A small correction was made for the effect of the U^{235} content of the uranium sample.

RESULTS

Six determinations of the ratio $\nu_{Th^{232}}/\nu_{U^{238}}$ were made. These were averaged using the method of least squares to obtain the most probable value $\nu_{Th^{232}}/\nu_{U^{238}}=0.98 \pm 0.08$ ($E_n=1.4$ Mev). A "best value" of $\nu_{U^{238}}$ at $E_n=1.4$ Mev was obtained by least-squares fitting the linear expression $\nu_{U^{238}}=a+bE_n$ to the existing experimental values of $\nu_{U^{238}}$. This linear fit to the experimental data is shown in Fig. 1. Using the "best value" of $\nu_{U^{238}}=2.63$, this experiment yielded $\nu_{Th^{232}}=2.58\pm0.20$ where the error pertains only to uncertainties in this measurement and does not reflect inaccuracies in the requisite value of $\nu_{U^{238}}$.

The result of this experiment is compared in the figure with the measurements at 3.5 and 14.2 Mev. It is evident that either $\nu_{Th^{232}}$ is not linearly dependent on incident neutron energy, or at least one of the experimental measurements is in error. It is perhaps interesting to point out that this experiment and the work at 14.2 Mev were carried out using essentially monoenergetic neutron sources. The results of both of these experiments indicate that $\nu_{Th^{232}}\simeq\nu_{U^{238}}$. The measurement of $\nu_{Th^{232}}$ at an "effective" neutron energy of 3.5 Mev utilized a continuous spectrum from a fast reactor.

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Mirror Nuclei Radii Utilizing Self-Energy Term and Nonuniform Charge Distributions

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The nature of the self-energy term in the mirror nucleus energy-difference formula is investigated. Two approaches are used. In the first this self-energy term is assumed to be a constant equal to the Coulomb self-energy of a single proton, and in the second a more refined quantum mechanical approach based on the Swamy and Green Coulomb exchange energy calculations is used. Both approaches yield r_0 values which possess the correct general trend with increasing A, but which disagree with theoretical values for very low A. The effect of nonuniform charge distributions on the values of nuclear radii obtained from mirror nuclei is investigated, and expressions for the Coulomb energy for various charge distributions are given. A direct comparison between the mirror nucleus radii and those obtained from electron scattering is made in the few cases where this is possible. Finally, the possible validity of a suggested value of 0.58 Mev for the Coulomb self-energy of the proton is discussed briefly.

THE use of the Coulomb energy differences between mirror nuclei for determining nuclear radii is well known. The method requires the assumption of charge symmetry, and in addition the adoption of some particular model of nucleon or charge distribution. Earlier workers assumed a uniform distribution of charge; later work assumed a uniform distribution of nucleons, but introduced the "exchange energy" term. Still more recently, calculations have been based on the assumption of more realistic nuclear models which reproduce shell features. The situation has been reviewed by Kofoed-Hansen,[1] where a complete list of references is given.

The present investigation originated as an attempt to provide a direct comparison between the mirror nuclei method and another method of nuclear radius determination, *viz.*, that of electron scattering. This method has been surveyed by Hofstadter,[2,3] and further articles of interest appear in Part 1 of *Reviews of Modern Physics* for April, 1958. The quantity which the electron scattering workers measure is the effective nuclear charge distribution. If, then, one calculates the classical Coulomb energy $W_C(A,Z)$ from the usual expression

$$W_C(A,Z)=16\pi^2\int_0^\infty\left[\int_0^r x^2\rho(x)dx\right]r\rho(r)dr, \quad (1)$$

one might expect that the relation

$$E_C(Z+1,Z)=W_C(A,Z+1)-W_C(A,Z) \quad (2)$$

would hold. E_C is the mirror nuclei energy difference [Kofoed-Hansen, Eq. (5), gives exact definitions of E_C], and $\rho(r)$ is the effective (non-normalized) nuclear charge distribution. Spherical symmetry is assumed. Since we are dealing with an effective charge distribution, and not with individual proton wave functions, the "exchange energy" term would not appear to be necessary. Closer examination, however, shows that (2) is incorrect, since $E_C(Z+1,Z)$ represents not the *total* Coulomb energy difference, but merely that part of the Coulomb energy which contributes to the *binding* energy. We therefore require an extra term to take care of the difference

[1] O. Kofoed-Hansen, Revs. Modern Phys. **30**, 449 (1958).
[2] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).
[3] R. Hofstadter, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Palo Alto, 1957), Vol. 7.

