Inelastic Final-State Interactions: K^- Absorption in Deuterium*

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In a reaction from which several strongly interacting particles emerge it is often possible to isolate the effects of forces between two of the outgoing particles. There are many cases in which this final-state interaction can produce inelastic reactions. The formalism that describes this situation is developed here, and the reaction $K^-+d \rightarrow \pi+\Sigma+N \rightarrow \pi+\Lambda+N'$ is studied in detail as an example. It is found that large Λ/Σ branching ratios can result and can be used to restrict the Σ-N and K-N interaction parameters. The gross features of the spectrum can be understood using a simple model. It does not seem possible to determine the parities of the strange particles from the momentum spectra. When the inelastic reaction in the final state is exothermic, as in the example, high partial waves may contribute.

I. INTRODUCTION

THE theory of final-state interactions[1] has been fruitfully applied to many elementary-particle processes. In the characteristic situation a "primary" reaction mechanism produces a set of outgoing particles which have strong mutual interactions. The final-state interaction concept is useful if the mutual interaction of one pair can be isolated from all other effects in this final state. This can happen if the relative momentum of this special pair is low compared to all other momenta in the state, and if the interaction of this pair with the other emergent particles is weak. When these conditions are fulfilled, the remaining particles escape quickly from the production region and have little effect on the pair which is left behind. The low-momentum pair can interact strongly for a longer time, producing important modifications in the energy spectrum and in the total reaction rate. If this final-state interaction is attractive, for instance, a low relative momentum will be favored and the reaction rate will be increased. Because the effect of other particles in the final state can be neglected, the scattering of the interacting pair can be related to the scattering properties of this pair when they are isolated. The final-state interaction concept will therefore be useful when there is only one important interaction in the final state; this should involve only low relative momentum and, for convenience, only a small number of angular momentum waves.

It is of interest to consider the application of this method to problems in which the strong final-state interaction can lead to inelastic processes. This can occur, for instance, when the primary reaction, which produces the final state, leads to antinucleons, K mesons with negative strangeness, or Σ hyperons. We shall be particularly interested in the reaction

$$K^-+d \rightarrow \pi+\Sigma+N \rightarrow \pi+\Lambda+N', \qquad (1)$$

proceeding through the strong Σ-N interaction, and shall see that the method can be adapted to this case. There are three principal differences from the conventional applications:

1. The diffuse structure of the initial state (the deuteron) permits the separation of the primary interaction from the final-state interaction. Since the range of the forces is small compared to the interparticle spacing, explicit many-particle effects involving the simultaneous interaction of more than two particles are expected to be small. The neglect of these effects is usually described as the "impulse approximation"[2] or the "multiple-scattering approximation." In this approximation we can describe *both* the primary interaction and the final-state interaction in our problem by parameters which can be determined from two-particle scattering and reaction data.

2. Even though the directly produced particles are in states of low relative momentum, the energy released in the inelastic process can lead to observed particle-pairs with high relative momenta. Many angular momentum waves in the final state may therefore contribute.

3. It is possible for a directly produced pair of particles to be in a (virtual) state such that their relative kinetic energy is negative. An inelastic final-state interaction can couple this state to a real final state provided the inelastic reaction is sufficiently exothermic. This part of the final-state spectrum cannot be directly related to free-particle reaction data since it involves the scattering matrix in an "unphysical" region.

To illustrate these facets of the final-state viewpoint we shall consider the strange-particle reaction (1) as an example. To isolate the final-state interaction between the Σ and the nucleon, we must neglect the interaction of the pion with these particles. If we do this, we have a problem which contains all the features of this method but in their simplest form: there are only three particles (the minimum number for our "many-particle"

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[1] K. M. Watson, Phys. Rev. 88, 1163 (1952).

[2] G. F. Chew and G. C. Wick, Phys. Rev. 85, 636 (1952); G. F. Chew and M. L. Goldberger, Phys. Rev. 87, 778 (1952).

system); there are only two outgoing-particle channels; and the large radius of the deuteron compared to the range of the two-particle forces enables us to use the "impulse approximation" with confidence.

Experimental evidence on this process has recently been reported.[3] Reactions following the capture of K^- mesons at rest by deuterium were studied. In particular, the spectrum of π^- mesons [Fig. 1(a)] associated with the reaction $K^-+d\rightarrow\pi^-+\Lambda+p$ has shown two peaks, one around 250 Mev/c and another near 190 Mev/c. These are just the momenta one would expect if the K meson were absorbed on a *single* nucleon to produce $\Lambda+\pi$ or $\Sigma+\pi$, respectively. The high-momentum peak can be explained by the direct production of $\Lambda+\pi$ on a single nucleon in the deuteron; the low peak can be interpreted as the production of $\Sigma+\pi$ on one nucleon, followed by the reaction $\Sigma+N\rightarrow\Lambda+N'$ on the second nucleon. In this latter case the pion momentum is characteristic of Σ production. The conversion reaction competes with elastic Σ-N scattering; it is this competition which we shall study. The directly produced Λ's may suffer some elastic scattering on the remaining nucleon,[4] but their inelastic collisions are of minor importance and will not be our concern here.

In our detailed study of these reactions, we shall, within the framework of the model discussed above, investigate the consequences of various assumptions concerning the initial state of the K meson and the character of the K-N and Σ-N interactions. Since only preliminary work on K-mesonic x-rays and on K- and Σ-particle absorption in hydrogen has been done, none of these parameters of our problem is known at the present time. The deuterium experiment does not seem to permit a detailed determination of the interactions, but its results will place restrictions upon them.

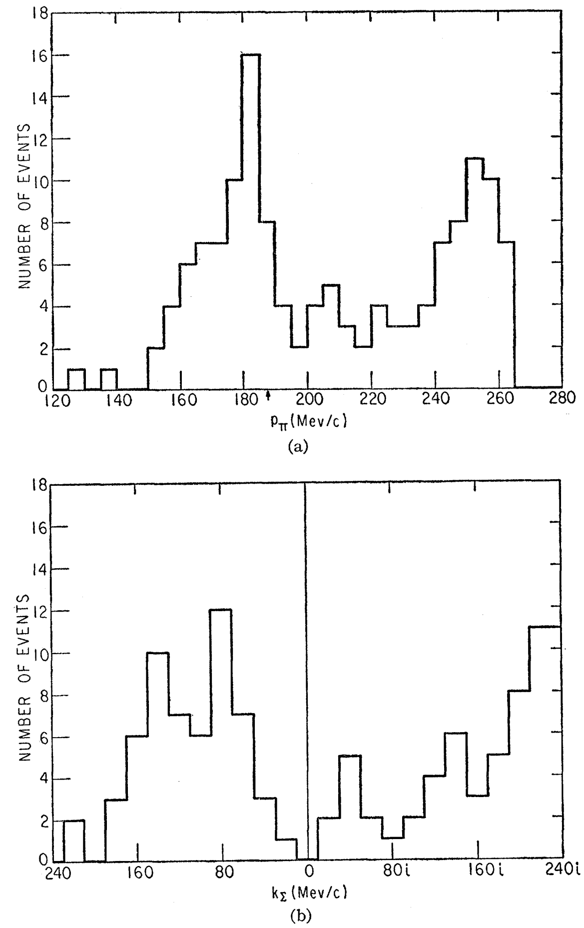

FIG. 1. Spectrum of $K^-+d\rightarrow\pi^-+\Lambda+p$: (a) as a function of the pion momentum and (b) as a function of k_Σ [see text, Eq. (11)]. Arrow denotes value of p_π for which $k_\Sigma=0$ ($p_\pi=188$ Mev/c). The k_Σ region depicted in Fig. 1(b) corresponds to $120<p_\pi<240$ (Mev/c).

II. FINAL-STATE INTERACTION FORMALISM

To understand the final-state interaction viewpoint, we shall examine what modifications in the exact matrix element for the process must be made to arrive at that formalism.

We first define the interactions in this problem. The K-nucleon interaction which interests us here is that part which leads to a state of $\Sigma+\pi$ and $\Lambda+\pi$; we denote this by V_K. (For simplicity we do not write the interactions with the individual nucleons, although this must be done in practice). We shall denote the complete Σ-N interaction, including both elastic and inelastic parts, by V_Y. This interaction vanishes if it acts on a state in which no hyperon is present. The remaining interactions will be denoted by V_r.

The exact matrix element for K^- absorption can be written in terms of ϕ_i, the initial state of a deuteron and a K meson bound in a Coulomb orbit, and $\psi_f^{(-)}$, the ingoing wave solution of the total Hamiltonian. $\psi_f^{(-)}$ is a solution of the integral equation

$$\psi_f^{(-)}=\phi_f+G_0^{(-)}(V_K+V_Y+V_r)\psi_f^{(-)}, \tag{2}$$

where ϕ_f is a free plane wave of the final nucleon, pion, and either Λ or Σ hyperon, and $G_0^{(-)}$ is the free-particle Green's function for total energy E_K+M_d satisfying ingoing-wave boundary conditions. In terms of these functions, the matrix element is[5]

$$M_{fi}=\langle\psi_f^{(-)}|V_K|\phi^4\rangle. \tag{3}$$

To relate this matrix element to other observable quantities, it is convenient to replace all potentials V_j by free-particle scattering operators[2,5,6] defined by the integral equation

$$T_j=V_j+V_jG_0^{(+)}T_j. \tag{4}$$

[3] N. Horwitz *et al.*, Bull. Am. Phys. Soc. **3**, 363 (1958). The theory for this process in the absence of final state interactions has been discussed by A. Fujii and R. E. Marshek, Nuovo cimento **8**, 643 (1958).

