k}\alpha}^*-a_{\mathbf{k}\alpha})/\sqrt{2}.\quad(5)$$

[8] Berger, Foldy, and Osborn, Phys. Rev. **87**, 1061 (1952).

Then it is straightforward to get

$$[d\text{-term},A_{\mathbf{k}\alpha}] = \sum_{\mathbf{k}'\mathbf{k}''\mathbf{k}'''}\{\sum_{\beta}[d/(2\omega\omega'\omega''\omega''')^{\frac{1}{2}}](a_{\mathbf{k}'\alpha}+a_{-\mathbf{k}'\alpha}^*)\times(a_{\mathbf{k}''\beta}+a_{-\mathbf{k}''\beta}^*)(a_{\mathbf{k}'''\beta}+a_{-\mathbf{k}'''\beta}^*)\}\times(\delta_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}-\delta_{-\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}), \quad (6)$$

$$[d\text{-term},B_{\mathbf{k}\alpha}] = i\sum_{\mathbf{k}'\mathbf{k}''\mathbf{k}'''}\{\cdots\}(\delta_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}+\delta_{-\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}),$$

where the vacant curly bracket stands for the corresponding part of the above expression.

According to Appendix I of reference 6, the physical nucleon expectation values of these commutators change signs if $\mathbf{k}$ is replaced by $-\mathbf{k}$. Upon examining the right-hand sides of (6), we see that $[d\text{-term},A_{\mathbf{k}\alpha}]$ has this property, while $[d\text{-term},B_{\mathbf{k}\alpha}]$ remains unchanged on replacing $\mathbf{k}$ by $-\mathbf{k}$. Thus the expectation value of the latter has to vanish. In regard to the former, we can conclude, because of its dependence on isospin and $\mathbf{k}$ and its parity and Hermiticity, that

$$\langle\psi|[d\text{-term},A_{\mathbf{k}\alpha}]|\psi'\rangle = [i\xi(k^2)/\mu\sqrt{\omega}]\langle u|(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_\alpha|u'\rangle, \quad (7)$$

where the ψ's describe physical nucleons and the u's are corresponding free Dirac spinors and we have defined a real scalar $\xi(k^2)$, a function of k^2 and proportional to d.

We now combine (7) with (19) of the previous work,[4] which leads to

$$\langle\psi|[H',A_{\mathbf{k}\alpha}]|\psi'\rangle = i\{[g+\xi(k^2)]/\mu\sqrt{\omega}\}\langle u|(\boldsymbol{\sigma}\cdot\mathbf{k})\tau_\alpha|u'\rangle, \quad (8)$$
$$\langle\psi|[H',B_{\mathbf{k}\alpha}]|\psi'\rangle=0.$$

It is seen that the renormalized P-wave coupling constant becomes energy-dependent, in general, due to the meson-meson scattering term. This energy dependence is, however, not significant according to Sec. VII, since it is shown there that ξ_1 in the expansion

$$\xi(k^2)=\xi_0+(k/\mu)^2\xi_1+\cdots \quad (9)$$

is small compared with $g+\xi_0$. The two terms g and $\xi(k^2)$ in (8) are schematized in Fig. 1.

We then construct double commutators,

$$[A_{\mathbf{k}'\beta},[d\text{-term},A_{\mathbf{k}\alpha}]] = \sum_{\mathbf{k}''\mathbf{k}'''}[d/2(\omega\omega'\omega''\omega''')^{\frac{1}{2}}][\delta_{\alpha\beta}\sum_{\gamma}(a_{\mathbf{k}''\gamma}+a_{-\mathbf{k}''\gamma}^*)\times(a_{\mathbf{k}'''\gamma}+a_{-\mathbf{k}'''\gamma}^*)+2(a_{\mathbf{k}''\alpha}+a_{-\mathbf{k}''\alpha}^*)\times(a_{\mathbf{k}'''\beta}+a_{-\mathbf{k}'''\beta}^*)]\{-\delta_{\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}+\delta_{-\mathbf{k}+\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}+\delta_{\mathbf{k}-\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}-\delta_{-\mathbf{k}-\mathbf{k}'+\mathbf{k}''+\mathbf{k}'''}\}, \quad (10)$$

and three others which include different linear combinations of the four δ-functions occurring in (10), but otherwise are the same as (10) except for a factor i.

