Green's Function Approximation Method. II. The Polaron*

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A method previously derived for the approximate construction of the nucleon Green's function is here applied to that of the polaron. After making an arbitrary translation of the phonon variables, a linear integral equation for the Green's function is derived by means of a symmetrical treatment and a noncorrelation assumption, in complete analogy to the nucleon problem. This equation is solved through the introduction of a spectral representation in the special case of zero total momentum. The lowest energy state of the system is calculated in terms of the arbitrary translations and then minimized with respect to them. Using the simplest nontrivial cutoff procedure to obtain the variational equation and its solution, results are obtained for values of the coupling parameter $\alpha \leqslant 3$, and are compared with those of Feynman.

1. INTRODUCTION

THIS paper will be another application of an approximation method that was derived for the case of a nucleon in interaction with its meson field.[1] As in that case, we shall concern ourselves with a Green's function with a known spectral representation and, by operating on it symmetrically, derive an integral equation for it. A noncorrelation assumption will be used to reduce this to a linear equation consistent with the spectral form, which form will then be used to obtain the solution. To suit this specific problem, the technique will be modified to include a variational calculation for the purpose of obtaining the lowest energy state of the system.

The situation here considered is an electron moving slowly in a crystal lattice. The physical assumptions are that the one-electron and perfect crystal approximations are valid (i.e., no correlations between electrons and no imperfections in the crystal lattice), but that there is a large dipole moment per unit volume, due to the longitudinal optical modes of vibration, with which the electron interacts. The frequency, ω, of these vibrations will be taken independent of their wave vector $\mathbf{k}$. All other electron-lattice interactions will be neglected (i.e., transverse optical and acoustical modes of vibration). Electrons bound to individual ions will be assumed to move with them without inertia. Electrical isotropy will be assumed and the crystal will be taken as a continuum. Finally, the effect of the periodic potential of the ions in their mean positions and the other electrons will be assumed to be entirely accounted for by an effective mass, m^*. We use units such that $\hbar=\omega=2m^*=1$. These approximations are discussed in detail by Schultz[2] who then derives (with slightly different notation) the Hamiltonian

$$H=\left[\mathbf{P}-\int\frac{(dk)}{(2\pi)^3}\mathbf{k}\tau^\dagger(\mathbf{k},t)\tau(\mathbf{k},t)\right]^2+(4\pi\alpha)^{\frac{1}{2}}\int\frac{(dk)}{(2\pi)^3}\frac{1}{k}[\tau^\dagger(\mathbf{k},t)+\tau(\mathbf{k},t)]+\int\frac{(dk)}{(2\pi)^3}\tau^\dagger(\mathbf{k},t)\tau(\mathbf{k},t)+M, \quad (1)$$

where $\mathbf{P}$ is the total momentum of the system, α measures the strength of the coupling between the electron (whose coordinates have been eliminated) and the phonons, $k\equiv|\mathbf{k}|$, the integrations are three-dimensional, and $\tau^\dagger$ and τ are the phonon creation and annihilation operators obeying

$$[\tau(\mathbf{k},t),\tau^\dagger(\mathbf{k}',t)]=\delta(\mathbf{k}-\mathbf{k}'). \quad (2)$$

The constant M has been introduced so that the lowest energy state will have $E=0$. It is analogous to the mass renormalizetion in I in that the uncoupled system ($\alpha=0$, $M=0$) has lowest energy $E_0=0$ and M represents the shift in the origin necessary for the lowest energy state to be unchanged when the coupling, α, is "switched on." In this case, however, M is finite and will be the ultimate object of our calculation. It is to be noted that $\mathbf{P}$ is a constant of the motion and may be taken as just a number.

2. DERIVATION OF THE EQUATION

Were it not for the cubic term in (1), the problem could be solved exactly by means of a translation of the phonon variables. In order to suit the treatment to this situation, an as yet undetermined translation will be introduced and will eventually be determined by a variational principle. Thus, let

$$\tau(\mathbf{k},t)=a(\mathbf{k},t)+(4\pi\alpha)^{\frac{1}{2}}A(\mathbf{k}),\qquad \tau^\dagger(\mathbf{k},t)=a^\dagger(\mathbf{k},t)+(4\pi\alpha)^{\frac{1}{2}}A(\mathbf{k}), \quad (3)$$

i.e., $A(\mathbf{k})$ is a real numerical function of $\mathbf{k}$.