[4] F. Crawford *et al.*, Phys. Rev. Letters **2**, 174 (1959).

[5] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

[6] K. M. Watson, Phys. Rev. **89**, 575 (1953); **105**, 1388 (1957).

The matrix elements of the operators T_j between plane-wave states are the exact scattering amplitudes for the corresponding two-body problem. The interaction between each pair of particles is described by the appropriate scattering operator.

If we perform this replacement, we find[6]

$$M_{fi}=\langle\psi_K^{(-)}|T_K|\phi i\rangle, \tag{5}$$

where

$$\psi_K^{(-)}=\phi_f+G_0^{(-)}\sum_{j\neq K}T_j^\dagger\psi_j^{(-)}, \tag{6}$$

with similar equations for the other $\psi_j^{(-)}$ ($j=Y$, K, and r). (The adjoint of T_j enters here because $\psi_K^{(-)}$ is the ingoing-wave solution.) The use of the scattering operator greatly reduces the number of terms which need be included to obtain an accurate result.

In this form $\psi_K^{(-)}$ has replaced $\psi_f^{(-)}$ as the wave function describing the final state. As we see, it differs by the removal of the most direct effects of the K-absorption mechanism. In our work we want to approximate this function by a wave function which includes only the effect of the Σ-N interaction. This function will be

$$_Y\chi^{(-)}=\phi_f+G_0^{(-)}V_{YY}\chi^{(-)}=\phi_f+G_0^{(-)}T_Y^\dagger\phi_f. \tag{7}$$

It is the product of a plane wave representing the pion and a two-body wave function describing the relative motion of the Σ-N system.

We can easily re-express $\psi_K^{(-)}$ in terms of this function. Using the definition of $\psi_K^{(-)}$ and $\psi_Y^{(-)}$ from Eq. (6), we have

$$\begin{aligned}\psi_K^{(-)}&=\phi_f+G_0^{(-)}T_Y^\dagger(\phi_f+G_0^{(-)}\sum_{j\neq Y}T_j^\dagger\psi_j^{(-)})\\&\qquad+G_0^{(-)}T_r^\dagger\psi_r^{(-)}\\&=\chi_Y^{(-)}+G_0^{(-)}T_Y^\dagger G_0^{(-)}T_K^\dagger\psi_K^{(-)}\\&\qquad+(1+G_0^{(-)}T_Y^\dagger)G_0^{(-)}T_r^\dagger\psi_r^{(-)}.\end{aligned}$$

Inserted into Eq. (5), this yields

$$\begin{aligned}M_{fi}=&\langle\chi_Y^{(-)}|T_K|\phi_i\rangle+\langle\psi_K^{(-)}|T_KG_0^{(+)}T_YG_0^{(+)}T_K|\phi_i\rangle\\&+\langle\psi_r^{(-)}|T_rG_0^{(+)}(1+T_YG_0^{(+)})|\phi^4\rangle.\end{aligned} \tag{8}$$

The first term includes the effects of the primary production mechanism and the Σ-N interaction. In the final-state interaction approach it is considered to be the dominant term:

$$M_{fi}\cong\langle\chi_Y^{(-)}|T_K|\phi_i\rangle. \tag{9}$$

This expression can be represented by Fig. 2.

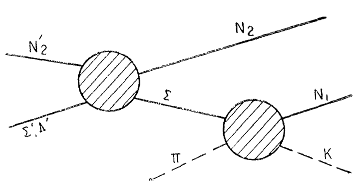

FIG. 2. Schematic representation of the process $K^-+d\rightarrow\pi+\Sigma+N_2\rightarrow\pi+\Sigma'+N_2'$ or $\pi+\Lambda+N_2'$.

The remaining terms in Eq. (8) represent higher order scattering corrections.[7] The second term contains "re-productive" processes in which the primary Σ is scattered, reabsorbed to yield a $K+N$, then must be reproduced; it is thus at least of third order in the production mechanism T_K. Figure 3 illustrates an example of such a process. These complicated processes are highly improbable, and we shall neglect them. The simplest contributions of the third term involve scattering of the pion by the Σ and the nucleon. These are obtained by letting $\psi_K^{(-)}\cong\chi_Y^{(-)}$. Examples are shown in Fig. 4(a) and 4(b).

III. FORMULATION OF $\Sigma-\Lambda$ CONVERSION PROBLEM

We now direct our attention toward the wave function $\chi_Y^{(-)}$ which contains the final state interaction. In a coordinate representation it has the form (with the 3-particle center of mass at rest and units $\hbar=c=1$)

$$\chi_Y^{(-)}(\mathbf{r}_\pi,\mathbf{R},\mathbf{r})=\frac{1}{(2\pi)^3}\exp[i\mathbf{p}_\pi\cdot(\mathbf{r}_\pi-\mathbf{R})]g^{(-)}(\mathbf{r}), \tag{10}$$

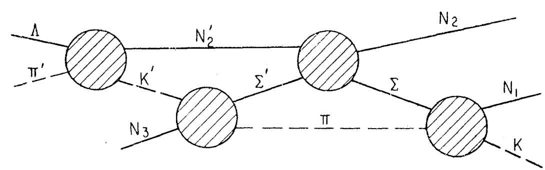

FIG. 3. Schematic representation of the process $K^-+d\rightarrow\pi+\Sigma+N_2\rightarrow\pi+\Sigma'+N_2'\rightarrow K'+N_3+N_2'\rightarrow\pi'+\Lambda+N_3$.

where $\mathbf{R}=(m_Y\mathbf{r}_Y+m_N\mathbf{r}_N)/(m_Y+m_N)$ is the center-of-mass coordinate of the hyperon-nucleon system and $\mathbf{r}=\mathbf{r}_Y-\mathbf{r}_N$ is the baryon relative coordinate. The momentum of the outgoing pion is $\mathbf{p}_\pi$.

The energy of the final state is given by

$$\frac{k_Y^2}{2\mu_{YN}}+\frac{p_\pi^2}{2(m_Y+m_N)}+(p_\Lambda^2+m_\pi^2)^{\frac{1}{2}}=E_K+m_d-m_N-m_Y=Q_Y \tag{11}$$

for either hyperon in the final state. $k_Y^2/2\mu_{YN}$ is the center-of-mass energy of the hyperon-nucleon system. The function $g^{(-)}(\mathbf{r})$ which describes the internal properties of the hyperon-nucleon system is a solution of the Schrödinger equation

$$(\nabla^2+k_Y^2-2\mu_{YN}V_Y)g^{(-)}(\mathbf{r})=0. \tag{12}$$

Since the interaction V_Y can change the identity of the hyperon ($\Sigma\leftrightarrow\Lambda$), it is convenient to represent $g^{(-)}(\mathbf{r})$

[7] Not all many-body effects are included in this formulation. We have assumed that all interactions can be represented by two-body potentials. In a correct field-theoretic description there are explicit three-body corrections arising from the exchange of virtual particles; such effects are not included in the present formulation. Consequently, any computation of the higher-order corrections within the framework presented here (which includes only those three-body processes which are successions of two-body effects) would necessarily be incomplete.

by a two-component wave function, one component being the Σ-N state and the other, the Λ-N state. In this notation V_Y is a 2×2 matrix, and k_Y and μ_{YN} are diagonal matrices.

There are two independent solutions of Eq. (12), corresponding to ingoing waves plus either Σ-particle plane waves or Λ-particle plane waves. Using Eqs. (7) and (12), these two solutions are

$$g_\Sigma^{(-)}(\mathbf{r})=\begin{pmatrix} g_{\Sigma\Sigma}^{(-)}(\mathbf{r}) \\ g_{\Lambda\Sigma}^{(-)}(\mathbf{r})\end{pmatrix} \text{ and } g_\Lambda^{(-)}(\mathbf{r})=\begin{pmatrix} g_{\Sigma\Lambda}^{(-)}(\mathbf{r}) \\ g_{\Lambda\Lambda}^{(-)}(\mathbf{r})\end{pmatrix}, \quad (13)$$

where

$$g_{\Sigma\Sigma}^{(-)}(\mathbf{r})=\exp(i\mathbf{k}_\Sigma\cdot\mathbf{r})-\frac{2\mu_{\Sigma N}}{4\pi}\int d\mathbf{r}'d\mathbf{r}'' \times\frac{\exp(-i\mathbf{k}_\Sigma|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}T_{\Sigma\Sigma}^\dagger(\mathbf{r}',\mathbf{r}'') \times\exp(i\mathbf{k}_\Sigma\cdot\mathbf{r}''), \quad (14a)$$

$$g_{\Lambda\Sigma}^{(-)}(\mathbf{r})=-\frac{2\mu_{\Lambda N}}{4\pi}\int d\mathbf{r}'d\mathbf{r}''\frac{\exp(-i\mathbf{k}_\Lambda<\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \times T_{\Lambda\Sigma}^\dagger(\mathbf{r}',\mathbf{r}'')\exp(i\mathbf{k}_\Sigma\cdot\mathbf{r}''), \quad (14b)$$

and

$$g_{\Sigma\Lambda}^{(-)}(\mathbf{r})=-\frac{2\mu_{\Sigma N}}{4\pi}\int d\mathbf{r}'d\mathbf{r}''\frac{\exp(-i\mathbf{k}_\Sigma|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} T_{\Sigma\Lambda}^\dagger(\mathbf{r}',\mathbf{r}'')\exp(i\mathbf{k}_\Lambda\cdot\mathbf{r}''), \quad (15a)$$

$$g_{\Lambda\Lambda}^{(-)}(\mathbf{r})=\exp(i\mathbf{k}_\Lambda\cdot\mathbf{r})-\frac{2\mu_{\Lambda N}}{4\pi}\int d\mathbf{r}'d\mathbf{r}'' \times\frac{\exp(-i\mathbf{k}_\Lambda|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}T_{\Lambda\Lambda}^\dagger(\mathbf{r}',\mathbf{r}'') \times\exp(i\mathbf{k}_\Lambda\cdot\mathbf{r}''). \quad (15b)$$

The Σ component in the first solution contains a free Σ wave and an ingoing wave of Σ particles; the Σ component of the second solution also contains an ingoing Σ wave. These components represent a Σ converging on the nucleon to yield either a Σ or a Λ with amplitudes given by $T_{\Sigma\Sigma}$ and $T_{\Sigma\Lambda}$, in addition to the free Σ wave. Similarly the Λ components represent the final-state interaction of a Λ particle converging on the nucleon to scatter or convert to a Σ.