TABLE I. Values of r_0 in units of 10^{-13} cm as calculated from Eq. (3) for various values of S_p. Data for E_c are obtained from Kofoed-Hansen.[a] Column 7 gives theoretical values of Sood and Green.[b]

A	Z	$E_c(Z+1, Z)$ in Mev	$S_p=0$	$S_p=1.29$ Mev	$S_p=0.58$ Mev	Theoretical	S_p from Eq. (8)
3	1	0.764±0.001	2.35	0.88	1.34	···	2.33
5	2	0.835±0.050	3.02	1.18	1.80	···	2.74
6	2	0.80 ±0.05	2.98	1.14	1.73	···	2.70
7	3	1.646±0.002	1.92	1.08	1.42	1.55	1.69
9	4	2.032±0.006	1.84	1.12	1.43	1.45	1.62
10	4	1.965±0.006	1.84	1.11	1.42	···	1.61
10	5	2.68 ±0.06	1.64	1.11	1.35	···	1.43
11	5	2.761±0.003	1.56	1.06	1.29	1.37	1.35
13	6	3.006±0.005	1.59	1.12	1.33	1.33	1.39
14	6	2.940±0.005	1.59	1.11	1.33	···	1.39
14	7	3.639±0.008	1.48	1.09	1.28	···	1.31
15	7	3.539±0.006	1.48	1.08	1.27	1.34	1.31
17	8	3.550±0.006	1.61	1.18	1.38	1.33	1.42
18	8.5	3.72 ±0.10	1.59	1.18	1.38	···	1.42
19	9	4.027±0.008	1.53	1.16	1.34	1.35	1.36
21	10	4.266±0.006	1.54	1.18	1.36	1.31	1.38
22	10	4.214±0.006	1.53	1.17	1.34	···	1.38
23	11	4.841±0.010	1.45	1.14	1.29	1.30	1.31
25	12	5.062±0.008	1.46	1.16	1.31	1.29	1.32
26	12	5.006±0.008	1.45	1.15	1.30	···	1.27
27	13	5.584±0.010	1.39	1.13	1.26	1.28	1.27
29	14	5.749±0.010	1.42	1.16	1.29	1.27	1.30
31	15	6.220±0.060	1.37	1.14	1.25	1.27	1.26
33	16	6.360±0.030	1.40	1.16	1.28	1.27	1.29
34	16	6.30 ±0.03	1.40	1.16	1.28	···	1.29
35	17	6.760±0.040	1.37	1.15	1.26	1.27	1.26
37	18	6.920±0.110	1.39	1.17	1.28	1.27	1.28
38	18	6.86 ±0.11	1.38	1.16	1.28	···	1.28
39	19	7.294±0.030	1.36	1.16	1.26	1.27	1.27
41	20	6.740±0.050	1.52	1.27	1.39	1.27	1.42
46	22	7.8 ±1.0	1.39	1.19	1.29	···	1.30
50	24	8.1 ±0.3	1.42	1.22	1.32	···	1.33
54	26	9.2 ±0.3	1.32	1.16	1.24	···	1.24

[a] See references 1 and 12.
[b] See reference 8.

between the self-energies of the protons in the two mirror nuclei. If we assume that protons retain their individual identities when inside a nucleus, this term is simply equal to the Coulomb self-energy S_p of a single proton, and we get

$$E_C(Z+1, Z)=W_C(A, Z+1)-W_C(A,Z)-S_p. \quad (3)$$

Equation (3) is of course equivalent to Eq. (4) of Kofoed-Hansen. Equation (2), which was used for so many years, is in fact incorrect even on a classical basis.