We first notice that all these are symmetric tensors of the second rank with respect to α and β. Therefore, their expectation values are proportional to $\langle u|\tau_\alpha\tau_\beta+\tau_\beta\tau_\alpha|u'\rangle$ or $\delta_{\alpha\beta}$, which enables us to replace

$$(a_{\mathbf{k}''\alpha}+a_{-\mathbf{k}''\alpha}^*)(a_{\mathbf{k}'''\beta}+a_{-\mathbf{k}'''\beta}^*)$$

by

$$\delta_{\alpha\beta}\sum_\gamma(a_{\mathbf{k}''\gamma}+a_{-\mathbf{k}''\gamma}^*)(a_{\mathbf{k}'''\gamma}+a_{-\mathbf{k}'''\gamma}^*)/3.$$

We next observe following even-odd and symmetry properties from explicit expressions of commutators,

	$\mathbf{k}\rightarrow-\mathbf{k}$	$\mathbf{k}'\rightarrow-\mathbf{k}'$;	interchange of $\mathbf{k}$ and $\mathbf{k}'$
$[A_{\mathbf{k}'\beta},[d\text{-term},A_{\mathbf{k}\alpha}]]$:	odd,	odd,	symmetric;
$[B_{\mathbf{k}'\beta},[d\text{-term},A_{\mathbf{k}\alpha}]]$:	odd,	even,	asymmetric;
$[A_{\mathbf{k}'\beta},[d\text{-term},B_{\mathbf{k}\alpha}]]$:	even,	odd,	asymmetric;
$[B_{\mathbf{k}'\beta},[d\text{-term},B_{\mathbf{k}\alpha}]]$:	even,	even,	symmetric.

(11)

According to a general argument (Appendix I of reference 6), the expectation values of these stay unchanged under simultaneous change of signs of $\mathbf{k}$ and $\mathbf{k}'$. We finally remark that these are Hermitian and scalar. We thus can conclude that

$$\langle\psi|[A_{\mathbf{k}'\beta},[d\text{-term},A_{\mathbf{k}\alpha}]]|\psi'\rangle = \eta(k^2,k'^2)\delta_{\alpha\beta}(\mathbf{k}\cdot\mathbf{k}')/\mu^3(\omega\omega')^{\frac{1}{2}}, \quad (12)$$
$$\langle\psi|[B_{\mathbf{k}'\beta},[d\text{-term},B_{\mathbf{k}\alpha}]]|\psi'\rangle=2\zeta(k^2,k'^2)\delta_{\alpha\beta}/(\omega\omega')^{\frac{1}{2}},$$

and the other two vanish. We have here defined two real scalars η and ζ which are both proportional to d and symmetric functions of k^2 and k'^2.

On combining (12) with (22) of the previous paper,[4] we finally get

$$\langle\psi|[A_{\mathbf{k}'\beta},[H',A_{\mathbf{k}\alpha}]]|\psi'\rangle = \eta(k^2,k'^2)\delta_{\alpha\beta}(\mathbf{k}\cdot\mathbf{k}')/\mu^3(\omega\omega')^{\frac{1}{2}},$$
$$\langle\psi|[A_{\mathbf{k}'\beta},[H',B_{\mathbf{k}\alpha}]]|\psi'\rangle=\langle\psi|[B_{\mathbf{k}\alpha},[H',A_{\mathbf{k}'\beta}]]|\psi'\rangle = \lambda(\omega'/\omega)^{\frac{1}{2}}\langle u|\epsilon_{\alpha\gamma\beta}\tau_\gamma|u'\rangle, \quad (13)$$
$$\langle\psi|[B_{\mathbf{k}'\beta},[H',B_{\mathbf{k}\alpha}]]|\psi'\rangle = 2[\lambda_0+\zeta(k^2,k'^2)]\delta_{\alpha\beta}/(\omega\omega')^{\frac{1}{2}},$$

with the same notation as before.[4] We see that the meson-meson scattering term gives rise to a new term [the first of (13)] which is evidently quadratic with respect to P-wave pions; it also affects one of the S-wave pion-pair terms, the isospin-independent one, making its net coupling constant energy-dependent. Diagrams

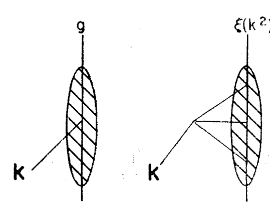

FIG. 1. Diagrams representing processes leading to g and $\xi(k^2)$ in (8). Both give rise to pure P-wave scattering. The shaded area represents the interaction with a physical nucleon.

corresponding to the various parameters are shown in Fig. 2. As for the energy dependence of η and ζ, Sec. VII shows that the expansions

$$\eta(k^2,k'^2)=\eta_0+[(k^2+k'^2)/\mu^2]\eta_1+\cdots,$$
$$\zeta(k^2,k'^2)=\zeta_0+[(k^2+k'^2)/\mu^2]\zeta_1+\cdots, \tag{14}$$

are rather poor, in contrast with (9).

