PHYSICAL REVIEW VOLUME 115, NUMBER 5 SEPTEMBER 1, 1959

Spin Wave Spectra for Canted Antiferromagnets and Ferromagnets*

R. ORBACH†
Department of Physics, University of California, Berkeley, California
(Received January 16, 1959)

The magnetic resonance conditions and the spin wave spectra are found for canted antiferromagnetic and ferromagnetic lattices, where the cant is produced by magnetocrystalline anisotropy fields which are noncollinear. The sublattices are thereby caused to cant towards each other in antiferromagnets and away from each other in ferromagnets. The cant of the antiferromagnetic sublattices may produce a net moment (weak ferromagnetism) but does not alter appreciably the usual antiferromagnetic spin wave spectrum in the presence of anisotropy. The static susceptibility parallel ($\chi_{\parallel}$) and at right angles ($\chi_{\perp}$) to the vector difference of the anisotropy fields is calculated, and is shown to be altered from the noncanted result. It is shown that in antiferromagnets with no apparent weak ferromagnetism, canted sublattices may still be present, and can be detected by a nonzero ratio of $\chi_{\parallel}$ to $\chi_{\perp}$ at 0°K. The ferromagnetic spin wave spectrum shows a sudden change from the normal spectrum as soon as one introduces the noncollinear anisotropy fields. An optical branch is formed and a high-frequency $\mathbf{k}=0$ magnetic resonance is expected. This resonance is a consequence of the two-sublattice character of the canted ferromagnet and may be termed an exchange resonance.

I. INTRODUCTION

RECENTLY Dzialoshinsky[1] showed that the weak ferromagnetism observed[2] in antiferromagnetic α-Fe_2O_3 and other materials is plausibly explained by noncollinear magnetocrystalline anisotropy fields. These fields serve to turn the antiferromagnetically coupled sublattices towards each other, producing a net magnetic moment. The angle of rotation may give rise to a ferromagnetic moment of 10^{-1} to 10^{-5} of the sublattice saturation moment. Dzialoshinsky also showed that some antiferromagnets may be canted by the anisotropy fields but will not show the weak ferromagnetism if the net moment of any two neighboring canted spins is cancelled by the moment of a nearby pair of spins which are canted in the opposite direction. This cancellation is a matter of magnetic crystalline symmetry.

It is the purpose of this paper to derive the magnetic resonance conditions and spin wave dispersion relations for antiferromagnets subject to noncollinear anisotropy fields. We shall also investigate the static magnetic properties of canted antiferromagnets and indicate how both the magnitude and angular direction of the anisotropy fields can be detected by ordinary susceptibility measurements. It should be noted that Vonsovsky and Turov[3] have recently considered this problem. However, they have apparently neglected the magnetocrystalline anisotropy fields $H_{A\parallel}$ which correspond to the normal anisotropy fields found in antiferromagnets. Such an omission from the resonance equations drastically changes their form. As it seems physically reasonable that such anisotropy fields should be present in a canted antiferromagnet, it is necessary that the calculation be done with these terms included.

The canting of a ferromagnetic spin system can arise from two noncollinear magnetocrystalline anisotropy fields, one field acting on one set of spins, the other on their nearest neighbors. These fields cause the ferromagnetic lattice to be broken up into two ferromagnetic sublattices, rotated away from each other in the plane containing the anisotropy fields. Here too the angle of rotation is assumed small. We shall compute the spin wave spectrum for the canted ferromagnet and compare it with the usual ferromagnetic spin wave spectrum to see if the canting has any effect on the dispersion law.

For simplicity, we shall assume all the lattices with which we deal can be subdivided into two magnetic sublattices, such that the nearest magnetic neighbors of any given magnetic atom on one sublattice shall all lie on the other sublattice.

II. CANTED ANTIFERROMAGNET

A. Zero External Field

In this section we compute the zero-field magnetic resonance ($\mathbf{k}=0$) frequency and also the spin wave spectrum of an antiferromagnet subject to the noncollinear anisotropy fields shown in Fig. 1. The anisotropy fields $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ act on the spins of the l and m sublattices, respectively. For our purposes, we assume

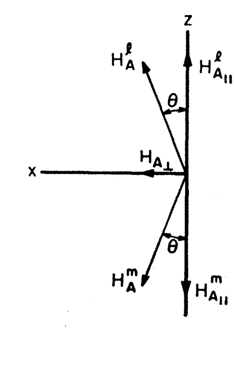

FIG. 1. The noncollinear anisotropy fields. $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ act on the l and m sublattices, respectively.