In the particular problem under consideration it is possible to distinguish experimentally between Λ's which are produced directly and those produced indirectly (except in a small region of overlap in momentum space). Directly produced Λ's do not induce many inelastic reactions (very few of them are sufficiently energetic to produce Σ's) and we shall henceforth ignore them and the interaction $T_{K\Lambda}$ which produces them. We shall concentrate on the fate of Σ's which are produced by the primary interaction $T_{K\Sigma}$; their subsequent motion is described by $g_{\Sigma\Sigma}^{(-)}(\mathbf{r})$ and $g_{\Sigma\Lambda}^{(-)}(\mathbf{r})$.

The matrix $T_{ij}(\mathbf{r}',\mathbf{r}'')$ is directly related to the observed matrix element for free-particle scattering:

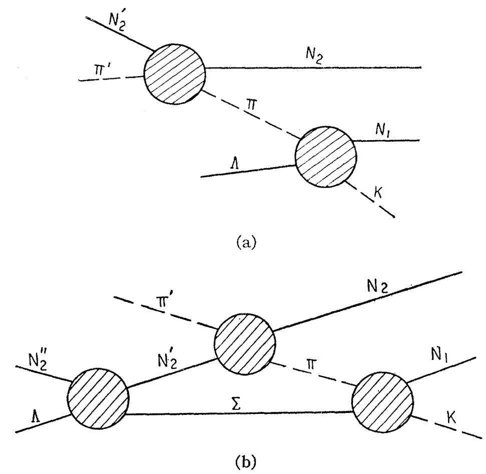

FIG. 4. Schematic representation of final-state interactions of the pion.

$$T_{ij}(\mathbf{k}_i,\mathbf{k}_j)=\int d\mathbf{r}'d\mathbf{r}''\, e^{-i\mathbf{k}_i\cdot\mathbf{r}'}T_{ij}(\mathbf{r}',\mathbf{r}'')e^{i\mathbf{k}_j\cdot\mathbf{r}''}, \quad (16)$$

where $\mathbf{k}_i$ and $\mathbf{k}_j$ are the corresponding center-of-mass momenta. A spin matrix element is also implied here. Since each $T_{ij}(\mathbf{k}_i,\mathbf{k}_j)$ represents the amplitude of an individual outgoing scattered wave at $t=\infty$, it must satisfy certain requirements in order that the total outgoing current shall equal the total ingoing current. These conditions can be obtained from the general principles of unitarity or by imposing current conservation on Eq. (14) and (15). Time-reversal invariance implies that $T_{ij}(\mathbf{k}_i,\mathbf{k}_j)=T_{ji}(-\mathbf{k}_j,-\mathbf{k}_i)$; parity conservation implies that $T_{ij}(\mathbf{k}_i,\mathbf{k}_j)=T_{ij}(-\mathbf{k}_i,-\mathbf{k}_j)$. Together they yield $T_{ij}(\mathbf{k}_i,\mathbf{k}_j)=T_{ji}(\mathbf{k}_j,\mathbf{k}_i)$. Using this and the Hermiticity of V_Y (which implies that both the diagonal and nondiagonal matrix elements of the current operator vanish on a surface at infinity, i.e., that the eigenstates of this Hamiltonian carry no net current), we obtain the following result:[8]

$$\mathrm{Im} f_{ij}(\mathbf{k}_i,\mathbf{k}_j)=\sum_m k_m\int\frac{d\Omega_m}{4\pi}f_{im}^\dagger(\mathbf{k}_i,\mathbf{k}_m)f_{mj}(\mathbf{k}_m,\mathbf{k}_j); \quad f_{ij}(\mathbf{k}_i,\mathbf{k}_j)=-(\mu_i/2\pi)T_{ij}(\mathbf{k}_i,\mathbf{k}_j) \quad (17)$$

is the usual scattering amplitude on the "energy shell," and

$$d\sigma_{ij}/d\Omega=(v_i/v_j)|f_{ij}(\mathbf{k}_i,\mathbf{k}_j)|^2$$

is the differential cross section. The generalization of Eq. (17) to many-channel problems and to include the spin degree of freedom is straightforward. It may be used to obtain the usual phase shift expansion. If the forces are spin-independent, one can obtain the

[8] R. Glauber and V. Schomaker, Phys. Rev. **89**, 667 (1953).

expansion

$$f_{ij}(\mathbf{k}_i,\mathbf{k}_j)=\frac{1}{k_j}\left(\frac{v_j}{v_i}\right)^{\frac{1}{2}}\sum_l (2l+1)f_{ij}{}^{(l)}P_l\left(\frac{\mathbf{k}_i\cdot\mathbf{k}_j}{k_ik_j}\right), \quad (18)$$

with $f_{ij}{}^{(l)}=f_{ji}{}^{(l)}$ and

$$\mathrm{Im}f_{ij}{}^{(l)}=\Sigma_m f_{im}{}^{(l)*}f_{mj}. \quad (19)$$

If there are only two states, as in the present example, these equations can be solved:

$$f_{ii}{}^{(l)}=(1/2i)(X_l\exp(2i\delta_i{}^{(l)})-1),$$
$$f_{ij}{}^{(l)}=\tfrac{1}{2}(1-X_l{}^2)^{\frac{1}{2}}\exp[i(\delta_i{}^{(l)}+\delta_j{}^{(l)})]. \quad (20)$$

X_l is a real parameter between zero and one; $\delta_i{}^{(l)}$ is the real phase shift.

With these restrictions, one can insure that the wave function of Eqs. (13)–(15) satisfy the conservation laws. In general this is not easy to do since the unitarity restriction, while relatively simple in momentum space, becomes complicated when applied to the spatial representation $T_{ij}(\mathbf{r}',\mathbf{r}'')$. We have avoided this difficulty by assuming zero-range interactions, for which the connection between coordinate space and momentum space is especially simple.

If spin dependences are inserted, the possibility of different structures for $T_{\Sigma\Lambda}$ depending on the Σ-Λ parity must be considered. In general the spectrum, angular correlations, and polarizations will depend on this parity. We are here going to examine only the spectrum and in our case this has only a weak dependence on the Σ-Λ parity. The Σ-Λ mass difference is so large that the Λ always has a large momentum which does not vary greatly over the width of the peaks in the spectra. It is only from a detailed knowledge of the incident Σ-N partial waves and a high experimental resolution that information on the parity could be obtained. In the absence of these conditions we shall assume that the parity is even and that the interaction is primarily spin independent.

Another characteristic of the inelastic final-state interaction problem is well illustrated in our example. We may distinguish two regions of pion momentum. When

$$p_\pi{}^2/2(m_\Sigma+m_N)+(p_\pi{}^2+m_\pi{}^2)^{\frac{1}{2}}<Q_\Sigma \quad (p_\pi<188\text{ Mev}/c),$$

both Σ's and Λ's may be produced. When

$$p_\pi{}^2/2(m_\Sigma+m_N)+(p_\pi{}^2+m_\pi{}^2)^{\frac{1}{2}}>Q_\Sigma \quad (p_\pi>188\text{ Mev}/c),$$

only Λ's may emerge from the deuteron. The pion may be emitted with such large momentum that not enough energy remains to produce a real Σ. It is still possible, however, for a virtual Σ to be created by the primary mechanism and to convert into a Λ through the final-state interaction. Such a virtual Σ can be considered to have negative kinetic energy and imaginary momentum through Eq. (11):

$$k_\Sigma{}^2/2\mu_{\Sigma N}=Q_\Sigma-p_\pi{}^2/2(m_\Sigma+m_N)-(p_\pi{}^2+m_\pi{}^2)^{\frac{1}{2}}<0.$$

The correct analytic continuation to this unphysical region is obtained by requiring that the wave function Eq. (13) remain finite for all r, which implies that $k_\Sigma\to -i\kappa_\Sigma(\kappa_\Sigma=|k_\Sigma|)$ in this function. (In more familiar applications one deals with outgoing waves, for which the correct continuation is $k_\Sigma\to i\kappa_\Sigma$).

The possibility of obtaining Λ's in a momentum range where Σ production is impossible shows another feature of this type of problem: The number of observed conversion Λ's is not simply related to the number of Σ's produced by the primary interaction. The unitarity conditions imply only that there is no net current carried out of the interaction volume; there are still local currents within the region of interaction. The behavior of these currents is described by the familiar remark that while real particles can propagate to infinite distances, virtual particles are constrained to remain in the region of interaction. These local virtual currents can, however, give real effects if they are intercepted by a scattering center and converted into real waves. Thus, while unitarity assures that one hyperon is produced for each K meson absorbed, the number of K mesons reacting *per unit time* depends not only on the energy-conserving transitions induced by V_K and observed in K-hydrogen experiments, but also on the additional reaction channels made possible by the final-state interaction. The production of Λ particles, the scattering of Σ particles from virtual to real states, and the complicated interference effects between the primary and scattered Σ's affect the reaction rates and modify the branching ratios.