It is instructive to give an explicit expression of the effective interaction Hamiltonian which gives the same expectation values as (8) and (13):

$$H'=(\bar{g}+\bar{\xi}-\bar{\eta}_0)(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\boldsymbol{\tau}\cdot\boldsymbol{\phi}/\mu+\eta_0(\mathrm{grad}\boldsymbol{\phi})^2/2\mu^3 + (\lambda_0+\zeta_0)\boldsymbol{\phi}^2+\bar{\lambda}\boldsymbol{\tau}\cdot\boldsymbol{\phi}\times\boldsymbol{\pi}, \tag{15}$$

where barred parameters are the unrenormalized forms of the unbarred ones; (15) is equivalent to (4), insofar as low-energy pion-nucleon scattering is analyzed in terms of the one-meson approximation of the Chew-Low-Wick formalism.[1] (15) is correct only in the lowest energy region: As the energy gets higher, we have to insert Δ-operators wherever η_0 and ζ_0 appear, corresponding to the energy dependence of η and ζ. It is added that the P-wave coupling constant is $g+\xi_0$ alone, even though $\bar{\eta}_0$ occurs in the first term of (15): The second term also contributes to (8) and cancels out a term with $\bar{\eta}_0$ in the first term.

IV. S-WAVE PION-NUCLEON SCATTERING

We have seen in the previous section that the analysis of S-wave scattering due to Drell *et al.*[6] is modified by the meson-meson scattering term only through the replacement of λ_0 by $\lambda_0+\zeta(k^2,k'^2)$. This energy dependence, however, makes it very complicated to solve the integral equation with $\zeta(k^2,k'^2)$ included. In this paper only qualitative considerations are given below.

We recall a result obtained in the previous work[4] that the *ps-ps* coupling term alone gives, as long as the renormalization is dropped, too large a λ_0 to fit the empirical value determined by Drell *et al.*[6] in the case of the Foldy transformation, while the theoretical value of λ seems reasonable. This situation would be improved if the meson-meson scattering term gives a negative ζ_0, since λ_0 is positive and the d-term does not modify λ at all. According to Table III in Sec. VII, this is the case if d is positive.

We then notice that, according to the calculation by Drell *et al.*,[6] both empirical S-wave phase shifts deviate from the theoretical predictions to the negative side rather soon as the energy gets higher. We see in Sec. VII that ζ_1 in the expansion (14) is opposite in sign to ζ_0 and that ζ_1 is not at all negligible compared with ζ_0. Therefore, the breakdown of the simple S-wave model[6] at comparatively low energy might be explained as due to the energy dependence of $\zeta(k^2,k'^2)$, since $\lambda_0+\zeta(k^2,k'^2)$ increases as the energy in the case when $\zeta_0<0$, because of the opposite sign of ζ_1 to ζ_0. The increase of $\lambda_0+\zeta(k^2,k'^2)$ is equivalent to the increase of the isospin-independent repulsive force, thus making both S-wave phase shifts deviate towards the negative side.

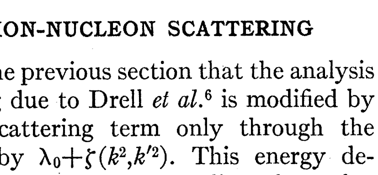

FIG. 2. Diagrams representing processes leading to λ_0, λ and $\eta(k^2,k'^2)$, $\zeta(k^2,k'^2)$ in (13). The shaded area represents the interaction with a physical nucleon.

According to Sec. VII, it seems possible to conclude that the meson-meson scattering term with sign and magnitude which fit low-energy P-wave scattering cancels appreciably, though not perfectly, the strong meson-pair term which is due to the *ps-ps* coupling term. It is, however, important to notice that this cancellation takes place only at the lowest energy, which means that S-wave mesons interact weakly with nucleons only when the wave vector k is smaller than, say, μ; otherwise they interact with nucleons much more strongly.

V. P-WAVE PION-NUCLEON SCATTERING

The main concern here is to see how the new term with $\eta(k^2,k'^2)$ in (13) modifies the analysis of P-wave scattering due to Chew and Low.[1] Again the energy dependence of η induces a great complexity. We, therefore, simply put $\eta=\eta_0$ and try to see how it modifies the conventional Chew-Low plot of the δ_{33}-phase shift in the lowest energy region.