* This research was supported in part by the National Science Foundation.
† National Science Foundation Predoctoral Fellow.
[1] I. Dzialoshinsky, J. Phys. Chem. Solids 4, 241 (1958).
[2] F. J. Morin, Phys. Rev. 78, 819 (1950); T. Smith, Phys. Rev. 8, 721 (1916); R. Chevallier and S. Mathieu, Ann. Physik 18, 258 (1943); L. Neél, Ann. Physik 4, 249 (1949).
[3] S. V. Vonsovsky and E. A. Turov, Proceedings of the Fourth Symposium on Magnetism and Magnetic Materials, J. Appl. Phys. 30, 9S (1959).

$|\mathbf{H}_A{}^l|=|\mathbf{H}_A{}^m|$. The z axis is taken parallel to $\mathbf{H}_A{}^l-\mathbf{H}_A{}^m$. The x axis is directed parallel to $\mathbf{H}_A{}^l+\mathbf{H}_A{}^m$ and θ is the angle $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ make with the z axis. For purposes of computation, we shall take

$$\begin{aligned} H_{A\perp}&=H_A{}^l\sin\theta=H_A{}^m\sin\theta,\\ H_{A\parallel}{}^l&=H_A{}^l\cos\theta=H_{A\parallel},\\ H_{A\parallel}{}^m&=H_A{}^m\cos\theta=-H_{A\parallel},\end{aligned}\tag{1}$$

where $H_{A\perp}$ acts along the x direction on both the l and m sublattices. $H_{A\parallel}{}^l$ and $H_{A\parallel}{}^m$ act in the $+z$ direction on the l sublattice and in the $-z$ direction on the m sublattice, respectively. The exact quantum mechanical equation of motion for a spin on the l sublattice is

$$\frac{d\mathbf{S}_l}{dt}=\frac{2J}{\hbar}[\mathbf{S}_l\times(\sum_m\mathbf{S}_m)]+\gamma(\mathbf{S}_l\times\mathbf{H}_l),\tag{2}$$

and similarly for a spin on the m sublattice. The sum goes over all spins $\mathbf{S}_m$ which are nearest neighbors to a given spin $\mathbf{S}_l$ on the l sublattice; $\mathbf{H}_l$ is the resultant magnetic field acting on the l sublattice, including $\mathbf{H}_{A\parallel}{}^l$, $\mathbf{H}_{A\perp}{}^l$, and the external field $\mathbf{H}_0$. We take $\mathbf{H}_0$ zero in this section. The equations of motion become

$$\frac{dS_l{}^x}{dt}=\frac{2J}{\hbar}S_l{}^y(\sum_m S_m{}^z)-\frac{2J}{\hbar}(\sum_m S_m{}^y)S_l{}^z+\gamma S_l{}^yH_{A\parallel},\tag{3a}$$

$$\frac{dS_l{}^y}{dt}=\frac{2J}{\hbar}S_l{}^z(\sum_m S_m{}^x)-\frac{2J}{\hbar}(\sum_m S_m{}^z)S_l{}^x+\gamma S_l{}^zH_{A\perp}-\gamma S_l{}^xH_{A\parallel},\tag{3b}$$

$$\frac{dS_l{}^z}{dt}=\frac{2J}{\hbar}S_l{}^x(\sum_m S_m{}^y)-\frac{2J}{\hbar}(\sum_m S_m{}^x)S_l{}^y-\gamma S_l{}^yH_{A\perp},\tag{3c}$$

$$\frac{dS_m{}^x}{dt}=\frac{2J}{\hbar}S_m{}^y(\sum_l S_l{}^z)-\frac{2J}{\hbar}(\sum_l S_l{}^y)S_m{}^z-\gamma S_m{}^yH_{A\parallel},\tag{4a}$$

$$\frac{dS_m{}^y}{dt}=\frac{2J}{\hbar}S_m{}^z(\sum_l S_l{}^x)-\frac{2J}{\hbar}(\sum_l S_l{}^z)S_m{}^x+\gamma S_m{}^zH_{A\perp}+\gamma S_m{}^xH_{A\parallel},\tag{4b}$$

$$\frac{dS_m{}^z}{dt}=\frac{2J}{\hbar}S_m{}^x(\sum_l S_l{}^y)-\frac{2J}{\hbar}(\sum_l S_l{}^x)S_m{}^y-\gamma S_m{}^yH_{A\perp}.\tag{4c}$$