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[1] D. S. Falk, preceding paper [Phys. Rev. **115**, 1069 (1959)]. Hereafter to be referred to as I.

[2] T. D. Schultz, Massachusetts Institute of Technology Solid-State and Molecular Theory Group, Technical Report No. 9, 1956 (unpublished). See also H. Fröhlich, Advances in Phys. **3**, 325 (1954).

Introduce states $\langle 0t|$ and $|0t'\rangle$ such that

$$a(\mathbf{k},t')|0t'\rangle=0=\langle 0t|a^\dagger(\mathbf{k},t). \tag{4}$$

The field equations are then given by

$$i\frac{\partial}{\partial t}\langle 0t|0t'\rangle=\langle 0t|H(t)|0t'\rangle. \tag{5}$$

Let

$$G(t,t')=i\langle 0t|0t'\rangle\theta(t,t'), \tag{6}$$

where

$$\begin{aligned}\theta(t,t')&=1 \quad \text{for} \quad t>t'\\ &=0 \quad \text{for} \quad t<t.'\end{aligned} \tag{7}$$

Then with

$$U(\mathbf{k})\equiv(1-2\mathbf{P}\cdot\mathbf{k}+k^2)A(\mathbf{k})+\frac{1}{k}+2(4\pi\alpha)\int\frac{(dk')}{(2\pi)^3}\mathbf{k}\cdot\mathbf{k}'A^2(\mathbf{k}')A(\mathbf{k}), \tag{8}$$

and

$$\begin{aligned}M'\equiv M&+(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}(1-2\mathbf{P}\cdot\mathbf{k}+k^2)A^2(\mathbf{k})\\&+2(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{A(\mathbf{k})}{k}\\&+(4\pi\alpha)^2\left[\int\frac{(dk)}{(2\pi)^3}\mathbf{k}A^2(\mathbf{k})\right]^2,\end{aligned} \tag{9}$$

the equation for G becomes

$$\begin{aligned}\left[P^2+M'-i\frac{\partial}{\partial t}\right]&G(t,t')-\delta(t-t')\\&=-i(4\pi\alpha)^{\frac{1}{2}}\int\frac{(dk)}{(2\pi)^3}U(\mathbf{k})\langle 0t|a(\mathbf{k},t)|0t'\rangle\theta(t-t')\\&\quad-i(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\mathbf{k}\cdot\mathbf{k}'A(\mathbf{k})A(\mathbf{k}')\\&\qquad\times\langle 0t|a(\mathbf{k},t)a(\mathbf{k}',t)|0t'\rangle\theta(t-t').\end{aligned} \tag{10}$$

As only the domain of small P is of interest, expand M' in a Taylor series as

$$M'=M_0+M_1P^2+\cdots, \tag{11}$$

and (10) becomes

$$\begin{aligned}\left[P^2(1+M_1)-i\frac{\partial}{\partial t}\right]&G(t,t')-\delta(t-t')=M_0G(t,t')\\&-i(4\pi\alpha)^{\frac{1}{2}}\int\frac{(dk)}{(2\pi)^3}U(\mathbf{k})\langle 0t|a(\mathbf{k},t)|0t'\rangle\theta(t-t')\\&-i(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\mathbf{k}\cdot\mathbf{k}'A(\mathbf{k})A(\mathbf{k}')\\&\qquad\times\langle 0t|a(\mathbf{k},t)a(\mathbf{k}',t)|0t'\rangle\theta(t-t').\end{aligned} \tag{12}$$