We first define, as usual,[1] scattering amplitudes $h_\alpha(\omega)$, which are related to the phase shifts $\delta_\alpha(\omega)$ in three substates of total isotopic and ordinary angular momenta,

$$h_\alpha(\omega)=\exp[i\delta_\alpha(\omega)]\sin\delta_\alpha(\omega)/k^3. \tag{16}$$

Then we can show, as long as we put $\xi=\xi_0$ and $\eta=\eta_0$, that $h_3(\omega)$ satisfies

$$h_3(\omega)=[(g+\xi_0)^2/3\pi\mu^2\omega]-(\eta_0/12\pi\mu^3) +\frac{1}{\pi}\int k'^3d\omega'\left\{\frac{|h_3(\omega)|^2}{\omega'-\omega-i\epsilon}+\sum_\beta A_{3\beta}\frac{|h_\beta(\omega)|^2}{\omega'+\omega}\right\}, \tag{17}$$

where the matrix $A_{\alpha\beta}$ is defined in an earlier paper.[1]

To follow the argument which led to the conventional Chew-Low plot of δ_{33},[1] we define a complex function $g(z)$ by

$$g(z)=[(g+\xi_0)^2/3\pi\mu^2z][h_3(z)+(\eta_0/12\pi\mu^3)], \tag{18}$$

where $h_3(z)$ becomes $h_3(\omega)$ if z goes to $\omega+i\epsilon$, ϵ being an infinitesimal positive number. It is not difficult to show that almost the same arguments as those due to Chew and Low[1] are applicable to $g(z)$, eventually leading to a modified Chew-Low plot:

$$(k^3\cot\delta_{33})/\omega=[1+(\eta_0/12\pi)\langle k^3\cot\delta_{33}/\mu^3\rangle_0] \times[3\pi\mu^2/(g+\xi_0)^2][1-(\omega/\omega_0)+\cdots], \tag{19}$$

where $\langle\ \ \rangle_0$ implies the zero-energy limiting value of the quantity inside the bracket, which is known to be 5 or 6 according to a recent experiment.[9] It is added that (19) is just the low-energy limiting form of the new effective-range expansion: We found that the new plot resulting from (18) seems to improve appreciably the deviation from experimental data of the conventional Chew-Low plot[1] at and above the resonance energy. According to Sec. VII, however, the approximation $\eta=\eta_0$ is not justified around the resonance energy; thus, only the low-energy limiting form (19) has a sound theoretical basis.

We now reach the most important conclusion of the present paper: the renormalized P-wave coupling constant $(g+\xi_0)^2/4\pi$ and the one which is determined by the conventional Chew-Low plot, $[(g+\xi_0)^2/4\pi]_{\text{C-L}}$, are related by

$$(g+\xi_0)^2/4\pi=[(g+\xi_0)^2/4\pi]_{\text{C-L}} \times[1+(\eta_0/12\pi)\langle k^3\cot\delta_{33}/\mu^3\rangle_0]. \quad (20)$$

According to recent determinations,[10]

$$[(g+\xi_0)^2/4\pi]_{\text{C-L}}=0.095\pm0.006, \quad (21)$$

and

$$(g+\xi_0)^2/4\pi=0.08. \quad (22)$$

The latter figure is due to relativistic dispersion relations: It is quite adequate to identify $(g+\xi_0)^2/4\pi$ as the coupling constant determined through relativistic dispersion relations, since it would automatically include the effect of the meson-meson scattering term.

The difference between values (21) and (22) is not very large compared with the uncertainties involved in these determinations. However, if we admit that the conventional Chew-Low plot does give a larger value than do dispersion relations, then (20) uniquely determines the sign of η_0; namely η_0 should be negative, since $\langle k^3\cot\delta_{33}/\mu^3\rangle_0$ is positive.[9] According to Table III in Sec. VII, a negative η_0 corresponds to a positive d. It is interesting to recall that a positive d was also required to have favorable effects on low-energy S-wave scattering. Equations (20), (21), and (22) are used in Sec. VII to estimate the magnitude of d.

VI. THRESHOLD PHOTOPRODUCTION OF MESONS

We apply the same method when the electromagnetic interaction is introduced. We assume the static approximation after having applied the Foldy transformation to the γ_5 Hamiltonian (1) with the electromagnetic interaction included. Then, because the electromagnetic field does not interact with the meson-meson scattering term, we get an additional Hamiltonian which is the same as (24) of reference 4. As for threshold photoproduction of mesons, it can be simplified[4] to

$$H_{\mathbf{A}}'=-(e\bar{g}_\gamma/\mu)(\boldsymbol{\sigma}\mathbf{A})(\boldsymbol{\tau}\times\boldsymbol{\phi})_3 -e\int\mathbf{A}(\phi_2\nabla\phi_1-\phi_1\nabla\phi_2)d\mathbf{x}, \quad (23)$$

where $\bar{g}_\gamma$ is the unrenormalized form of g_γ which is defined by (28) of reference 4. As was remarked there,[4] the same Hamiltonian follows also when the electromagnetic interaction was introduced directly into the static Hamiltonian (4).