To solve this set of equations, we perform a rotation of coordinates in the same manner as in Keffer and Kittel.[4] We rotate about the y axis to two new frames of reference in which the $z_{l'}$ and $z_{m'}$ directions are taken as the static magnetization directions for each sublattice under the influence of $\mathbf{H}_{A\parallel}$, $\mathbf{H}_{A\perp}$, and the exchange field, $\mathbf{H}_e$. The latter may be taken as $2JzS_m/\gamma\hbar$ acting on the l sublattice and $-2JzS_l/\gamma\hbar$ acting on the m sublattice, where S_m and S_l are the magnitudes of the spins on the m and l sublattices, respectively, and z is the number of nearest neighbors of each spin. For an antiferromagnet, $\mathbf{S}_m=-\mathbf{S}_l$. The rotation angles φ_l and φ_m may be found by requiring that the sum of the torques be zero for each sublattice. This results in the pair of equations

$$\begin{aligned} H_e\sin(\varphi_l+\varphi_m)+H_{A\parallel}\sin\varphi_l&=H_{A\perp}\cos\varphi_l,\\ H_e\sin(\varphi_l+\varphi_m)+H_{A\parallel}\sin\varphi_m&=H_{A\perp}\cos\varphi_m.\end{aligned}\tag{5}$$

For small angles the solutions are

$$\varphi=\varphi_l=\varphi_m\cong H_{A\perp}/2H_e,\tag{6}$$

where we have assumed $H_e\gg H_{A\perp}$, $H_{A\parallel}$. As $H_{A\perp}/H_e\sim10^{-1}$ to 10^{-5}, we see that the small angle approximation is justified. We rotate to the new coordinate systems labelled l' and m' by making the transformations

$$\begin{aligned} S_{l'}{}^x&=S_l{}^x\cos\varphi_l-S_l{}^z\sin\varphi_l,\\ S_{l'}{}^y&=S_l{}^y,\\ S_{l'}{}^z&=S_l{}^z\cos\varphi_l+S_l{}^x\sin\varphi_l,\\ S_{m'}{}^x&=S_m{}^x\cos\varphi_m+S_m{}^z\sin\varphi_m,\\ S_{m'}{}^y&=S_m{}^y,\\ S_{m'}{}^z&=S_m{}^z\cos\varphi_m-S_m{}^x\sin\varphi_m.\end{aligned}\tag{7}$$

We now make the usual spin wave approximations of small amplitudes and write

$$\mathbf{S}_{l'}=(\epsilon_{lx},\epsilon_{ly},S_l)\cong(\epsilon_{lx},\epsilon_{ly},S),\tag{8a}$$

$$\mathbf{S}_{m'}=(\epsilon_{mx},\epsilon_{my},S_m)\cong(\epsilon_{mx},\epsilon_{my},-S),\tag{8b}$$

where the ϵ's are assumed small deviations of the magnetic moment from the z axis of the l' and m' coordinate systems. S_l represents the magnitude of the spin on the l sublattice, and is assumed approximately constant along the $z_{l'}$ direction; S_m represents the magnitude of the spin on the m sublattice, and is assumed approximately constant along the $z_{m'}$ direction. We take the ϵ's to be in the form of standing waves:

$$\begin{aligned} \epsilon_{lx}&=\epsilon_{lx}{}^0\sin k_xx\sin k_yy\sin k_zz\sin\omega t,\\ \epsilon_{ly}&=\epsilon_{ly}{}^0\sin k_xx\sin k_yy\sin k_zz\cos\omega t,\\ \epsilon_{mx}&=\epsilon_{mx}{}^0\sin k_xx\sin k_yy\sin k_zz\sin\omega t,\\ \epsilon_{my}&=\epsilon_{my}{}^0\sin k_xx\sin k_yy\sin k_zz\cos\omega t.\end{aligned}\tag{9}$$

The sum over nearest neighbors takes the form

$$\textstyle\sum_m\epsilon_{mx}=z\gamma_k\epsilon_{mx},\tag{10}$$

similarly for ϵ_{my}, ϵ_{lx}, ϵ_{ly}, and

$$\gamma_k=(1/z)\sum_{\boldsymbol{\rho}}\cos(\boldsymbol{\rho}\cdot\mathbf{k}),\quad|\boldsymbol{\rho}|=a.\tag{11}$$

Here, $\boldsymbol{\rho}$ is the vector between a given atom and its nearest neighbors, and the sum in (11) goes over all such vectors. Using (5), (6), (7), (8), (9), and neglecting terms quadratic in the ϵ's, (3) and (4) reduce to

[4] F. Keffer and C. Kittel, Phys. Rev. 85, 329 (1952).