Finally, the equation analogous to (I-10) is

$$\begin{aligned}\left\{\left[P^2(1+M_1)-i\frac{\partial}{\partial t}\right]\right.&G(t,t')-\delta(t-t')\quad\left.P^2(1+M_1)-i\frac{\overleftarrow{\partial}}{\partial t'}\right]=-M_0\delta(t-t')\\&+M_0{}^2G(t,t')+iM_0(4\pi\alpha)^{\frac{1}{2}}\int\frac{(dk)}{(2\pi)^3}U(\mathbf{k})\langle 0t|a^\dagger(\mathbf{k},t')|0t'\rangle\theta(t-t')\\&+iM_0(4\pi\alpha)^{\frac{1}{2}}\int\frac{(dk)}{(2\pi)^3}U(\mathbf{k})\langle 0t|a(\mathbf{k},t)|0t'\rangle\theta(t-t')\\&+i(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}U(\mathbf{k})U(\mathbf{k}')\langle 0t|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t')|0t'\rangle\theta(t-t')\\&+iM_0(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\mathbf{k}\cdot\mathbf{k}'A(\mathbf{k})A(\mathbf{k}')\langle 0t|a^\dagger(\mathbf{k},t')a^\dagger(\mathbf{k}',t')|0t'\rangle\theta(t-t')\\&+iM_0(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\mathbf{k}\cdot\mathbf{k}'A(\mathbf{k})A(\mathbf{k}')\langle 0t|a(k,t)a(k',t)|0t'\rangle\theta(t-t')\\&+i(4\pi\alpha)^{\frac{3}{2}}\int\frac{(dk)}{(2\pi)^3}\frac{(dk'')}{(2\pi)^3}\frac{(dk''')}{(2\pi)^3}U(\mathbf{k})\mathbf{k}''\cdot\mathbf{k}'''A(\mathbf{k}'')A(\mathbf{k}''')\langle 0t|a(\mathbf{k},t)a^\dagger(\mathbf{k}'',t')a^\dagger(\mathbf{k}''',t')|0t'\rangle\theta(t-t')\\&+i(4\pi\alpha)^{\frac{3}{2}}\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\frac{(dk'')}{(2\pi)^3}U(\mathbf{k}'')\mathbf{k}\cdot\mathbf{k}'A(\mathbf{k})A(\mathbf{k}')\langle 0t|a(k,t)a(k',t)a^\dagger(k'',t')|0t'\rangle\theta(t-t')\\&+i(4\pi\alpha)^2\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}\frac{(dk'')}{(2\pi)^3}\frac{(dk''')}{(2\pi)^3}(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}''\cdot\mathbf{k}''')A(\mathbf{k})A(\mathbf{k}')A(\mathbf{k}'')A(\mathbf{k}''')\\&\qquad\times\langle 0t|a(\mathbf{k},t)a(\mathbf{k}',t)a^\dagger(\mathbf{k}'',t')a^\dagger(\mathbf{k}''',t')|0t'\rangle\theta(t-t').\end{aligned} \tag{13}$$

We now introduce the noncorrelation assumption in a manner completely analogous to (I-22, 23, etc.). Thus, matrix elements of the form $\langle 0t|a^\dagger(\mathbf{k},t')|0t'\rangle$, $\langle 0t|a^\dagger(\mathbf{k},t')a^\dagger(\mathbf{k}',t')|0t'\rangle$, etc., are neglected; the M_0^2 term is deferred to a higher order; and the noncorrelation is written as

$$i\langle 0t|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t')|0t'\rangle\theta(t-t') \approx -iG(t,t')\mathcal{G}(\mathbf{k},\mathbf{k}';t,t') \quad (14)$$

and

$$i\langle 0t|a(\mathbf{k},t)a(\mathbf{k}',t)a^\dagger(\mathbf{k}'',t')a^\dagger(\mathbf{k}''',t')|0t'\rangle\theta(t-t') \approx -G(t,t')\mathcal{G}'(\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}''';t,t'). \quad (15)$$

The phonon Green's functions, $\mathcal{G}$ and $\mathcal{G}'$, are then replaced by their free-field values, i.e., their values when α, and thus $A(\mathbf{k})$, is set equal to zero. Under these circumstances the phonon vacuum states become time-independent and

$$i\frac{\partial}{\partial t}\langle 0|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t')|0\rangle = (1-2\mathbf{P}\cdot\mathbf{k}+k^2)\langle 0|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t')|0\rangle. \quad (16)$$