IV. EVALUATION OF MATRIX ELEMENTS: S-WAVE CASE†

There are two remaining functions in the matrix element of Eq. (9) which must be specified. The general form of the initial state is

$$\phi_i=\xi_K\left(\mathbf{r}_K-\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\right)\varphi_d(\mathbf{r}_1-\mathbf{r}_2) \quad (21)$$

in the rest frame of the system. $\varphi_d(\mathbf{r}_1-\mathbf{r}_2)$ is the internal wave function of the deuteron; $\xi_K(\mathbf{r}_K-(\mathbf{r}_1+\mathbf{r}_2)/2)$ is the orbital wave function of the K meson relative to the center of the deuteron.

In coordinate space the hyperon production interaction has the form

$$T_K(\mathbf{r}_\pi,\mathbf{r}_Y;\mathbf{r}_K,\mathbf{r}_1)=\delta(\mathbf{R}_2-\mathbf{R}_1)T_K(\boldsymbol{\varrho}_1,\boldsymbol{\varrho}_1), \quad (22)$$

where

$$\mathbf{R}_1=\frac{m_K\mathbf{r}_K+m_N\mathbf{r}_1}{m_K+m_N} \quad\text{and}\quad \mathbf{R}_2=\frac{m_Y\mathbf{r}_Y+m_\pi\mathbf{r}_\pi}{m_Y+m_\pi}$$

are the center-of-mass coordinates, and the δ-function arises from the Galilean invariance and translation invariance of the interaction (the latter leads to conservation of total momentum in momentum space).

† See reference ‡ in Sec. VII.

$\boldsymbol{\rho}_1=\mathbf{r}_K-\mathbf{r}_1$ and $\boldsymbol{\rho}_2=\mathbf{r}_\pi-\mathbf{r}_Y$ are the relative coordinates. $T_K(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1)$ is related to the production cross section for K mesons on free nucleons in the same way as $T_{ij}(\mathbf{r}',\mathbf{r}'')$ is related to the cross section for hyperons on nucleons [Eq. (16) and succeeding discussion].

Inserting Eqs. (10), (21), and (22) into Eq. (9), we obtain

$$M_{fi}=\frac{1}{(2\pi)^j}\int d\mathbf{s}d\mathbf{r}d\boldsymbol{\rho}_1d\boldsymbol{\rho}_2 \times\exp\left[-i\mathbf{p}_\pi\cdot\left(\boldsymbol{\rho}_2+\frac{m_N}{m_N+m_Y}\mathbf{r}\right)\right]g^{(-)*}(\mathbf{r}) \times\delta\left(\mathbf{r}-\mathbf{s}-\frac{m_K}{m_K+m_N}\boldsymbol{\rho}_1+\frac{m_\pi}{m_Y+m_\pi}\boldsymbol{\rho}_2\right)T_K(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1) \times\zeta_K(\boldsymbol{\rho}_1+\tfrac{1}{2}\mathbf{s})\varphi_d(\mathbf{s}). \quad (23)$$

The δ-function links the four relative coordinates which are most convenient in this problem. The extra dependences on ρ_1 and ρ_2 which occur because of this δ-function simply enforce the requirement that the matrix element can depend on r_2, the position of the "spectator" nucleon, only through the relative coordinate $\mathbf{r}_2-\mathbf{r}_1$. The "recoil" corrections which they generate can give angular correlations but do not appear to modify the spectra or conversion probabilities. For this reason we shall simplify the subsequent discussion by neglecting these corrections. M_{fi} then takes the form

$$M_{fi}=\frac{1}{(2\pi)^3}\int d\mathbf{r}d\boldsymbol{\rho}_1d\boldsymbol{\rho}_2\, e^{-i\mathbf{q}\cdot\mathbf{r}}g^{(-)*}(\mathbf{r}) \times\exp(-i\mathbf{p}_\pi\cdot\boldsymbol{\rho}_2)T_K(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1)\xi_K(\boldsymbol{\rho}_1+\tfrac{1}{2}\mathbf{r})\varphi_d(\mathbf{r}), \quad (24)$$

where $\mathbf{q}=[m_N/(m_N+m_Y)]\mathbf{p}_\pi$.

For nuclear capture from a bound atomic orbit, the K meson wave function is a hydrogenic Coulomb function about the center of mass of the deuteron. This latter condition follows from the fact that the nuclear motion is very much faster than the motion of a particle in a Coulomb orbit (so that any charge asymmetries are averaged out) and is consistent with the observed charge independence of the reaction. By analogy with the quite similar calculation of Brueckner, Serber, and Watson,[9] we assume that nuclear capture takes place from a low-lying atomic level since radiative and Auger-electron processes are expected to dominate for the higher levels. Since the Bohr radius $a_K=1/\mu_Ke^2$ is nearly fifteen times the deuteron radius, we can approximate the wave function by its amplitude near the origin. For an nS orbit this gives

$$\xi_K(\boldsymbol{\rho}_1+\tfrac{1}{2}\mathbf{s})\cong N_{nS}=(1/\pi n^3a_K^3)^{\frac{1}{2}}. \quad (25)$$

We use a Hulthén form for the deuteron wave function:

$$\varphi_d(r)=N_d(e^{-\alpha r}-e^{-\beta r})/r, \quad (26)$$

with

$$N_d=\left(\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right)^{\frac{1}{2}},\quad \alpha=(m_N\epsilon_d)^{\frac{1}{2}}\cong m_\pi/3,\quad \beta=7\alpha.$$

In practice we find that β may be permitted to become arbitrarily large without modifying our results. The short-range or high-momentum behavior of the deuteron function is important only when the Σ-N momentum k_Σ is large or when the final-state wave function becomes highly singular for small r.

To compute the probability of nuclear capture we must have a specific form for the K-N interaction. In this section we shall consider capture through the S-wave K-N channel. Making a zero-range approximation and assuming here a pseudoscalar K meson, we have

$$T_{K\Sigma}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1)=A_{SS}\delta(\boldsymbol{\rho}_2)\delta(\boldsymbol{\rho}_1). \quad (27)$$

Using Eq. (24) for the S-orbit, S-channel case, we find

$$M_{fi}=[1/(2\pi)^3]N_dN_{nS}A_{SS}\mathfrak{M}_{fi}{}^S, \quad (28)$$

with

$$\mathfrak{M}_{fi}{}^S=\int d\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}g^{(-)*}(\mathbf{r})\frac{e^{-\alpha r}}{r}. \quad (29)$$

We must now specify the form of the Σ-N interaction. In this section we shall study the properties of an S-wave interaction (both absorption and scattering); the case of higher partial waves is discussed later. We have assumed that the interaction has zero range, since estimates of the effect of a finite range indicate modifications of the order of 10% or less.

For this case Eqs. (14a) and (15a) become

$$g_{\Sigma\Sigma}{}^{(-)}(\mathbf{r})=\exp(i\mathbf{k}_\Sigma\cdot\mathbf{r})+f_{\Sigma\Sigma}{}^{(0)\dagger}(rk_\Sigma)^{-1}\exp(-ik_\Sigma r), \quad (30a)$$

$$g_{\Sigma\Lambda}{}^{(-)}(\mathbf{r})=f_{\Sigma\Lambda}{}^{(0)\dagger}(rk_\Lambda)^{-1}(v_\Lambda/v_\Sigma)^{\frac{1}{2}}\exp(-ik_\Sigma r). \quad (30b)$$

Inserting these into Eq. (29), we obtain

$$\mathfrak{M}_\Sigma{}^S=4\pi\left\{\frac{1}{(\mathbf{k}_\Sigma+\mathbf{q})^2+\alpha^2}+if_{\Sigma\Sigma}{}^{(0)}I_{00}\right\},$$

$$\mathfrak{M}_\Lambda{}^S=4\pi i(k_\Sigma M_\Sigma/k_\Lambda M_\Lambda)^{\frac{1}{2}}f_\Sigma{}^{(0)}I_{00}, \quad (31)$$

where

$$I_{00}=\int_0^\infty rdr\, h_0{}^{(1)}(k_\Sigma r)j_0(qr)e^{-\alpha r} =\frac{1}{2k_\Sigma q}\ln\frac{k_\Sigma+q+i\alpha}{k_\Sigma-q+i\alpha}. \quad (32)$$

Here $j_0(qr)$ and $h_0{}^{(1)}(k_\Sigma r)$ are the spherical Bessel function and spherical Hankel function of the first kind, respectively.[10]

[9] Brueckner, Serber, and Watson, Phys. Rev. **81**, 575 (1951).

[10] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1465.