$$
\begin{aligned}
(\omega/\gamma)\epsilon_{lx} &= \epsilon_{ly}(H_e \cos 2\varphi + H_{A\parallel} \cos\varphi + H_{A\perp} \sin\varphi) \\
&\qquad + \epsilon_{my} H_e \gamma_k, \\
(\omega/\gamma)\epsilon_{ly} &= \epsilon_{lx}(H_e \cos 2\varphi + H_{A\parallel} \cos\varphi + H_{A\perp} \sin\varphi) \\
&\qquad + \epsilon_{mx} H_e \gamma_k \cos 2\varphi, \\
(\omega/\gamma)\epsilon_{mx} &= -\epsilon_{my}(H_e \cos 2\varphi + H_{A\parallel} \cos\varphi + H_{A\perp} \sin\varphi) \\
&\qquad - \epsilon_{ly} H_e \gamma_k, \\
(\omega/\gamma)\epsilon_{my} &= -\epsilon_{mx}(H_e \cos 2\varphi + H_{A\parallel} \cos\varphi + H_{A\perp} \sin\varphi) \\
&\qquad - \epsilon_{lx} H_e \gamma_k \cos 2\varphi.
\end{aligned}
\tag{12}
$$

Solving this set of equations for ω/γ, we find

$$
\begin{aligned}
(\omega/\gamma)^2 = &\left[H_e(1\pm\gamma_k)\cos 2\varphi + H_{A\parallel}\cos\varphi + H_{A\parallel}\sin\varphi\right] \\
&\times\left[H_e(\cos 2\varphi \mp \gamma_k) + H_{A\parallel}\cos\varphi + H_{A\perp}\sin\varphi\right].
\end{aligned}
\tag{13}
$$

For small angles we may simplify this expression using (6). We find

$$
\begin{aligned}
(\omega/\gamma)^2 = &H_{A\parallel}(H_{A\parallel}+2H_e) + H_e^2(1-\gamma_k^2) \\
&+ \tfrac{1}{2}H_{A\perp}^2\gamma_k(\gamma_k \pm 1).
\end{aligned}
\tag{14}
$$

For $ka \ll 1$, $\gamma_k \approx 1-(k^2a^2/2)$, so that (14) reduces to

$$
(\omega_1/\gamma)^2 = H_{A\parallel}(H_{A\parallel}+2H_e) + (H_e^2 - \tfrac{1}{2}H_{A\perp}^2)k^2a^2, \tag{15a}
$$

$$
\begin{aligned}
(\omega_2/\gamma)^2 = &H_{A\parallel}(H_{A\parallel}+2H_e) + H_{A\perp}^2 \\
&+ (H_e^2 - \tfrac{3}{4}H_{A\perp}^2)k^2a^2.
\end{aligned}
\tag{15b}
$$

The $\mathbf{k}=0$ resonance frequencies (uniform precession) are similar to those predicted by Keffer and Kittel[4] and Nagamiya[5] for an external magnetic field $\mathbf{H}_0$ at right angles to the anisotropy axis of a normal antiferromagnet. Their $\mathbf{H}_0$ corresponds to our $\mathbf{H}_{A\perp}$ and their anisotropy field $\mathbf{H}_A$ to our $\mathbf{H}_{A\parallel}$. The spin wave spectrum of (15a) is similar to the usual spin wave spectrum of an uncanted antiferromagnet: quadratic in k for $H_e^2k^2a^2 < 2H_eH_{A\parallel}+H_{A\parallel}^2$ and linear in k for $H_e^2k^2a^2 > 2H_eH_{A\parallel}+H_{A\parallel}^2$. However, the spectrum of (15b) differs a little from the usual dispersion law in that it is quadratic in k for larger values of ka; i.e., until $H_e^2k^2a^2 > 2H_eH_{A\parallel}+H_{A\parallel}^2+H_{A\perp}^2$.

On solving the equations of motion for the rf susceptibilities, one finds that, if the rf field is in the usual y direction, only the frequencies (15a) will be excited. If the rf field is in the x direction (parallel to $H_{A\perp}$) the frequencies (15b) will be excited. The differential excitation of these modes is very similar to that obtained by Keffer and Kittel.[4]