THE VALUE OF THE SELF-ENERGY TERM

The question now arises as to what value is to be used for S_p. If the proton and the neutron are assumed to be identical particles, differing only in the protonic charge, then it would appear that

$$S_p=n-p \text{ mass difference}=1.29 \text{ Mev}. \quad (4)$$

This value of S_p does, however, lead to unsatisfactory results, as can be seen by reference to column five of Table I. Here values of r_0 for various A have been calculated from Eqs. (1), (3), and (4), assuming a uniform distribution of charge $\rho(r)$. It is seen that the r_0 obtained are too low (a mean of 1.15×10^{-13} cm results if the $A=3$ value is excluded). Electron scattering data and mirror nuclei calculations on a shell-model basis both give values of r_0 which decrease down to an approximate level of $r_0=1.24\times10^{-13}$ cm as $A\rightarrow30$ and beyond. The values in column five are not only too low; they also fail to show the expected decrease of r_0 as A increases. A few years ago this might have been regarded as desirable, but in the light of current ideas it is almost certainly incorrect.

We have therefore to turn to some other method of determining S_p, and we make use of the interesting argument of Peaslee[4] that the well-known "exchange energy" term is in fact to be interpreted as the self-energy of the protons. Peaslee's own determination of the value of the exchange term is, unfortunately, no longer of any use, since he determined the coefficients empirically with a view to obtaining a constant r_0. (It is interesting to note that Peaslee obtained an $S_p=1.11$ Mev by this method. This value is not too far from the $n-p$ mass difference of 1.29 Mev.) We shall accordingly fall back on the statistical evaluation of the exchange-energy term used by Cooper and Henley.[5] This gives a value of $-0.46Z^{\frac{1}{3}}e^2/r_0A^{\frac{1}{3}}$ for the "exchange energy," which in turn implies a self-energy contribution to the energy difference E_c of approximately

$$-0.613Z^{\frac{1}{3}}e^2/r_0A^{\frac{1}{3}}. \quad (5)$$

This value is not a constant, which would appear to be in contradiction with our intention of using it for S_p in (3). However, as Scott[6] has emphasized, the statistical evaluation is only valid for large A. In this case, $A\cong2Z$, and the contribution to E_c becomes equal to -0.583 Mev if we put $r_0=1.2\times10^{-13}$ cm. This method thus gives $S_p=0.58$ Mev for large A.[7] Since we assume that S_p is a constant, we shall assume that this value holds for all A, and Eq. (3) becomes

$$E_C(Z+1, Z)=W_C(A, Z+1) -W_C(A,Z)-0.58 \text{ Mev}. \quad (6)$$

In column six of Table I values of r_0 based on Eq. (6) are shown. A uniform charge distribution is again assumed, and it can be seen that the general features are consistent with the idea that r_0 decreases to an approximate level of $\sim1.26\times10^{-13}$ cm as A increases. (Secondary features such as the lower r_0 for $A=4n+3$ and the increases in r_0 after closed shells, are also evident, but we shall not discuss these here.) This provides rough agreement with both individual-particle model calculations[8] and electron scattering data.[2] In column seven we list the theoretical r_0 values of Sood and Green[8] for comparison, and it can be seen that the agreement is generally fairly satisfactory. Equation (6), and the S_p value of 0.58 Mev, can therefore be regarded as reasonable representations of the true situation. For low A, however (notably $A=7$, 11, and 15), discrepancies in

[4] D. C. Peaslee, Phys. Rev. **95**, 717 (1954).
[5] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).
[6] J. M. C. Scott, *Progress in Nuclear Physics* (Butterworths-Springer, London, 1956), Vol. 5.
[7] It should be emphasized that this value of S_p is only approximate, because (a) expression (5) is merely the first term in a Taylor expansion and (b) the value $r_0=1.2\times10^{-13}$ cm is arbitrarily chosen as being a suitable value for large A.
[8] P. C. Sood and A. S. Green, Nuclear Phys. **4**, 274 (1957).

the r_0 values are apparent, and in an attempt to eliminate these we shall consider one further method of estimating S_p.