From the viewpoint of the final-state interaction, it is clear that the interesting momentum is not the pion momentum but is k_Σ, the relative momentum of the colliding Σ-N system. In fact, it seems most convenient and illuminating to consider both the Σ spectrum and the Λ spectrum as functions of this variable. The transition rates that we consider are then

$$R_\Sigma(k_\Sigma)=\frac{dw_\Sigma}{dk_\Sigma}=\int|M_\Sigma|^2\rho(k_\Sigma)d\Omega_k,$$

$$R_\Lambda(k_\Sigma)=\frac{dw_\Lambda}{dk_\Sigma}=R_\Lambda(k_\Lambda)\frac{dk_\Lambda}{dk_\Lambda}=\int|M_\Lambda|^2\rho(k_\Lambda)\frac{dk_\Lambda}{dk_\Sigma}d\Omega_k, \quad (33)$$

$$\rho(k_i)=8\pi^2k_i^2p_\pi\left(\frac{1}{E_\pi}+\frac{1}{m_i+m_N}\right)^{-l},\quad \frac{dk_\Lambda}{dk_\Sigma}=\frac{v_\Sigma}{v_\Lambda}.$$

It is convenient also to plot the Λ spectrum as a function of k_Σ, even though this variable becomes imaginary for values of p_π such that Σ production is forbidden; part of the experimental spectrum of Fig. 1(a) is displayed in this way in Fig. 1(b). The spectra differ most strikingly in the region of $k_\Sigma=0$; the k_Σ spectrum must vanish at this point due to the phase space factor.

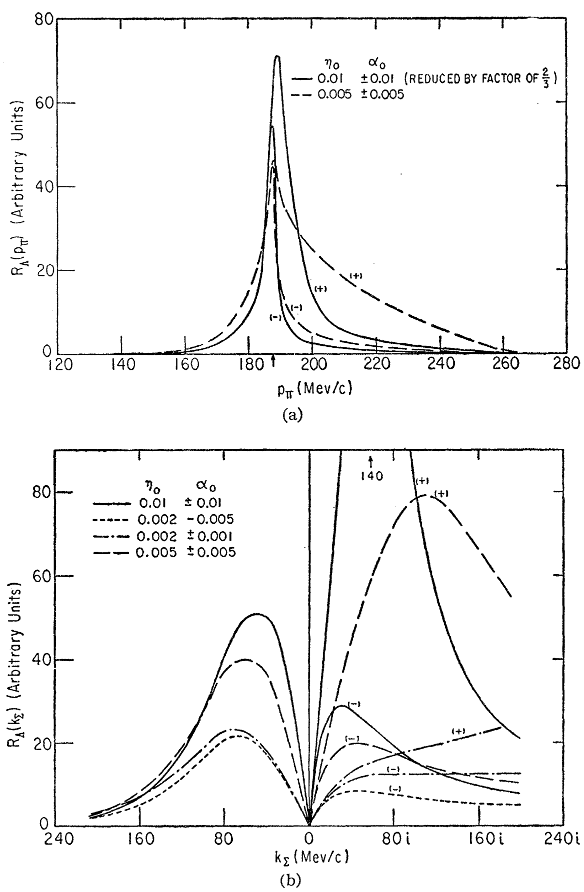

FIG. 5. Λ spectrum as a function of the S-wave Σ-N interaction parameters for the "S-wave" case. $\tan\delta_\Sigma^{(0)}=-k_\Sigma(a_0-i\eta_0)$ is used. η_0 and a_0 are measured in units of $(\text{Mev}/c)^{-1}$. The labels on the curves in the imaginary region denote the sign of a_0; the curves in the real region are independent of this sign.

We shall represent the absorption and scattering amplitudes by a scattering length formula

$$\tan\delta_\Sigma^{(0)}=-k_\Sigma(a_0-i\eta_0), \quad (34)$$

where $\delta_\Sigma^{(0)}$ is the complex S-wave Σ-N phase shift $\{f_{\Sigma\Sigma}^{(0)}=[1/2i][\exp(2i\delta_\Sigma^{(0)})-1]$, etc.$\}$. In this approximation

$$f_{\Sigma\Sigma}^{(0)}=\frac{-k_\Sigma a_0+i[k_\Sigma\eta_0(1+k_\Sigma\eta_0)+(k_\Sigma a_0)^2]}{(1+k_\Sigma\eta_0)^2+(k_\Sigma a_0)^2},$$
$$|f_{\Sigma\Lambda}^{(0)}|^2=\frac{k_\Sigma\eta_0}{(1+k_\Sigma\eta_0)^2+(k_\Sigma a_0)^2}. \quad (35)$$

In the region in which only Λ's may be produced, we use the assumption that the amplitudes $f_{ij}^{(l)}$ are analytic functions of the energy in the upper half of the complex energy plane and can thus be analytically continued from the region in which k_Σ is real to that in which $k_\Sigma=i\kappa_\Sigma$. The result is that

$$|f_{\Sigma\Lambda}^{(0)}|^2=\frac{\kappa_\Sigma\eta_0}{(1-\kappa_\Sigma a_0)^2+(\kappa_\Sigma\eta_0)^2}. \quad (36)$$

We now have all the material needed to compute the Λ and Σ spectra. The numerical results are presented in Fig. 5 and 6 and in Table I. They illustrate the dependence of the conversion ratio $\mathcal{R}=w_\Lambda/w_\Sigma$ and the spectrum shape on the Σ-N interaction parameters. To compute the ratio we must integrate over the Σ spectrum and the conversion $-\Lambda$ spectrum. In doing this we have defined conversion Λ's to be those for which $k_\Sigma^2>-(200\ \text{Mev}/c)^2$.

When a_0 is negative, representing an attractive Σ-N force, **R** is less than one but can still be quite large. Making a_0 more negative (increasing the attraction) increases the rate for producing Σ's but does not greatly affect the Λ rate, since this depends primarily on η_0.[11] On the other hand, when a_0 is positive, **R** can be quite large, and the Λ spectrum takes on a characteristic shape: it rises sharply in the "unphysical" region. A positive a_0 corresponds to the existence of a bound state with binding energy $E_B\sim(1/2\mu_{\Sigma N})(1/a_0)^2$ in the Σ-N system. Because of the coupling to the Λ, this state decays quickly and has a width $\delta\kappa_\Sigma\sim1/\eta_0$ or $\delta E\sim(a_0/\eta_0)E_B$. For $a_0\gtrsim0.005(\text{Mev}/c)^{-1}$, corresponding to a binding energy $E_B\lesssim20$ Mev, there will be a peak in this region.

[11] From the point of view of a fundamental theory, this distinction is of course not physically meaningful, since both η_0 and a_0 are determined simultaneously from the equations of motion. If there is a net attraction, the Σ-N wave function will be increased for small r with the consequence that inelastic processes will be correspondingly larger.

V. DEPENCENCE ON K-NUCLEON INTERACTION PARAMETERS

There is a large probability for nuclear capture of the K meson from a P orbit. Capture from this orbit can take place through the S-wave K-N channel because of the finite size of the deuteron. Even though the K meson is in a P orbit with respect to the center of mass of the deuteron, it has an S-wave component with respect to either nucleon when that nucleon is displaced from the center of mass. At the same time there will be considerable P wave present, so that if there is a P-wave interaction it will lead to capture as well.

We assume that there is an interaction in the S- and P-wave incident channels

$$T_{K\Sigma}(\boldsymbol{\varrho}_2,\boldsymbol{\varrho}_1)=A(\boldsymbol{\varrho}_2,\rho_1{}^2)+\mathbf{B}(\boldsymbol{\varrho}_2,\rho_1{}^2)\cdot\mathbf{p}_1. \qquad (37)$$

$\mathbf{p}_1$ denotes the gradient operator on $\boldsymbol{\varrho}_1$, and the functions $A(\boldsymbol{\varrho}_2,\rho_1{}^2)$ and $\mathbf{B}(\boldsymbol{\varrho}_2,\rho_1{}^2)$ depend on the intrinsic parity of the K-N-Σ system. If K is pseudoscalar relative to N-Σ, then A must be a scalar and B a polar vector

TABLE I. Conversion ratio $\mathcal{R}=w_\Lambda/w_\Sigma$ for the "S-wave" case (see text, Sec. IV) as a function of S-wave Σ-N interaction parameters. Scattering lengths are in units of $(\text{Mev}/c)^{-1}$.

η_0	a_0	$\mathcal{R}$
0.01	−0.01	0.60
0.005	−0.005	0.53
0.002	−0.005	0.22
0.002	−0.001	0.38
0.01	+0.01	3.0
0.005	+0.005	2.0
0.002	+0.005	1.9
0.002	+0.001	0.57

which we write

$$A_-(\boldsymbol{\varrho}_2,\rho_1{}^2)=A_{SS}F_{SS}(\rho_2{}^2,\rho_1{}^2),$$
$$\mathbf{B}_-(\boldsymbol{\varrho}_2,\rho_1{}^2)=B_{PP}F_{PP}(\rho_2{}^2,\rho_1{}^2)\mathbf{p}_2+B_{SO}F_{SO}(\rho_2{}^2,\rho_1{}^2)\boldsymbol{\sigma}\times\mathbf{p}_2. \qquad (38-)$$

If K is scalar, then A must be a pseudoscalar and B an axial vector:

$$A_+(\boldsymbol{\varrho}_2,\rho_1{}^2)=A_{PS}F_{PS}(\rho_2{}^2,\rho_1{}^2)\boldsymbol{\sigma}\cdot\mathbf{p}_2,$$
$$\mathbf{B}_+(\boldsymbol{\varrho}_2,\rho_1{}^2)=B_{SP}F_{SP}(\rho_2{}^2,\rho_1{}^2)\boldsymbol{\sigma}+B_{DP}F_{DP}(\rho_2{}^2,\rho_1{}^2)\times[p_2{}^2\boldsymbol{\sigma}-3(\boldsymbol{\sigma}\cdot\mathbf{p}_2)\mathbf{p}_2]. \qquad (38+)$$

In the zero-range limit which we shall use, all the scalar functions become products of δ-functions, $F_{ij}(\rho_2{}^2,\rho_1{}^2)\rightarrow\delta(\boldsymbol{\varrho}_2)\delta(\boldsymbol{\varrho}_1)$. The detailed structure of these functions is unimportant for this problem since the actual range is small compared to the deuteron size.