B. External Field Perpendicular to $\mathbf{H}_A{}^l - \mathbf{H}_A{}^m$

In this case, the external field is assumed perpendicular to $\mathbf{H}_A{}^l - \mathbf{H}_A{}^m$, i.e., perpendicular to the z axis. The treatment is similar to section IIA, except that $\mathbf{H}_{A\perp}$ is replaced by $\mathbf{H}_{A\perp}+\mathbf{H}_0$. If $\mathbf{H}_0$ is not parallel to $\mathbf{H}_{A\perp}$, but in the plane perpendicular to the z axis, we may merely take the vector resultant of $\mathbf{H}_{A\perp}+\mathbf{H}_0$, redefine this direction as the x direction, and perform the same analysis as in Sec. IIA using $|\mathbf{H}_{A\perp}+\mathbf{H}_0|$ for $\mathbf{H}_{A\perp}$. The angle of rotation about the new y axis will be given by

$$
\varphi = \varphi_l = \varphi_m = |\mathbf{H}_{A\perp}+\mathbf{H}_0|/2H_e. \tag{16}
$$

The susceptibility perpendicular to the z axis at 0°K, $\chi_\perp$, assuming $\mathbf{H}_0$ parallel to $\mathbf{H}_{A\perp}$, is given by

$$
\chi_\perp = (1+H_{A\perp}/H_0)M_s/H_e, \tag{17}
$$

where M_s is the sublattice magnetization. If H_e is assumed equal to λM_s (i.e., the molecular field approximation), then

$$
\chi_\perp = (1+H_{A\perp}/H_0)/\lambda. \tag{18}
$$

In canted antiferromagnets where there is no weak ferromagnetism, $\mathbf{H}_{A\perp}$ must reverse directions for alternate pairs of spins. If $\mathbf{H}_{A\perp}$ and $\mathbf{H}_0$ are in opposite directions, the perpendicular susceptibility becomes

$$
\chi_\perp = (1-H_{A\perp}/H_0)/\lambda. \tag{19}
$$

The sum of the susceptibilities, (18) and (19), is $\chi_\perp = 1/\lambda$, the usual result for an antiferromagnet. We have taken care of the factor of two by noting that, for the case of no weak ferromagnetism in canted antiferromagnets, (18) and (19) refer to half the total magnetization.

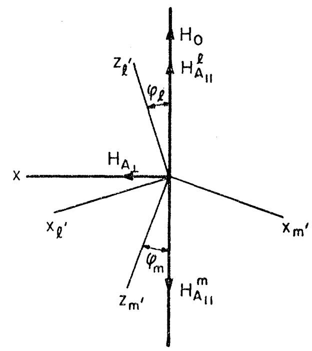

FIG. 2. The rotated antiferromagnetic sublattice coordinate systems. $\mathbf{H}_0$ is parallel to the z axis.

C. External Field Parallel to $\mathbf{H}_A{}^l - \mathbf{H}_A{}^m$

In this case the external field is parallel to $\mathbf{H}_A{}^l - \mathbf{H}_A{}^m$, i.e., parallel to the z axis, and φ_l no longer equals φ_m, as is shown in Fig. 2. One finds, in place of (5), by requiring the sum of the torques to be zero for each sublattice,

$$
\begin{aligned}
H_e \sin(\varphi_l+\varphi_m) + (H_{A\parallel}+H_0)\sin\varphi_l &= H_{A\perp}\cos\varphi_l, \\
H_e \sin(\varphi_l+\varphi_m) + (H_{A\parallel}-H_0)\sin\varphi_m &= H_{A\perp}\cos\varphi_m.
\end{aligned}
\tag{20}
$$

This set of equations may be solved by assuming φ_l and $\varphi_m \ll 1$. We find

$$
\begin{aligned}
\varphi_l &= \frac{H_{A\perp}(H_{A\parallel}-H_0)}{H_{A\parallel}^2 - H_0^2 + 2H_eH_{A\parallel}} \cong \frac{H_{A\perp}(H_{A\parallel}-H_0)}{2H_eH_{A\parallel}}, \\
\varphi_m &= \frac{H_{A\perp}(H_{A\parallel}+H_0)}{H_{A\parallel}^2 - H_0^2 + 2H_eH_{A\parallel}} \cong \frac{H_{A\perp}(H_{A\parallel}+H_0)}{2H_eH_{A\parallel}}.
\end{aligned}
\tag{21}
$$

[5] T. Nagamiya, Progr. Theoret. Phys. (Kyoto) **6**, 350 (1951).

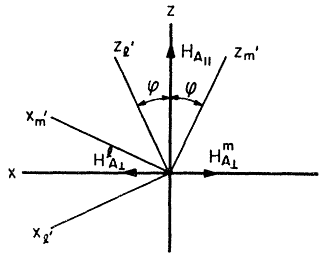

FIG. 3. The rotated ferromagnetic sublattice coordinate systems.