From (2) it follows that

$$\langle 0|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t)|0\rangle=\delta(\mathbf{k}-\mathbf{k}'). \quad (17)$$

Hence, the free-field value of $\mathcal{G}$, written with the delta function removed, is

$$\delta(\mathbf{k}-\mathbf{k}')\mathcal{G}_k^0(t,t')\equiv i\langle 0|a(\mathbf{k},t)a^\dagger(\mathbf{k}',t')|0\rangle\theta(t-t') =\delta(\mathbf{k}-\mathbf{k}')i\exp[-i(k^2-2\mathbf{P}\cdot\mathbf{k}+1)(t-t')] \times\theta(t-t'). \quad (18)$$

In a similar manner, $\mathcal{G}'$ is replaced by

$$[\delta(k-k'')\delta(k'-k''')+\delta(k-k''')\delta(k'-k'')] \times\mathcal{G}_{kk'}^{\prime 0}(t,t') \quad (19)$$

where

$$\mathcal{G}_{kk'}^{\prime 0}(t,t')=\mathcal{G}_k^0(t,t')\mathcal{G}_{k'}^0(t,t'). \quad (20)$$

With all this, the approximate equation here becomes

$$\left\{\left[P^2(1+M_1)-i\frac{\partial}{\partial t}\right]G(t,t')-\delta(t-t')\right\}\left[P^2(1+M_1)-i\frac{\overleftarrow{\partial}}{\partial t'}\right] =-M_0\delta(t-t')-i(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}U^2(\mathbf{k})G(t,t')\mathcal{G}_k^0(t,t') -2(4\pi\alpha)^2\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}(\mathbf{k}\cdot\mathbf{k}')^2A^2(\mathbf{k})A^2(\mathbf{k}') \times G(t,t')\mathcal{G}_{kk'}^{\prime 0}(t,t'). \quad (21)$$

This is the equation analogous to (I-17, 27). There the technique of solution involved the use of the spectral form of G, and we shall follow the same procedure here.

3. THE SPECTRAL COEFFICIENTS

The Green's function may be written as

$$G(t,t')=i\int d\omega\,\rho(P^2,\omega)e^{-i\omega(t-t')}\theta(t-t'), \quad (22)$$

where the ρ are non-negative, as they represent the squares of the absolute values of matrix elements. Inserting this in (21), multiplying by $e^{i\omega'(t-t')}$, integrating with respect to $(t-t')$ from $-\infty$ to ∞, and noting that[3]

$$\frac{1}{\omega-i\epsilon}=i\int_0^\infty e^{-i\omega t}dt, \quad \epsilon\to 0+, \quad (23)$$

we get, using (18) and (20),

$$\int d\omega\frac{\rho(P^2,\omega)}{\omega-\omega'-i\epsilon}[P^2(1+M_1)-\omega']^2-[P^2(1+M_1)-\omega'] =-M_0+(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}U^2(\mathbf{k}) \times\int d\omega\frac{\rho(P^2,\omega-k^2+2\mathbf{k}\cdot\mathbf{P}-1)}{\omega-\omega'-i\epsilon} +2(4\pi\alpha)^2\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}(\mathbf{k}\cdot\mathbf{k}')A^2(\mathbf{k})A^2(\mathbf{k}') \times\int d\omega\frac{\rho(P^2,\omega-k^2-k'^2+2(\mathbf{k}+\mathbf{k}')\cdot\mathbf{P}-2)}{\omega-\omega'-i\epsilon}. \quad (24)$$