It is one of the characteristics of the general static model resulting from the *ps-ps* meson theory that g_γ is not the same as g. The reason is that the static Hamiltonian (4) is highly nonlinear as regards S-wave mesons. It was, however, shown[4] that, when the Foldy transformation is assumed, g and g_γ are nearly the same. We, therefore, simply put g_γ equal to g in this paper.

Since (23) is not modified, it follows[4,6] that threshold photoproduction of mesons depends solely on g except for the effect of rescattering the photoproduced meson by the nucleon: The meson-meson scattering term affects threshold photoproduction of mesons only through the S-wave pion-nucleon scattering amplitude appearing in the integral equation. According to Sec. IV, the S-wave analysis due to Drell *et al.*[6] is not modified by the meson-meson scattering term, as far as the low-energy limit is concerned. Thus the coupling constant determined through the conventional photomeson extrapolation is $g^2/4\pi$, which is not the same as the renormalized P-wave coupling constant $(g+\xi_0)^2/4\pi$.

Turning to the experimental side, the most recent analysis[11] reports 0.073. If we further take into account the effect of S-wave rescattering according to Drell *et al.*,[6] this is reduced by 15% to

$$g^2/4\pi=0.062. \quad (24)$$

Though the photomeson extrapolation is still quite ambiguous, it seems that it gives smaller values than $(g+\xi_0)^2/4\pi$ or 0.08. If we assume that this discrepancy is real, this also determines the sign of d: According to Table IV, Sec. VII, a positive ξ_0 is attained by a positive d. This sign is again the same as was determined by low-energy P- and S-wave pion-nucleon scattering, independently. The numerical values of $g^2/4\pi$, corresponding to our final estimates of the magnitude of d, are given in Sec. VII.

We should, however, compare our result with the relativistic dispersion relation.[2] According to (22.6) of the last paper of reference 2, threshold photoproduction of mesons depends, except for effects due to nucleon magnetic moments, on the renormalized P-wave coupling constant and on a small correction which is there denoted as $N^{(-)}$. This $N^{(-)}$ is the correction due

[9] S. Barnes, *Report on the Seventh Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, New York, 1957).
[10] G. Puppi and A. Stanghellini, Nuovo cimento 5, 1305 (1957).
[11] Cini, Gatto, Goldwasser, and Ruderman, Nuovo cimento 70, 243 (1958).

to rescattering of the photoproduced meson. It is, therefore, seen that our result does not agree with the relativistic dispersion relation.

To see the origin of this difference, let us consider the Kroll-Ruderman theorem.[12] This theorem is also satisfied by our static Hamiltonian (23) plus (4) as long as the line current is properly assumed so that the total Hamiltonian satisfies the gauge invariance requirements[6]; thus threshold photoproduction of mesons in the zero-meson-mass limit determines $g+\xi_0$, and not g. Therefore our result is not inherent in our static approximation but is entirely due to the one-meson approximation.

If there were no meson-meson scattering term, our result would be consistent with the relativistic dispersion relation, since the same g appears throughout and $N^{(-)}$, the effect of rescattering, is also included as an additive correction in our calculation.[6]

We, therefore, would have to conclude that the one-meson approximation of the Chew-Low-Wick formalism[1] is not adequate when the meson-meson scattering term is included, if the conventional photomeson extrapolation does really give the same coupling constant as the relativistic dispersion relation.

VII. NUMERICAL ESTIMATION

To estimate numerically the various parameters, we need knowledge concerning the structure of a physical nucleon. We here assume that the Tomonaga intermediate-coupling approximation is good enough and also that only S- and P-wave mesons interact with a nucleon. Then we may expand

$$a_{\mathbf{k}\alpha}=(2\pi/\omega)^{\frac{3}{2}}[fY_{00}(\theta\varphi)a_\alpha -i(g/\mu)\textstyle\sum_m kY_{1m}(\theta\varphi)b_{\alpha m}], \quad (25)$$

where a_α and $b_{\alpha m}$ are annihilation operators of S- and P-wave mesons with configurations $f(k)=fk/\omega\sqrt{\omega}$ and $g(k)=gk^2/\mu\omega\sqrt{\omega}$, respectively, being normalized as $\int f^2(k)dk=\int g^2(k)dk=1$.