The susceptibility parallel to the z axis at 0°K, $\chi_\parallel$, may be computed from (21):

$$\chi_\parallel = M_s H_{A\perp}^2/2H_e^2 H_{A\parallel}. \tag{22}$$

It is to be noted that the 0°K parallel susceptibility is not zero, as it is in the noncanted antiferromagnet. The ratio $\chi_\parallel/\chi_\perp$ for the canted antiferromagnet becomes

$$\frac{\chi_\parallel}{\chi_\perp} = \frac{H_{A\perp}^2}{2H_e H_{A\parallel}(1+H_{A\perp}/H_0)}. \tag{23}$$

For $H_{A\perp} > H_{A\parallel}$, i.e., the angle between $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ less than 90°, this ratio may possess a significant value. It is to be noted that even if the canted antiferromagnet does not show any weak ferromagnetism, the lattices will show the nonzero ratio of susceptibilities, given by (22):

$$\frac{\chi_\parallel}{\chi_\perp} = \frac{H_{A\perp}^2}{2H_e H_{A\parallel}}. \tag{24}$$

Thus, a lattice cant can be detected even if no observable weak ferromagnetic moment is present.

We return to our equations of motion (2). We assume $\mathbf{H}_0$ parallel to the z direction, use (20), (21), and solve for (ω/γ):

$$\begin{aligned}(\omega/\gamma)^2 = {} & H_e^2(1-\gamma_k^2) + H_{A\parallel}(H_{A\parallel}+2H_e) \\ & + H_0^2 + \tfrac{1}{2}H_{A\perp}^2\gamma_k^2 \pm \{4H_e^2H_0^2(1-\gamma_k^2) \\ & + 8H_eH_{A\parallel}H_0^2 - 4(H_eH_{A\perp}^2H_0^2/H_{A\parallel})(1-\gamma_k^2) \\ & + 4H_{A\parallel}^2H_0^2 - H_0^2H_{A\perp}^2(7-\gamma_k^2) \\ & + (H_{A\perp}^2H_0^2/H_{A\parallel}^2)(H_{A\perp}^2-H_0^2)(1-\gamma_k^2) \\ & + \tfrac{1}{4}H_{A\perp}^4\gamma_k^2\}^{\frac{1}{2}}. \end{aligned} \tag{25}$$

We have assumed $H_e \gg H_{A\parallel}$, $H_{A\perp}$, and H_0. It is seen for $ka \ll 1$ that the spin wave spectrum of (25) is similar to that calculated in Sec. IIA for a canted antiferromagnet with zero external field. The $k=0$ uniform precessional mode has the frequencies

$$\begin{aligned}(\omega/\gamma)^2 = {} & H_{A\parallel}(H_{A\parallel}+2H_e) + H_0^2 + \tfrac{1}{2}H_{A\perp}^2 \\ & \pm\{8H_eH_{A\parallel}H_0^2 + 4H_{A\parallel}^2H_0^2 - 6H_0^2H_{A\perp}^2 \\ & + \tfrac{1}{4}H_{A\perp}^4\}^{\frac{1}{2}}. \end{aligned} \tag{26}$$

As in the noncanted antiferromagnet,[4] an rf field in the y direction will excite both resonance modes.

III. CANTED FERROMAGNET

In this section we compute the zero-field magnetic resonance ($\mathbf{k}=0$) frequency and also the spin wave spectrum of a ferromagnet subject to noncollinear anisotropy fields as shown in Fig. 3. The ferromagnetic lattice is subdivided into two magnetic sublattices, l and m, such that the magnetic nearest neighbors of a spin on the l sublattice lie on the m sublattice and *vice versa*. The anisotropy fields $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ act as before on the spins of the l and m sublattices, respectively. We assume $|\mathbf{H}_A{}^l| = |\mathbf{H}_A{}^m|$ and take the z direction parallel to $\mathbf{H}_A{}^l + \mathbf{H}_A{}^m$. The x axis is directed parallel to $\mathbf{H}_A{}^l - \mathbf{H}_A{}^m$ and φ is the angle $\mathbf{H}_A{}^l$ and $\mathbf{H}_A{}^m$ make with the z axis. For purposes of computation, we shall take

$$\begin{aligned} H_{A\parallel} &= H_A{}^l\cos\varphi = H_A{}^m\cos\varphi, \\ H_{A\perp} &= H_A{}^l\sin\varphi = H_{A\perp}, \\ H_{A\perp}{}^m &= H_A{}^m\sin\varphi = -H_{A\perp}, \end{aligned} \tag{27}$$