As with the Σ-Λ parity, it seems to be impossible to determine the K parity simply from a study of the conversion-Λ spectrum. When these forms are inserted into Eq. (24), $\mathbf{p}_2$ is replaced by the pion momentum $\mathbf{p}_\pi$.

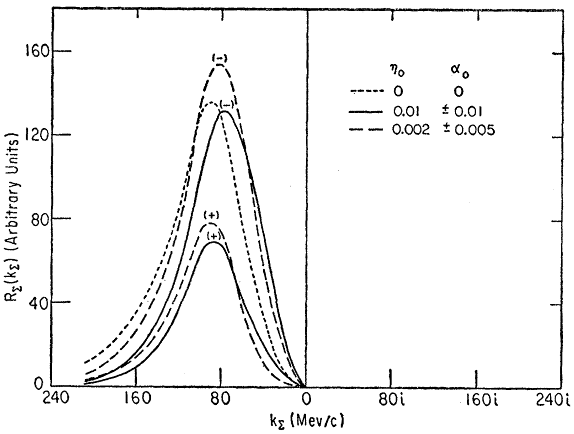

FIG. 6. Σ spectrum as a function of the S-wave Σ-N interaction parameters for the "S-wave" case. $\tan\delta_\Sigma{}^{(0)}=-k_\Sigma(a_0-i\eta_0)$ is used. η_0 and a_0 are measured in units of $(\text{Mev}/c)^{-1}$. The labels on the curves denote the sign of a_0. $\eta_0=a_0=0$ represents no final-state interaction.

This momentum changes little over this spectrum and the dependence on it, which distinguishes the two cases, is difficult to determine. Only polarization measurements seem capable of deciding the K parity in this experiment.

For an S orbit we expect the P channel contribution to be extremely small since the K-mesonic Bohr radius, which determines the variation in the Coulomb wave function, is very much larger than the range of the absorptive interaction. The calculation in the previous section will then be nearly independent of the K-nucleon interaction—as long as there is some S-wave K-N absorption, as seems to be indicated by experiment.[12]

For an nP orbit, we write the wave function as

$$\xi_K(\boldsymbol{\varrho}_1+\tfrac{1}{2}\mathbf{s})\cong N_{nP}(\boldsymbol{\varrho}_1+\tfrac{1}{2}\mathbf{s})\cdot\mathbf{e}_i, \qquad (39)$$

where $N_{nP}=[(n^2-1)/3\pi n^5a_K{}^5]^{\frac{1}{2}}$ and $\mathbf{e}_i$ specifies the orientation of the P orbit. If we insert this into Eq. (24), we find

$$M_{fi}{}^P=[1/(2\pi)^3]N_DN_{nP}[\tfrac{1}{2}A\mathfrak{M}_{fi}{}^P-i(\mathbf{B}\cdot\mathbf{e}_i)\mathfrak{M}_{fi}{}^S], \qquad (40)$$

where

$$\mathfrak{M}_{fi}{}^P=\int d\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}g^{(-)*}(\mathbf{r})\frac{e^{-\alpha r}}{r}(\mathbf{r}\cdot\mathbf{e}_i), \qquad (41)$$

and $A=A_{SS}$, $\mathbf{B}=B_{PP}\mathbf{p}_\pi+B_{SO}\boldsymbol{\sigma}\times\mathbf{p}_\pi$ for a pseudoscalar K; $A=A_{PS}\boldsymbol{\sigma}\cdot\mathbf{p}_\pi$, $\mathbf{B}=B_{SP}\boldsymbol{\sigma}+B_{DP}[p_\pi{}^2\boldsymbol{\sigma}-3(\boldsymbol{\sigma}\cdot\mathbf{p}_\pi)\mathbf{p}_\pi]$ for a scalar K.

To study the effect of the K-N parameters, we have considered the model of a zero-range S-wave Σ-N interaction, as before. The results for absorption in other partial waves are similar.

[12] M. F. Kaplon, *Proceedings of the* 1958 *Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958).

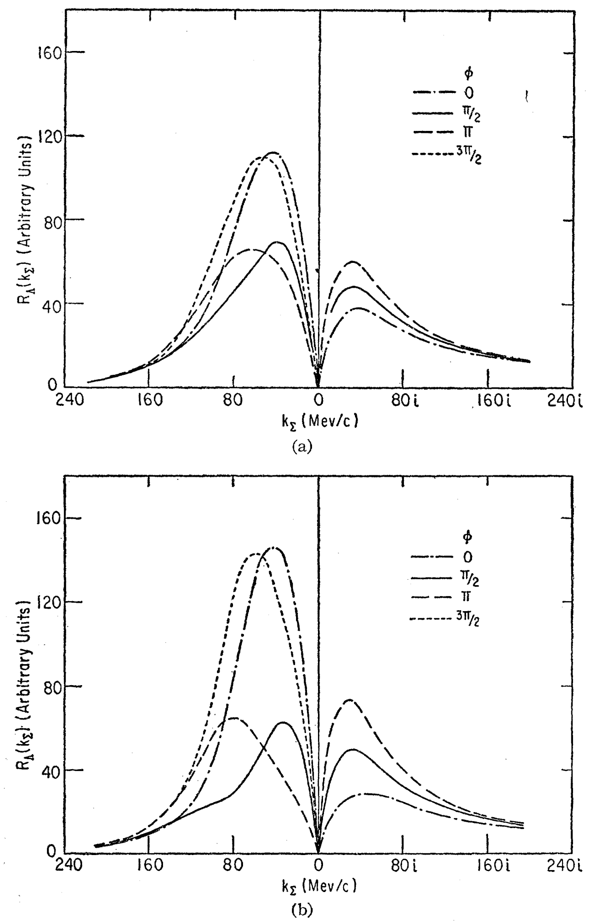

FIG. 7. Λ spectrum for capture from a P orbit as a function of the K-N interaction parameters with $\eta_0=a_0=-0.01\,(\text{Mev}/c)^{-1}$. The ratio of the S- and P-channel amplitudes is: (a) $A_{SS}/B_{PP}=\frac{1}{2}e^{i\phi}$; (b) $A_{SS}/B_{PP}=e^{i\phi}$; the units are $(\mu_\pi c)^2$. In the imaginary region, $R_\Lambda(k_\Sigma)$ is independent of $\text{Im}(A_{SS}/B_{PP})$.

In this case

$$\mathfrak{M}_\Sigma^P=-4\pi i\left\{\frac{2(\mathbf{k}_\Sigma+\mathbf{q})\cdot\mathbf{e}_i}{[(\mathbf{k}_\Sigma+\mathbf{q})^2+\alpha^2]^2}+if_{\Sigma\Sigma}^{(0)}\frac{(\mathbf{q}\cdot\mathbf{e}_i)}{q}J_{01}\right\},$$
$$\mathfrak{M}_\Lambda^P=4\pi(k_\Sigma M_\Sigma/k_\Lambda M_\Lambda)^{\frac{1}{2}}f_{\Sigma\Lambda}^{(0)}\frac{(\mathbf{q}\cdot\mathbf{e}_i)}{q}J_{01} \qquad (42)$$

where

$$J_{01}=\int_0^\infty r^2dr\,h_0^{(1)}(k_\Sigma r)j_1(qr)e^{-\alpha r}=\frac{1}{q}I_{00}-\frac{1}{k_\Sigma q}\frac{k_\Sigma+i\alpha}{(k_\Sigma+i\alpha)^2-q^2}. \qquad (43)$$

The integral $\mathfrak{M}_{fi}^S$ which first occured for S capture from an S orbit enters here also for P capture [the terms multiplied by B in Eq. (40)] from a P orbit; these both belong to a general "S-wave" case. In this case the relative angular momentum of the Σ-N wave emerging from the primary production center is largely S-wave; it would be pure S-wave if the Σ recoil, represented by $-\mathbf{q}$, were zero. The "P-wave" case, when the Σ-N wave is largely P-wave, occurs for S-channel capture from a P orbit.

The numerical results obtained using the above formulas are presented in Fig. 7 and Table II. The calculations were performed for the case of a pseudoscalar K meson. We have set $B_{SO}=0$ since only the combinations $|A_{SS}|^2$, $B_{PP}A_{SS}^*$, and $|B_{PP}|^2+2|B_{SO}|^2$ enter into the spectrum and conversion ratio.

In the model considered here, there is only S-wave Σ-N absorption, so that the P-wave case yields a small conversion ratio. As Table II shows, a significant contribution of the S channel to capture from the P orbit decreases $\mathfrak{R}$.

VI. Σ-N ABSORPTION IN HIGHER PARTIAL WAVES

We have seen that capture from a P orbit can lead to an intermediate Σ-N state which has only a small amount of S wave. There must therefore be a considerable amount of P and higher partial waves present in this state. Hence it is important to consider the effect of Σ-N absorption through higher partial waves. The various partial waves do not interfere in the total reaction rate, so we may consider them separately.