The imaginary part of (24) is

$$\rho(P^2,\omega)[P^2(1+M_1)-\omega]^2 =(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}U^2(\mathbf{k})\rho(P^2,\omega-k^2+2\mathbf{k}\cdot\mathbf{P}-1) +2(4\pi\alpha)^2\int\frac{(dk)}{(2\pi)^3}\frac{(dk')}{(2\pi)^3}(\mathbf{k}\cdot\mathbf{k}')^2A^2(\mathbf{k})A^2(\mathbf{k}') \times\rho(P^2,\omega-k^2-k'^2+2(\mathbf{k}+\mathbf{k}')\cdot\mathbf{P}-2), \quad (25)$$

and the real part, in conjunction with (25), yields

$$\int\rho(P^2,\omega)d\omega=1, \quad (26)$$

and

$$\int\omega\rho(P^2,\omega)d\omega=M_0+P^2(1+M_1). \quad (27)$$

Equation (27) then determines the change in energy and effective mass once the ρ are determined from (25) and normalized by (26).

[3] J. Schwinger, Phys. Rev. **82**, 664 (1951).

4. THE SOLUTION FOR $P^2=0$

For simplicity, only the case $P^2=0$ will be considered. This will be sufficient for the calculation of the lowest energy shift, M. Thus, for $P^2=0$,

$$A(\mathbf{k})\equiv A^0(k^2);\quad U(\mathbf{k})\equiv U^0(k^2);\quad \rho(0,\omega)\equiv\rho(\omega);\quad M'=M_0. \tag{28}$$

The functions A^0 and U^0 depend only on k^2, as that is the only scalar that can be constructed from $\mathbf{k}$ with $P^2=0$. Some of the integrals in (8) and (9) then vanish by symmetry, leaving

$$U^0(k^2)=(1+k^2)A^0(k^2)+1/k, \tag{29}$$

and

$$M_0=M+(4\pi\alpha)\int\frac{(dk)}{(2\pi)^3}\left[(1+k^2)A^0(k^2)+\frac{2}{k}A^0(k^2)\right]. \tag{30}$$

Equations (26) and (27) become

$$\int\rho(\omega)d\omega=1, \tag{31}$$

and

$$\int\omega\rho(\omega)d\omega=M_0. \tag{32}$$

Upon performing the angular integrations in (25), we get

$$\omega^2\rho(\omega)=\frac{2}{\pi}\alpha\int_0^\infty dk\,k^2[U^0(k^2)]^2\rho(\omega-k^2-1)+\frac{2}{3}\frac{4}{\pi^2}\alpha^2\int_0^\infty dkdk'\,k^4k'^4[A^0(k^2)]^2\times[A^0(k'^2)]^2\rho(\omega-k^2-k'^2-2). \tag{33}$$

It is to be recalled that, by the definition of M, the lowest energy state is taken as zero. This is equivalent to the statement

$$\rho(P^2,\omega)=0,\quad \omega<0. \tag{34}$$

Now, in (33), choose $\omega<1$. The right-hand side then vanishes, yielding

$$\omega^2\rho(\omega)=0,\quad \omega<1. \tag{35}$$

The only solution of (35) compatible with (30) and (32) may be seen to be[4]

$$\rho(\omega)=\rho_1\delta(\omega),\quad \omega<1. \tag{36a}$$

Now, let $1\leqslant\omega<2$, then (33) becomes

$$\rho(\omega)=\frac{2}{\pi}\alpha\rho_1\frac{1}{\omega^2}\int_0^\infty dk\,k^2[U^0(k^2)]^2\delta(\omega-k^2-1)=\frac{\alpha}{\pi}\rho_1\frac{(\omega-1)^{\frac{1}{2}}}{\omega^2}[U^0(\omega-1)]^2,\quad 1\leqslant\omega<2, \tag{36b}$$

since for $1\leqslant\omega<2$, it follows that $\omega-k^2-1<1$. In a similar manner, for $2\leqslant\omega<3$, the results (36) may be utilized to yield

$$\rho(\omega)=\frac{\alpha}{\pi}\rho_1[U^0(\omega-1)]^2\frac{(\omega-1)^{\frac{1}{2}}}{\omega^2}+\frac{2}{\pi^2}\alpha^2\rho_1\int_0^{(\omega-2)^{\frac{1}{2}}}dk\Big\{k^2[U^0(k^2)]^2[U^0(\omega-2-k^2)]^2\times\frac{(\omega-2-k^2)^{\frac{1}{2}}}{\omega^2(\omega-k^2-1)}+\frac{2}{3}k^4\frac{(\omega-2-k^2)^{\frac{3}{2}}}{\omega^2}[A^0(k^2)]^2\times[A^0(\omega-2-k^2)]^2\Big\},\quad 2\leqslant\omega<3. \tag{36c}$$