The assumption that S_p is the (constant) Coulomb self-energy of a single proton, rather than the difference between the Coulomb self-energies of the protons in the $(Z+1)$ nucleus and those in the Z nucleus, may be the cause of the trouble. This can be eliminated by utilizing a quantum mechanical calculation of S_p. We again follow Peaslee in assuming that the "exchange energy" term is just the Coulomb self-energy of the protons, and we make use of the recent calculations of Swamy and Green[9] of this term. On the basis of the individual-particle model they obtained an expression $\frac{3}{5}c(Z^{\frac{4}{3}}/r_0A^{\frac{1}{3}})$ for this term, where c is a parameter which tends to 0.7636 as Z becomes large (see the Cooper-Henley evaluation). This in turn implies a self-energy contribution to the energy difference of approximately

$$-\tfrac{4}{5}ce^2Z^{\frac{1}{3}}/r_0A^{\frac{1}{3}}. \tag{7}$$

Using this value for S_p, we get

$$E_C(Z+1,Z)=W_C(A,Z+1)-W_C(A,Z) -\tfrac{4}{5}ce^2Z^{\frac{1}{3}}/r_0A^{\frac{1}{3}}. \tag{8}$$

In column eight of the table values of r_0 are given based on Eq. (8), a uniform charge distribution, and the use of the approximate expression for the parameter c given by Swamy and Green, *viz.*,

$$c=0.7636-e^{-0.38Z}. \tag{9}$$

Comparison with column seven values shows that again we have fairly good agreement for the higher A. Again, however, we get disagreement in the region $A<20$. This time our values of r_0 tend to be higher than those of Sood and Green, whereas the values from Eq. (6) tended to be lower. If anything, the simpler Eq. (6) gives a better fit than the quantum mechanically more refined Eq. (8).

NONUNIFORM CHARGE DISTRIBUTIONS

Direct comparison of both the theoretical and mirror r_0 values with the electron scattering radii is obviously desirable. Sood and Green show that their theoretical values are in rough agreement with the electron scattering data, and the same can therefore be said for our values from Eqs. (6) and (8). These are, however, based on a uniform charge distribution, which electron scattering tells us is unrealistic, and if direct comparison with electron scattering data is desired the shape of the charge distribution must be taken into account. Even when this has been done, other minor discrepancies will remain (e.g., the quadrupole and pairing effects; see Kofoed-Hansen[1]), and exact agreement with electron scattering values cannot be expected without a more refined theoretical treatment.

The first charge distribution used was the trapezoidal distribution, which might be expected to be valid for nuclei with $A\sim40$ and higher. Hofstadter,[2] Eq. (56), gives the required expression for $\rho(r)$. Use of Eq. (1) then gives us[10]

$$W_C=\frac{Z^2e^2}{140}\left[\frac{13(c+z)^7-(c-z)^4(13c^3+143c^2z+95cz^2+29z)^3}{8c^2z^2(c^2+z^2)^2}\right]. \tag{10}$$

We also[11] find

$$a^2=(c^2+3z^2)(3c^2+z^2)/5(c^2+z^2). \tag{11}$$

A reasonable value of z (the skin-thickness parameter) was assumed ($z=1.0\times10^{-13}$ cm was used), and calculations were made on the basis of Eq. (6) and the experimental values of $E_c(Z+1,Z)$. Data for $E_c(Z+1,Z)$ have been taken from Kofoed-Hansen,[1] and (for the even-A "isotopic spin" mirrors) from Kofoed-Hansen.[12] The results are disappointing, inasmuch as they give values of r_0 very little different from those given by the uniform model (column six, Table I). The differences are mostly 0.01×10^{-13} cm, and occasionally 0.02×10^{-13} cm. In view of the inadequacies of the present calculation these differences can hardly be regarded as significant, and it has not been thought worthwhile to quote the results. Use of $z=0.5$ and 1.5×10^{-13} cm does not alter the situation appreciably.

The other charge distributions which have been used are the "modified exponential" and the "hollow exponential" types. Hofstadter[2] (Table I, models VI and XII) gives $\rho(r)/Ze$ for both of these. From Eq. (1) we then get

$$W_c(A,Z)=K(Z^2e^2/a), \tag{12}$$

where $K=0.512$ for the hollow exponential model and $K=0.522$ for the modified exponential model. We choose these distributions because they have been used with some success[13,14] on the electron scattering data for Li^6, Li^7, and Be^9, and these are the only nuclei thus far studied by electron scattering where we have a *direct* check with mirror-nucleus data. Generalization from nuclei such as He^4, C^{12}, Mg^{24}, Si^{28}, S^{32}, and Ca^{40} (for which electron scattering data are available) would be highly dangerous. They are all $4n$ nuclei, whereas the

[9] N. V. V. Swamy and A. S. Green, Phys. Rev. **112**, 1719 (1958).