Let us expand $|\psi\rangle$ in terms of zero-, one-, and two-meson eigenstates $|\psi_i\rangle$ as

$$|\psi\rangle=\sum_{i=1}^{5} C_i|\psi_i\rangle, \quad (26)$$

where $|\psi_i\rangle$ is constructed in terms of $a_\alpha{}^*$ and $b_{\alpha m}{}^*$ in a well-known manner.[13] After straightforward calculations, we get from (7) and (12)

$$\begin{aligned}\xi(k^2)&=(10d/3\sqrt{2})[C_2C_3F_\xi+\sqrt{3}C_2C_4G_\xi],\\ \eta(k^2,k'^2)&=(10d/3)[(6^{\frac{1}{2}}C_1C_3+2C_3{}^2)F_\eta\\ &\quad+(3\sqrt{2}C_1C_4+C_2{}^2+2C_4{}^2+2C_5{}^2)G_\eta], \quad (27)\\ \zeta(k^2,k'^2)&=(5d/3\mu)[(6^{\frac{1}{2}}C_1C_3+2C_3{}^2)F_\zeta\\ &\quad+(3\sqrt{2}C_1C_4+C_2{}^2+2C_4{}^2+2C_5{}^2)G_\zeta],\end{aligned}$$

where

$$\begin{aligned}F_\xi&=\frac{2f^2g}{\pi}\int\int\frac{k'k''dk'dk''}{(k'^2+k''^2+2\mu^2)}K(2k'^2,k^2)\\ &\qquad\times\log\frac{(k'+k'')^2+2\mu^2}{(k'-k'')^2+2\mu^2},\\ G_\xi&=-\frac{g^3}{\pi\mu^2}\int\int\frac{k'k''(k'^2-k''^2)dk'dk''}{(k'^2+k''^2+2\mu^2)}K(2k'^2,k^2)\\ &\qquad\times\log\frac{(k'+k'')^2+2\mu^2}{(k'-k'')^2+2\mu^2},\\ K(k'^2,k^2)&=\left(\frac{1}{3\omega'^2}+\frac{2\mu^2}{3\omega'^4}\right)\\ &\qquad+k^2\left(\frac{1}{15\omega'^4}+\frac{8\mu^2}{15\omega'^6}-\frac{8\mu^4}{5\omega'^8}\right),\\ F_\eta&=4f^2\mu^3\int dk''\left[\left(\frac{k''^2}{\omega''^6}-\frac{4\omega''^4}{3\omega''^8}\right)\right.\\ &\qquad\left.-2(k^2+k'^2)\left(\frac{k''^2}{\omega''^8}-\frac{4k''^4}{\omega''^{10}}+\frac{16k''^6}{5\omega''^6}\right)\right],\\ G_\eta&=4g^2\mu\int dk''\left[\left(\frac{5k''^4}{3\omega''^6}-\frac{4k''^6}{3\omega''^8}\right)\right.\\ &\qquad\left.-2(k^2+k'^2)\left(\frac{7k''^4}{3\omega''^8}-\frac{28k''^6}{5\omega''^{10}}+\frac{16k''^8}{5\omega''^2}\right)\right],\\ F_\zeta&=2f^2\mu\int dk''\left[\frac{k''^2}{\omega''^4}-(k^2+k'^2)\left(\frac{k''^2}{\omega''^6}-\frac{4k''^4}{3\omega''^8}\right)\right],\\ G_\zeta&=\frac{2g^2}{\mu}\int dk''\left[\frac{k''^4}{\omega''^4}-(k^2+k'^2)\left(\frac{5k''^4}{3\omega''^6}-\frac{4k''^6}{3\omega''^8}\right)\right].\end{aligned} \quad (28)$$

The double integrals in F_ξ and G_ξ were evaluated numerically. The numerical results are summarized in Table I for three choices of $k_{\max}$, the cutoff momentum, where

$$\begin{aligned}F_\xi&=F_\xi{}^0+(k^2/\mu^2)F_\xi{}^1,\\ G_\xi&=G_\xi{}^0+(k^2/\mu^2)G_\xi{}^1,\\ F_\eta&=F_\eta{}^0+[(k^2+k'^2)/\mu^2]F_\eta{}^1,\\ G_\eta&=G_\eta{}^0+[(k^2+k'^2)/\mu^2]G_\eta{}^1,\\ F_\zeta&=F_\zeta{}^0+[(k^2+k'^2)/\mu^2]F_\zeta{}^1,\\ G_\zeta&=G_\zeta{}^0+[(k^2+k'^2)/\mu^2]G_\zeta{}^1.\end{aligned} \quad (29)$$

Table I shows that the energy-dependent coefficients ξ_1, η_1, ζ_1 have always opposite signs to ξ_0, η_0, ζ_0. This implies that all three parameters ξ, η, ζ become less effective as the energy gets higher. It is also noticed

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[12] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).