This process may be carried on to yield the value of $\rho(\omega)$ for any value of ω. This method of solution is the same inductive method used in obtaining the solution (I-43). The physical basis is also the same, unity here being the energy necessary to create a phonon, and consequently the analog of the meson mass μ.

5. THE VARIATIONAL PRINCIPLE

The value of $\rho(\omega)$ given by (36c) contains the first example of the contribution from the "two-phonon term" in (33), i.e., the α^2 term. As an extremely simple-minded approximation, let it be assumed that (36c) is valid for all values of $\omega\geqslant 2$, i.e., neglect the iteration of these terms in (33). This is the most elementary non-trivial assumption one might make, since if the series were terminated at (36b) the results would simply be lowest order perturbation theory. This approximation may be expected to be poorest for large α since it is just the higher powers of α that are here neglected. Granting this, to evaluate M, we first determine ρ_1 from (31), and then M from (32) and (30). Still to be determined, of course, are the translations $A^0(k^2)$. Thus, inserting (30) and (36) in (31) and (32),

$$1=\rho_1+\frac{\alpha}{\pi}\rho_1\int_1^\infty d\omega[U^0(\omega-1)]^2\frac{(\omega-1)^{\frac{1}{2}}}{\omega^2}+\frac{2}{\pi^2}\alpha^2\rho_1\int_2^\infty d\omega\int_0^{(\omega-2)^{\frac{1}{2}}}dk\Big\{k^2[U^0(k^2)]^2\times[U^0(\omega-2-k^2)]^2\frac{(\omega-2-k^2)}{\omega^2(\omega-k^2-1)}+\frac{2}{3}k^4\frac{(\omega-2-k^2)^{\frac{3}{2}}}{\omega^2}[A^0(k^2)]^2[A^0(\omega-2-k^2)]^2\Big\} \tag{37}$$

[4] It should be noted that for $P^2\neq 0$ one obtains, in place of (36a), for $1>1-\omega\geqslant P^2\geqslant 0$, $\rho(P^2,\omega)=\rho_1(P^2)\delta(\omega-P^2(1+M_1))$. Thus the effective mass is $\frac{1}{2}(1+M_1)^{-1}$.

and

$$M=-\frac{2}{\pi}\alpha\int_0^\infty dk\,k^2\Big\{(1+k^2)[A^0(k^2)]^2+\frac{2}{k}A^0(k^2)\Big\}$$
$$+\frac{\alpha}{\pi}\int_1^\infty d\omega\,[U^0(\omega-1)]^2\frac{(\omega-1)^{\frac{1}{2}}}{\omega}+\frac{2}{\pi^2}\alpha^2\rho_1$$
$$\times\int_2^\infty d\omega\int_0^{(\omega-2)^{\frac{1}{2}}}dk\Big\{k^2[U^0(k^2)]^2[U^0(\omega-2-k^2)]^2$$
$$\times\frac{(\omega-2-k^2)^{\frac{1}{2}}}{\omega(\omega-k^2-1)}+\frac{2}{3}\frac{k^4(\omega-2-k^2)^{\frac{3}{2}}}{\omega}[A^0(k^2)]^2$$
$$\times[A^0(\omega-2-k^2)]^2\Big\}. \quad (38)$$