where $H_{A\parallel}$ acts along the z direction on both the l and m sublattices. $H_{A\perp}{}^l$ and $H_{A\perp}{}^m$ act in the $+x$ direction on the l sublattice and in the $-x$ direction on the m sublattice, respectively. The equations of motion are similar to (3), except that the changes appropriate to (27) are made. We perform the rotation (7), the sublattices now being rotated about the y axis away from the same $+z$ direction, as shown in Fig. 3. Requiring the sum of the torques be zero for each sublattice, we find the angle of rotation φ becomes $\cong H_{A\perp}/2H_e$. We have made the small-angle approximation and assumed $H_e \gg H_{A\perp}$. H_e is the exchange field, defined in the ferromagnetic case to be $2JzS_m/\gamma\hbar$ acting on the l sublattice, and $2JzS_l/\gamma\hbar$ acting on the m sublattice. For this case we take

$$\begin{aligned} \mathbf{S}_{l'} &= (\epsilon_{lx},\epsilon_{ly},S_l) \cong (\epsilon_{lx},\epsilon_{ly},S), \\ \mathbf{S}_{m'} &= (\epsilon_{mx},\epsilon_{my},S_m) \cong (\epsilon_{mx},\epsilon_{my},S). \end{aligned} \tag{28}$$

We insert the above into our equations of motion and solve for (ω/γ):

$$\begin{aligned}(\omega/\gamma)^2 = {} & H_e^2(1\mp\gamma_k)^2 + 2H_eH_{A\parallel}(1\mp\gamma_k) \\ & + H_{A\parallel}^2 \pm \tfrac{1}{2}H_{A\perp}^2\gamma_k(1\mp\gamma_k). \end{aligned} \tag{29}$$

For $ka \ll 1$,

$$(\omega_1/\gamma)^2 = H_{A\parallel}^2 + (H_eH_{A\parallel} + \tfrac{1}{4}H_{A\perp}^2)k^2a^2 + \tfrac{1}{4}H_e^2k^4a^4; \tag{30a}$$

$$\begin{aligned}(\omega_2/\gamma)^2 = {} & 4H_e(H_e+H_{A\parallel}) + H_{A\parallel}^2 - H_{A\perp}^2 \\ & - (2H_e^2 + H_eH_{A\parallel} - \tfrac{3}{4}H_{A\perp}^2)k^2a^2 + \tfrac{1}{4}H_e^2k^4a^4. \end{aligned} \tag{30b}$$

Equation (30a) is the usual ferromagnetic resonance expression except for the term in $H_{A\perp}^2$ which arises from the canting field. Equation (30b) however, represents an optical branch for the ferromagnet. This optical mode arises from the two sublattice character of the canted ferromagnet and is similar to the exchange mode found in ferrites. It can be shown that a uniform rf field in either the x or y direction will excite only the lower, or acoustic mode, (30a), at $k=0$.[6] The optical

[6] It is to be noted that the uniform rf field can excite spin waves of nonzero $\mathbf{k}$ if the end pins are pinned, as shown by C. Kittel, Phys. Rev. **110**, 1295 (1958). As for $\mathbf{k}=0$, only spin waves in the acoustic mode will be excited by rf fields in the x or y directions.

mode may be excited either because of a difference in the g values of the two sublattices or by an rf field parallel to $\mathbf{H}_A{}^l+\mathbf{H}_A{}^m$ (z direction). The former corresponds to the exchange resonance in ferrites predicted by Kaplan and Kittel[7] and need not be discussed here. The latter excitation, however, is possible also for identical sublattices. The rf susceptibility for a field in the direction of $\mathbf{H}_A{}^l+\mathbf{H}_A{}^m$ (z direction) is found to be, at $k=0$,

$$\chi_z=\frac{M_s\sin^2\varphi[H_e(1+\cos2\varphi)+H_{A\parallel}\cos\varphi+H_{A\perp}\sin\varphi]}{[(\omega/\gamma)^2-(4H_e{}^2+4H_eH_{A\parallel}+H_{A\parallel}{}^2-H_{A\perp}{}^2)]},\tag{31}$$

where M_s is the sublattice saturation magnetization. For small angles φ and a resonance line width of ΔH the susceptibility at resonance becomes

$$\chi_z{}^{\text{res}}=(M_s/\Delta H)\varphi^2=(M_s/\Delta H)(H_{A\perp}/2H_e).^2\tag{32}$$

Here $\chi_z{}^{\text{res}}$ is the usual paramagnetic susceptibility multiplied by the square of the cant angle. From comparison with the canted antiferromagnet we may take $\varphi\sim10^{-2}$. Then assuming $M_s\sim10^3$ and $\Delta H\sim10^2$, we find $\chi_z{}^{\text{res}}\sim10^{-3}$. This susceptibility may be large enough to allow detection by present day infrared equipment.