For absorption in the lth partial wave through a zero-range interaction, Eq. (15a) becomes

$$g_{\Sigma\Lambda}^{(-)}(\mathbf{r};l)=(1/i^{l+1})k_\Sigma f_{\Sigma\Lambda}^\dagger h_l^{(2)}(k_\Sigma r)P_l(\mathbf{k}_\Lambda\cdot\mathbf{r}/k_\Lambda r), \qquad (44)$$

where, from Eq. (18),

$$k_\Sigma f_{\Sigma\Lambda}=(\mu_{\Sigma N}k_\Sigma/\mu_{\Lambda N}k_\Lambda)^{\frac{1}{2}}(2l+1)f_{\Sigma\Lambda}^{(l)}.$$

Consider the dependence of the reaction rate $R_\Lambda(k_\Sigma)$ on the momentum k_Σ near $k_\Sigma=0$. For a short-range interaction $f_{\Sigma\Lambda}$ behaves as k_Σ^l for $k_\Sigma\rightarrow 0$. This dependence comes from the fact that the scattering amplitude is the plane-wave matrix element Eq. (16) of the interaction and hence contains the usual angular momentum barrier associated with the fact that

TABLE II. Conversion ratio $\mathfrak{R}=w_\Lambda/w_\Sigma$ for capture from a P orbit as a function of K-N interaction parameters. $\tan\delta_\Sigma^{(0)}=-k_\Sigma(a_0-i\eta_0)$ with $\eta_0=-a_0=0.01\,(\text{Mev}/c)^{-1}$. Define $A_{SS}/B_{PP}=\chi e^{i\phi}$ with χ in units of $(\mu_\pi c)^2$.

χ	ϕ	$\mathfrak{R}$
$\frac{1}{2}$	0	0.58
	$\pi/2$	0.40
	π	0.42
	$3\pi/2$	0.60
1	0	0.43
	$\pi/2$	0.23
	π	0.24
	$3\pi/2$	0.45
∞		0.005

$j_l(k_\Sigma r) \rightarrow (k_\Sigma r)^l$ as $k_\Sigma \rightarrow 0$. The free particle Σ-absorption cross-section Eq. (17) is then proportional to k_Σ^{2l-1}. In our problem, however, the Σ-N wave is represented, not by a spherical Bessel function, but by the spherical Hankel function $h_l^{(2)}(k_\Sigma r)$ of Eq. (44). The Hankel function behaves as $(1/k_\Sigma r)^{l+1}$, with the consequence that $g_{\Sigma\Lambda}^{(-)}(\mathbf{r};l)$ is independent of k_Σ for small k_Σ. Equation (33) ff. shows that $\rho(k_\Lambda)dk_\Lambda/dk_\Sigma$ is proportional to k_Σ. Since M_Λ is proportional to the overlap integral Eq. (24), and hence to $g_{\Sigma\Lambda}^{(-)}(\mathbf{r};l)$, we find that $R_\Lambda(k_\Sigma)$ is proportional to k_Σ for all values of l.

By contrast, $f_{\Sigma\Sigma}$ is proportional to k_Σ^{2l}, so that $R_\Sigma(k_\Sigma)$ behaves as k_Σ^{2l+1} for small k_Σ. That is, the angular momentum barrier does limit the elastic scattering in the final state to low partial waves.[13] In the inelastic channel of the process the outgoing particle is a high-energy Λ particle, for which the angular momentum barrier does not become significant until l is very large. The important feature here is that there is no angular momentum barrier in the intermediate state. If the inelastic reaction is sufficiently exothermic, therefore, many partial waves can make significant contributions.

This fact has an interesting relation with our earlier discussion. The real part of the spherical Hankel function is finite for small k_Σ, and is in fact simply the spherical Bessel function $j_l(k_\Sigma r)$. The singularity is in the imaginary part, the same imaginary part that generates the "virtual" currents we discussed previously. It is these currents which are responsible for the contribution of high partial waves in the intermediate state.

Let us now consider P-wave absorption. The most favorable conditions for P-wave Λ conversion occur in what we called the "P-wave" case: S-channel K-N absorption from a P orbit [Eq. (40) with $B=0$]. The Σ emerges from the first center predominantly in a P wave with respect to the second nucleon. For these conditions we find (neglecting spin-dependent effects)

$$\mathfrak{M}_\Lambda{}^P = -(4\pi/3)k_\Sigma f_{\Sigma\Lambda}\{\mathcal{P}_0 J_{01} + \sqrt{2}\mathcal{P}_2 J_{12}\}, \tag{45}$$

where

$$\mathcal{P}_0 = \frac{\mathbf{k}_\Lambda\cdot\mathbf{e}_i}{k_\Lambda} \quad \text{and} \quad \mathcal{P}_2 = \frac{1}{\sqrt{2}}\left[\frac{\mathbf{k}_\Lambda\cdot\mathbf{e}_i}{k_\Lambda} - 3\left(\frac{(\mathbf{k}_\Lambda\cdot\mathbf{p}_\pi)(\mathbf{p}_\pi\cdot\mathbf{e}_i)}{k_\Lambda p_\pi{}^2}\right)\right]$$

TABLE III. Conversion ratio $\mathfrak{R} = w_\Lambda/w_\Sigma$ for the "P-wave" case as a function of P-wave Σ-N interaction parameters. Scattering lengths are in units of $(\text{Mev}/c)^{-1}$.

η_1	a_1	$\mathfrak{R}$
0.01	−0.01	0.70
0.007	−0.007	0.38
0.005	−0.005	0.24
0.0035	−0.0035	0.03
0.01	0.01	0.91

[13] See M. Ruderman and R. Karplus, Phys. Rev. **102**, 247 (1956), where this barrier effect is the dominant momentum dependence.

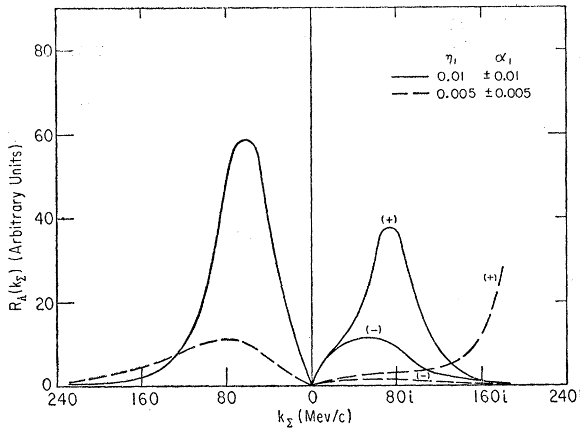

FIG. 8. Λ spectrum as a function of the P-wave Σ-N interaction parameters. $\tan\delta_\Sigma{}^{(1)} = -k_\Sigma{}^3(a_1{}^3 - i\eta_1{}^3)$ is used. η_1 and a_1 are in units of $(\text{Mev}/c)^{-1}$.

are the projection operators for a P-wave Λ and S- and D-wave pion, respectively.

The integrals which enter here are

$$\begin{aligned} J_{12} &= (3/q)I_{11} - J_{10}, \\ I_{11} &= \frac{k_\Sigma{}^2+q^2+\alpha^2}{2k_\Sigma q}I_{00} - i\frac{k_\Sigma+i\alpha}{2k_\Sigma{}^2 q}, \\ J_{10} &= \frac{1}{k_\Sigma}I_{00} + \frac{1}{k_\Sigma}\frac{1}{(k_\Sigma+i\alpha)^2-q^2}. \end{aligned} \tag{46}$$

The scattering length approximation in this case becomes

$$\begin{aligned} &\tan\delta_\Sigma{}^{(1)} = -k_\Sigma{}^3(a_1{}^3 - i\eta_1{}^3), \\ &|f_{\Sigma\Lambda}{}^{(1)}|^2 = \frac{(k_\Sigma\eta_1)^3}{[1+(k_\Sigma\eta_1)^3]^2+(k_\Sigma a_1)^6} \quad \text{for real } k_\Sigma, \\ &\qquad = \frac{(\kappa_\Sigma\eta_1)^3}{[1-(\kappa_\Sigma a_1)^3]^2+(\kappa_\Sigma\eta_1)^6} \quad \text{for } k_\Sigma = i\kappa_\Sigma. \end{aligned} \tag{47}$$

The results of calculations using these forms are shown in Fig. 8 and Table III. Comparison of these with Fig. 5 and Table I, the results for S-wave Σ-N absorption, shows that the P-wave conversion ratio is indeed comparable with the S-wave ratio when the interaction parameters (the scattering lengths) are comparable. As expected, there is no tendency toward suppression of low momenta, as one would expect for free-particle absorption in a high angular momentum state. The spectral shapes are in fact quite similar, with the differences arising from the stronger dependences of the wave function and scattering amplitude on momentum in the higher partial waves. The more rapid falloff with high momentum is a result of our scattering-length approximation for the phase shift, and is probably not significant.

The higher partial waves yield similar results and can make significant contributions. There will eventually be a cutoff due to the limited angular momentum carried by the outgoing π-meson, represented by $j_l(qr)$ in the overlap integrals, and due to the angular-momentum barrier encountered by the outgoing Λ particle (this effect appears in $f_{\Sigma\Lambda}^{(l)}$).

It should be possible to determine the most important partial waves in the Σ-N absorption process by examining the angular correlation between $\mathbf{k}_\Lambda$ and $\mathbf{p}_\pi$. For instance, if the Σ-Λ parity is even and there is no spin-orbit force, Eq. (45) yields a Λ-π correlation in the P-wave case of the form

$$1-\tfrac{3}{4}\cos^2(\mathbf{k}_\Lambda,\mathbf{p}_\pi).$$

VII. COMPARISON WITH EXPERIMENT

In this section we shall discuss the available experimental information to show that the theory we have described does seem capable of accounting for several features of the observations on deuterium: the large conversion ratio for $\Sigma\rightarrow\Lambda$ and the shape of the spectrum.