[10] In spite of its different appearance, Eq. (10) agrees with the expression for B_c given by W. D. Gunter and R. A. Hubbs, Phys. Rev. **113**, 252 (1959).

[11] a is the usual root mean square radius, defined by

$$a^2=\frac{4\pi}{Ze}\int_0^\infty r^4\rho(r)dr.$$

[12] O. Kofoed-Hansen, Nuclear Phys. **2**, 441 (1956).

[13] G. R. Burleson and R. Hofstadter (private communication).

[14] U. Meyer-Berkhout (private communication, provisional data).

TABLE II. Values of r_0 in units of 10^{-13} cm for Li^6, Li^7, and Be^9 obtained from mirror nucleus and electron scattering data. In the calculations based on Eq. (8) the values of the parameter c were taken from Table I in Swamy and Green[a] for Li^7 and Be^9, and from relation (11) in Swamy and Green for Li^6. The electron scattering data for Li^7 appears to be very uncertain.

	Mirror data				Electron scattering		
	Modified exponential		Hollow exponential		Modified exponential	Hollow exponential	Theoretical values[b]
Nucleus	Eq. (8)	Eq. (6)	Eq. (8)	Eq. (6)			
Li^6	3.06	1.94	2.99	1.90	No fit	1.99	1.97
Li^7	1.97	1.60	1.92	1.56	1.83(?)	?	1.55, 1.45, 1.49
Be^9	1.78	1.61	1.74	1.57	No fit	1.74	1.45, 1.54

[a] See reference 9.
[b] See references 8, 12, and 15.

mirror data covers only $4n+1$, $4n+2$, and $4n+3$ data, and discrepancies on the basis of this classification alone are to be expected. In Table II we give values of r_0 for Li^6, Li^7 and Be^9 as calculated from Eqs. (6) and (8). The data of Kofoed-Hansen have again been used for $E_C(Z+1, Z)$. Values from electron scattering and from various theoretical calculations[8,12,15] are given for comparison. Apart from the r_0 for Be^9 obtained from Eq. (8) and the r_0 for Li^6 obtained from Eq. (6), the agreement is not good, and the treatment we have given is quite clearly not refined enough to be of any assistance in deciding (a) between different charge distributions allowed by the electron scattering workers and (b) between the relative merits of Eqs. (6) and (8). The r_0 values from the theoretical models appear to be low; this one would expect, since they have not allowed for the type of charge distribution which is valid at this low A.

CONCLUSIONS

The situation can be summarized as follows: (a) Equation (2) obviously requires modification by means of a self-energy correction term. (b) Equations (6) and (8) provide correction terms, both of which yield values of r_0 with a satisfactory general trend (tending down to $r_0 \sim 1.26$ as $A \rightarrow 40$). (c) Both (6) and (8) do, however, appear to be unsatisfactory for very low A (<15 or 20), and it is impossible at this stage to say which gives the better fit.

If the value of 0.58 Mev for the Coulomb self-energy of a single proton is approximately correct [as is implied by Eq. (6)], then this result will be of great significance. The fact that Eq. (6) yields approximately the desired trend in r_0 [whereas Eq. (4) does not] suggests that 0.58 Mev is at any rate a better approximation to the proton's Coulomb self-energy than the n-p mass difference of 1.29 Mev. The crux of the matter seems to lie in the question as to whether or not one would expect the S_p term to be a constant. If so, Eq. (6) will be valid, and the value of 0.58 Mev, although approximate, should be reasonably close to the true proton Coulomb self-energy. If not, Eq. (8) should be roughly correct, and it does possess the merit of a quantum mechanical foundation. These aspects of the problem are currently being investigated.

Elucidation of the exact form of the self-energy correction term is obviously highly desirable, and to this end further electron scattering data would be most useful—for example, a knowledge of the charge distributions and radii of nuclei such as B^{11}, F^{19}, Na^{23}, Al^{27}, Si^{29}, P^{31}, Cl^{35}, and K^{39} would be particularly valuable for comparison with mirror data.

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[15] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).