[13] We here use the same notations as A. Kanazawa and M. Sugawara, Progr. Theoret. Phys. (Kyoto) **16**, 95 (1956).

TABLE I. Numerical values of integrals defined by (28) and (29) for three values of cutoff momentum, $k_{\max}$, together with corresponding values of f and g.

$k_{\max}$	F_ξ^0	F_ξ^1	G_ξ^0	G_ξ^1	F_η^0	F_η^1	G_η^0	G_η^1	F_ζ^0	F_ζ^1	G_ζ^0	G_ζ^1	f	g
4μ	0.0950	−0.00354	0.0512	−0.00405	0.239	−0.072	0.160	−0.0305	0.970	−0.119	0.708	−0.0798	0.943	0.406
5μ	0.0806	−0.00262	0.0451	−0.00257	0.199	−0.061	0.101	−0.0180	0.887	−0.100	0.592	−0.0506	0.866	0.311
6μ	0.0706	−0.00198	0.0401	−0.00176	0.175	−0.054	0.070	−0.0118	0.827	−0.088	0.513	−0.0350	0.815	0.254

that the expansions (14) are rather poor. On the other hand, the expansion (9) is much better.

We now assign numerical values to the C_i's in the expansion (26). To obtain a general idea, let us assume four assignments as given in Table II. These were taken from the present authors' calculations[14] on the meson cloud around a physical nucleon, which have a similar basis to that adopted here. Cases I and II correspond to columns A_2 and A_6 of Table I of reference 13, respectively. Since the meson-pair term seems to have been suppressed too much in that paper,[13] we add in Table II two extra cases III and IV, in which C_3 is strengthened, with the corresponding depression of C_1 and C_2, as compared with cases I and II, respectively. All these assignments are, however, quite arbitrary. It is added that C_1 and C_2 are zero- and one-meson amplitudes, C_3 is the amplitude including two S-wave mesons and C_4 and C_5 are two independent amplitudes corresponding to two P-wave mesons.

We need also to fix $k_{\max}$ in Table I. We remark that the F's in (27) are connected with C_3 or the contributions from S-wave mesons, while the G's are due to P-wave mesons. Therefore, let us assume $k_{\max}=6\mu$ in Table I for all G's, while we cut off at $k_{\max}=4\mu$ for all F's, since such a smaller cutoff was required by Drell *et al.* in their analysis of S-wave pion-nucleon scattering.[6] This choice is again quite arbitrary and has no firm basis. These assignments give, from (27), (29), and Tables I and II, the values summarized in Table III.

We see from Table III that, if d is positive, ξ_0 is positive and the other parameters are negative, which has to be the case according to the previous sections. This important conclusion is entirely due to the fact that the first terms containing C_3 linearly on the right-hand sides of (27) are always the dominant ones for C_3 not quite negligibly small. Therefore, a repulsive meson-pair term with at least appreciable magnitude is definitely necessary to infer the positive sign of d.

TABLE II. Four assignments for the C_i's in the expansion (26) normalized to unity.

	C_1	C_2	C_3	C_4	C_5
I	0.7746	−0.5196	−0.3178	0.0775	0.1517
II	0.8562	−0.4243	−0.2757	0.0447	0.0949
III	0.7616	−0.4472	−0.4359	0.0775	0.1549
IV	0.8173	−0.4231	−0.3768	0.0447	0.0949

[14] See reference 13. It is remarked that there was an error in the paper; the numerical values in Table I of reference 13 are probabilities, normalized to unity, while the signs are those of probability amplitudes. The negative sign of C_2 is related to a positive *ps-ps* coupling constant, the sign of which is entirely arbitrary.

We now estimate the magnitude of d from (20), (21), (22) and an empirical value[9] of $\langle k^3 \cot\delta_{33}/\mu^3\rangle_0$. These are summarized in Table IV, together with other parameters.

From the values in Table IV, we conclude finally that

$$d \approx 4 \ (\hbar = c = 1), \tag{30}$$

though this could be wrong by a factor of, say, 2, since the estimation of ξ, η, and ζ depends strongly on the detail of the structure of a physical nucleon which we know only poorly. It is seen that η_0 is almost -1, while ξ_0 is roughly a 10 to 20% correction to g which is almost $+1$.

TABLE III. Numerical values of ξ_0, η_0, and ζ_0 defined by (9), (14), (27), and (29), corresponding to the four assignments in Table II.