To apply the results of our calculation, we must first identify the atomic orbit from which capture takes place. Day and Snow[14] have pointed out that the nuclear capture rate from the $2P$ orbit exceeds the radiation rate by a factor of ten even in the absence of final state interactions. Nuclear capture from a D orbit proceeds too slowly to result in appreciable competition with radiation. Auger transitions are negligible. Unfortunately the observation of the mesic x-ray yield from light elements is inconsistent with this simple picture of the atomic processes and suggests that the $2P$ orbit is not occupied as frequently as one would expect from the calculated transition rates. We have, therefore, described the theory for both S-orbit and P-orbit capture.‡

Next, we must recognize that many of the capture processes will result in Σ-nucleon systems in the isotopic spin $I=\frac{3}{2}$ state. Since the Λ conversion can take place only from the $I=\frac{1}{2}$ state, it is necessary to make an isotopic spin analysis of the observed branching ratios.[15] Eigenstates of the baryon isotopic spin are useful only in our model, in which final-state interactions other than those within the two-baryon system are neglected. Otherwise this variable is not a constant of the motion and it would not be possible to define a "conversion ratio" unambiguously. In terms of the two amplitudes $\beta_{\frac{1}{2}}(p_\pi)$ and $\beta_{\frac{3}{2}}(p_\pi)$ that refer to the two possible states, the branching ratios as functions of p_π are

$$\begin{aligned}(\pi^+\Sigma^-n):(\pi^-\Sigma^+n):(\pi^0\Sigma^0n):(\pi^0\Sigma^-p):(\pi^-\Sigma^0p)\\ =|\beta_{\frac{3}{2}}|^2:\tfrac{1}{9}|\beta_{\frac{3}{2}}-2\beta_{\frac{1}{2}}|^2:\tfrac{1}{9}|2\beta_{\frac{3}{2}}-\beta_{\frac{1}{2}}|^2:\\ (2/9)|\beta_{\frac{3}{2}}+\beta_{\frac{1}{2}}|^2:(2/9)|\beta_{\frac{3}{2}}+\beta_{\frac{1}{2}}|^2.\end{aligned}\quad(48)$$

We must also introduce the amplitude β_Λ for the production of Λ particles by the *indirect* process. The branching ratio for the two Λ channels is

$$(\pi^-\Lambda p):(\pi^0\Lambda n)=\tfrac{1}{3}|\beta_\Lambda|^2:\tfrac{2}{3}|\beta_\Lambda|^2.\quad(49)$$

The ratio $\mathcal{R}$ of converted Λ particles to observed Σ-N systems in the $I=\frac{1}{2}$ state is

$$\mathcal{R}=\frac{w_\Lambda}{w_\Sigma(I=\frac{1}{2})}=\int|\beta_\Lambda|^2dp_\pi\Big/\int|\beta_{\frac{1}{2}}|^2dp_\pi.\quad(50)$$

It is found experimentally to be $\mathcal{R}=0.6\pm0.1$.

In Tables I–III there are a number of cases in which this ratio is attained. If we consider the observed spectral distribution in Fig. 1, it is possible to eliminate those cases in which the real scattering length is positive (bound state) because they give rise to a spectrum substantially displaced towards higher pion momenta (Fig. 5). We see that in the remaining alternatives a combination of several favorable factors must enter involving the orbit from which the K particle is captured, the mechanism of the K-nucleon interaction, and the mechanism of the Σ-nucleon interaction. This conclusion is summarized in Table IV.

TABLE IV. Interaction channels leading to a large conversion ratio $\mathcal{R}$.

Orbit of K meson	$K+N\rightarrow\Sigma+\pi$	$\Sigma+N\rightarrow\Lambda+N'$
S	S	S
P	P	S
P	S	P

We note that there is no evidence for contributions of bound Σ-N hyperfragments to Λ production.[16] As Day and Snow[14] have pointed out, this does not imply that such hyperfragments do not exist, since the Σ-N system may be primarily in a partial wave that does not have a bound state.

We still must discuss the nature of the approximations that have been made. The hyperon-nucleon final-state interaction which has been studied in the previous sections is only the most important of many effects that enter into our three-body problem. There are field-theoretic three-body forces as well as multiple-scattering

[14] T. B. Day and G. A. Snow, Phys. Rev. Letters **2**, 59 (1959).

‡ *Note added in proof.*—Day, Snow, and Sucher [Phys. Rev. Letters **3**, 61 (1959)] have shown that the K-meson orbital wave function in liquid hydrogen and deuterium has a significant S-wave component that leads to the observed nuclear captures. For comparison with experiment, therefore, we must use a wave function ξ_k [Eq. (21)] which does not vanish at the origin. Except for a constant factor that determines the absolute capture rate, the observed situation is the one treated in Sec. IV and in the first line of Table IV.

[15] The branching ratios of the reaction K^-+d into the various allowed final states are consistent with the requirements of isotopic spin conservation. We shall therefore assume that I is conserved in the reaction.

[16] It has been pointed out by G. Chew (private communication) that a Σ-N hyperfragment with $I=\frac{3}{2}$ could contribute to the Λ-production process. Such a state is not stationary because of the Σ mass differences. After about 10^{-21} sec the isotopic spin would change to $I=\frac{1}{2}$ and permit the conversion reaction to take place rapidly.

effects of the several two-body forces (Figs. 3 and 4) that will contribute when all three particles are in a small region of space. Such corrections, however, will give a fairly uniform spectrum to the outgoing pions, because the momentum range in which the Σ and nucleon have a low relative momentum has no special importance for these effects.

There are, nevertheless, two ways in which the three-body effects can appreciably modify the peaked pion spectrum. First, the interference with the Λ amplitude produced by the simple conversion process depends on the magnitude and phase of that amplitude; these quantities vary rapidly in the critical momentum region (near $k_\Sigma=0$). Second, an intermediate pion-nucleon interaction in the resonant $(\frac{3}{2},\frac{3}{2})$ state can take place[17]; since the momentum of the pion relative to the nucleon varies between 140 and 220 Mev/c, a rapid variation of this part of the Λ-production amplitude might be expected. A simple estimate of these effects shows that both can modify the Λ-production rate, and that the former can also broaden the spectrum appreciably. We are not including a quantitative discussion here because the present experimental situation does not justify the introduction of the many additional parameters.

If we ignore these corrections, we can turn to the problem of relating the reaction amplitudes in deuterium to the amplitudes in hydrogen. We therefore introduce the usual isotopic singlet and triplet amplitudes α_0 and α_1 for the K-nucleon interactions and their linear combinations

$$\beta_{\frac{1}{2}}'=-(\tfrac{1}{6})^{\frac{1}{2}}\alpha_0+\alpha_1,$$
$$\beta_{\frac{3}{2}}'=(\tfrac{1}{6})^{\frac{1}{2}}\alpha_0+\tfrac{1}{2}\alpha_1, \tag{51}$$
$$\left(\sum_{\tau=\frac{1}{2}}^{\frac{3}{2}}(2\tau+1)|\beta_\tau'|^2=\sum_{t=0}^{1}(2t+1)|\alpha_t|^2\right),$$

which are the amplitudes for production of a Σ on one nucleon of the $I=0$ deuteron system such that the Σ is in the $I=\frac{1}{2}$ or $I=\frac{3}{2}$ state with respect to the other nucleon. In our model the β_τ differ from the β_τ' by the final-state interaction. Since this difference will in general depend on the Σ-nucleon relative momentum, detailed Σ spectra are necessary to extract information about α_0 and α_1 from the present experiment. We do not know either whether the same K-N channels are participating in hydrogen and deuterium. Thus, the striking difference between the Σ branching ratios in the two cases[12,18] could be due to both the occurrence of different incident channels and the final-state interaction. It is to be hoped that independent measurement of the K-hydrogen parameters will lead to a better understanding of the final-state interaction in this problem.

[17] We note that such an interaction is not possible once the Λ has been produced, since the pion-nucleon system must then have isotopic spin $\frac{1}{2}$.

[18] R. D. Tripp, *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN*, edited by B. Ferretti (CERN, Geneva, 1958), p. 184.

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Green's Function Approximation Method. I. The Nucleon*

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A method for the approximate construction of the nucleon Green's function is presented. The development is such that the approximate Green's function automatically has the same analytical properties as the exact one. This method involves the symmetrical treatment of the Green's function and, ultimately, the assumption that certain particles behave in an uncorrelated manner. The approximation results in a linear integral equation for the Green's function which is completely renormalized. This equation is solved exactly through the use of the spectral representation which, by construction, is consistent with the approximation.

1. INTRODUCTION

ONE criterion that might reasonably be demanded of an approximation method is that any approximate solution should have the same analytical properties which the exact solution is known to possess. Accepting this criterion, it is natural to consider the single particle Green's function, for its analytical properties are well known.[1] Further, previous attempts at approximating it have either failed at just this requirement,[2] or have had to artificially patch up the approximation in order to meet it.[3] It is clearly desirable to develop an approximation method which auto-

* Based in part on a Ph.D. thesis submitted to Harvard University, January, 1959.

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[2] Ning Hu, Phys. Rev. **80**, 1109 (1950); K. A. Brueckner, Phys. Rev. **91**, 761 (1953); S. Kamefuchi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **9**, 529 (1953); G. Feldman, Proc. Roy. Soc. (London) **A223**, 112 (1954).

[3] P. J. Redmond, Phys. Rev. **112**, 1404 (1958).