For $H_{A\perp}$ different from zero, the number of spin waves in the acoustic branch is less than the number present for $H_{A\perp}$ zero; the remaining spin waves, their number proportional to the Boltzmann factor between the two branches, lie in the optical branch. It is to be noted that both branches follow the $\omega\sim k^2$ law for ka sufficiently large such that $H_ek^2a^2\gg H_{A\parallel}$, $H_{A\perp}$. For smaller ka the behavior is more complex, but in general follows the k^2 relation.

IV. CONCLUSIONS

The magnetic resonance experiments of Anderson Merritt, Remeika, and Yager[8]; and of Kumagai, Abe, Ono, Hayashi, Shimada, and Iwanaga[9] on α-Fe_2O_3 are difficult to interpret using the analysis of Sec. II because at the time they were performed, the resonance was thought to be a ferromagnetic one. Hence the experimentors paid little attention to the direction of $\mathbf{H}_0$, the external field, relative to the weak moment in the basal plane. As the canted spins lie in the basal plane in α-Fe_2O_3, interpreting their results necessitates a knowledge of the angle between $\mathbf{H}_0$ and the vector sum of the anisotropy fields, $\mathbf{H}_A{}^l+\mathbf{H}_A{}^m$, the direction of the weak moment. Once this angle is determined, the results of Sec. II can then be used to explain the experimental results. The observed anisotropy of the ferromagnetic moment out of the basal plane can be explained by the analysis of Sec. IIB, in which the external field $\mathbf{H}_0$ is directed normal to $\mathbf{H}_{A\parallel}$ and must be added vectorially to $\mathbf{H}_{A\perp}$. The vector sum of $\mathbf{H}_0$ and $\mathbf{H}_{A\perp}$ determines the direction the net magnetization makes with the basal plane. The anisotropy in the basal plane can be explained using the results of Sec. IIC. The field $\mathbf{H}_0$ in the basal plane must be broken into its components along $H_{A\perp}$ and $H_{A\parallel}{}^l$. If these components be labelled $H_{0\perp}$ and $H_{0\parallel}$, the angle of rotation in the basal plane of the weak ferromagnetic moment from its zero field value becomes $(H_{A\perp}+H_{0\perp})H_{0\parallel}/2H_eH_{A\parallel}$.

As Dzialoshinsky has shown, magnetic sysmmetry considerations predict that some canted antiferromagnetic lattices may not show any weak ferromagnetism. The cant in such materials can be detected, however, by the ratio of the static susceptibilities as defined in Secs. IIB and C, $\chi_\parallel/\chi_\perp$ at 0°K. This ratio will differ from zero as $H_{A\perp}{}^2$, as shown in (24). $H_{A\perp}$ may thereby be estimated from this ratio if H_e and $H_{A\parallel}$ are determined from the resonance data.

It is to be further noted that the appearance of a net moment resulting from a canting of the antiferromagnetic sublattices does not appreciably alter the spin wave spectrum of the antiferromagnet. This result seems surprising at first glance, as the change from an antiferromagnet to a ferromagnet produces a violent change in the dispersion law for $H_ek^2a^2\gg H_A(H_A+2H_e)$. It is seen from the preceding calculations, however, that the spins are still compensated in a canted antiferromagnet, such that the antiferromagnetic character of the dispersion law is preserved.

Observation of the ferromagnetic canting has yet to be made. Its occurrence will depend upon the local magnetocrystalline anisotropy fields and the differences in local environment of the two sublattices. Observation of the optical branch will serve to confirm the existence of the canted ferromagnet. It should also be pointed out that the character of the ferromagnetic dispersion law for the canted ferromagnet is not appreciably altered, a result which now seems reasonable in light of the results for the canted antiferromagnet.

V. ACKNOWLEDGMENTS

The author is indebted to Professor C. Kittel for suggesting this investigation and for his advice during its progress. The author also wishes to thank Professor A. Portis for his interest in this paper and for helpful discussions concerning the solutions. Appreciation is expressed to Professor F. Keffer for his comments concerning the solution of rotated antiferromagnetic spin systems. This research was carried out during the tenure of a National Science Foundation Predoctoral Fellowship.

[7] J. Kaplan and C. Kittel, J. Chem. Phys. **21**, 760 (1953).
[8] Anderson, Merritt, Remeika, and Yager, Phys. Rev. **93**, 717 (1954).
[9] Kumagai, Abe, Ono, Hayashi, Shimada, and Iwanaga, Phys. Rev. **99**, 1116 (1955).