The determination of A^0 is now accomplished by means of the variational principle which minimizes the lowest energy state, *viz.*,

$$\delta M/\delta A^0(k^2)=0. \quad (39)$$

This results in an integral equation for $A^0(k^2)$ which, noting that (37) makes ρ_1 a functional of A^0, we find to be

$$A^0(k^2)\Big[\frac{2}{\pi}\alpha(1-\rho_1)k(1+k^2)\Big]$$
$$=\frac{-1}{k(1+k^2)}\Big[\frac{2}{\pi}\alpha(1-\rho_1)k(1+k)^2\Big]$$
$$+\frac{2}{\pi^2}\alpha^2\int_{k^2+2}^\infty d\omega\Big[U^0(k^2)[U^0(\omega-2-k^2)]^2\frac{k(\omega-2-k^2)^{\frac{1}{2}}}{(\omega-k^2-1)}$$
$$+\frac{4}{3}\frac{k^3(\omega-2-k^2)^{\frac{3}{2}}}{\omega}A^0(k^2)[A^0(\omega-2-k^2)]^2\Big]$$
$$-\rho_1{}^2\Big\{\frac{2}{\pi}\alpha U^0(k^2)\frac{k}{1+k^2}+\frac{2}{\pi^2}\alpha^2\int_{k^2+2}^\infty d\omega$$
$$\times\Big[U^0(k^2)[U^0(\omega-2-k^2)]^2\frac{k(\omega-2-k^2)^{\frac{1}{2}}}{\omega(\omega-k^2-1)}$$
$$+\frac{4}{3}\frac{k^2(\omega-2-k^2)^{\frac{3}{2}}}{\omega^2}A^0(k^2)[A^0(\omega-2-k^2)]^2\Big]\Big\}$$
$$\times\Big\{\frac{\alpha}{\pi}\int_1^\infty d\omega\,U^0(\omega-1)\frac{(\omega-1)^{\frac{1}{2}}}{\omega}$$
$$+\frac{2}{\pi^2}\alpha^2\int_2^\infty d\omega\int_0^{(\omega-2)^{\frac{1}{2}}}dk\Big[k^2[U^0(k^2)]^2$$
$$\times[U^0(\omega-2-k^2)]^2\frac{(\omega-2-k^2)^{\frac{1}{2}}}{\omega(\omega-k^2-1)}$$
$$+\tfrac{2}{3}k^4\frac{(\omega-2-k^2)^{\frac{3}{2}}}{\omega}[A^0(k^2)]^2[A^0(\omega-2-k^2)]^2\Big]\Big\}. \quad (40)$$

TABLE I. Self-energy, M, as a function of coupling, α, with $\hbar\omega=1$.

α	0.100	0.500	1.00	1.50	2.00	2.50	3.00	3.50
M	0.1003	0.506	1.02	1.54	2.06	2.58	3.09	3.59
Feynman's result	0.1001	0.503	1.01	1.52	2.05	2.59	3.13	3.69

As the crudest approximation to (40), assume that the terms involving $A^0(k^2)[A^0(\omega-2-k^2)]^2$ are negligible, i.e., neglect the two-phonon terms in (40). The solution is then simply

$$A^0(k^2)=\frac{-1}{k(1+k^2)}, \quad (41)$$

which implies, by (29),

$$U^0(k^2)=0. \quad (42)$$

Again, this sort of approximation may be expected to be especially bad for large α.

6. THE ENERGY SHIFT

If these results are now used in (37) and (38), one obtains

$$1/\rho_1=1+(0.175)\alpha^2, \quad (43)$$

and

$$M=\alpha+(0.0253)\rho_1\alpha^2, \quad (44)$$

or

$$M=\alpha+\frac{(0.0253)\alpha^2}{1+(0.175)\alpha^2}. \quad (45)$$

In Table I we compare the result (45) with those obtained by Feynman.[5] It will be seen that, despite the crudeness of (41) and the approximation which terminates at (36c), our results do not appreciably depart from those of Feynman's until as large a value as $\alpha=3$. For large values of α, Eq. (45) is expected to be invalid due to the nature of these approximations, not necessarily because of the basic approximations leading to (33). It would seem quite likely that a more adequate solution of (40), or the termination of the solution of (33) at a point somewhat later than (36c) would yield correspondingly better results.

7. ACKNOWLEDGMENT

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[5] R. P. Feynman, Phys. Rev. **97**, 660 (1955).