Case	ξ_0	η_0	$\mu\zeta_0$
I	$+0.0304d$	$-0.183d$	$-0.150d$
II	$+0.0231d$	$-0.254d$	$-0.377d$
III	$+0.0380d$	$-0.236d$	$-0.264d$
IV	$+0.0324d$	$-0.291d$	$-0.456d$

We see also that $g^2/4\pi$ in Table IV agrees well with the present experimental value (24).

It is also remarked that $\mu\zeta_0$ in Table IV is always larger than the empirical value (0.4) of $\mu(\zeta_0+\lambda_0)$ determined by Drell *et al.*[6] This implies that $\mu\zeta_0$, the contribution from the meson-meson scattering term, has, in fact, an appreciable effect on the S-wave pair-damping. It is also evident that $\mu\zeta_0$ *alone* can never explain the strong damping in question. It is suggested that the renromalization is vitally important in this respect.

VIII. SUMMARY AND CONCLUSION

The *ps-ps* coupling term gives a highly nonlinear static Hamiltonian, which is, however, almost equivalent to the conventional static models[1,6] thus far proposed, concerning low-energy pion-nucleon scattering and threshold photoproduction of mesons.[4]

Since the meson-meson scattering term commutes with the generating transformation adopted (the Foldy transformation), it need simply be added to the above Hamiltonian. This is true even when the electromagnetic field exists, because it does not interact with the meson-meson scattering term.

The net effect of this meson-meson scattering term, is, however, not so simple as it appears. According to Sec. III, its effect upon low-energy pion-nucleon scattering is well reproduced by the effective Hamiltonian (15). The second term, which is quadratic as regards P-wave mesons, is an essentially new term. Such a term is also expected as a nonstatic correction.[5] It is, however, important to remark that this second term is much larger in magnitude than is expected from a nonstatic correction.[5] It is remarked that (15) is correct only in the lowest energy region; terms with ξ, η, and ζ are in general energy-dependent (or include Δ-operators).

This extra P-wave term has an interesting consequence; it modifies the Chew-Low plot of the δ_{33}-phase shift, such that the renormalized P-wave coupling constant becomes smaller than is simply determined by the plot, for a positive coefficient of the meson-meson scattering term.

For this choice of the sign, it is shown that ξ_0 has the same sign as g, while ζ_0 is negative. This sign of ξ_0 implies that the net P-wave coupling constant effective in pion-nucleon scattering is larger than the ps-ps coupling constant alone. On the other hand, a negative ζ_0 contributes to cancelling λ_0 or contributes, partially, to the S-wave pair-damping.

TABLE IV. Final estimate of d and other related parameters, corresponding to the four cases of Table II.

Case	d	ξ_0	η_0	$\mu\zeta_0$	$g^2/4\pi$
I	5.9	+0.179	−1.08	−0.885	0.054
II	4.3	+0.099	−1.09	−1.62	0.065
III	4.6	+0.175	−1.09	−1.21	0.055
IV	3.7	+0.120	−1.08	−1.69	0.062

If the same procedure is applied to threshold photoproduction of mesons, it is concluded that the coupling constant determined through the conventional photomeson extrapolation is the ps-ps coupling constant, which is, therefore, smaller than the renormalized P-wave coupling constant. This also seems to agree with the present data, though not agreeing with the relativistic dispersion relation.

Although all these are very small effects, it seems possible to conclude that the empirical values of the Chew-Low plot and threshold photomeson extrapolation are consistent with the meson-meson scattering term with a coefficient of $\approx +4$ ($\hbar = c = 1$) and the renormalized P-wave coupling constant of 0.08, though this final estimate could be wrong by a factor of, say, 2. This result may be regarded as a positive support to the renormalizable meson theory.

As regards the S-wave pair-damping, the reduction induced by the meson-meson scattering term is appreciable but would not be sufficient by itself. The pair-damping would be explained in combination with the renormalization[15] which has been completely neglected in this work.

Numerical details and even the sign of d depend very much upon the assumption concerning the structure of a physical nucleon. To infer the positive sign of d, it is essential to assume a non-negligible negative amplitude corresponding to two S-wave mesons. This implies the existence of a repulsive meson-pair interaction of appreciable magnitude. It is added here that this is not in contradiction with the pair-damping, since the damping is just an accidental cancellation, according to the present work, which takes place only in the lowest energy limit; in most energy regions, S-wave mesons interact with nucleons more strongly. We thus reasonably expect an appreciable amplitude corresponding to two S-wave mesons.

What has been neglected in the present work are the effects of renormalization and also of strange particles.

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[15] A. Klein, Phys. Rev. 95, 1061 (1954).