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Remark on Recurrence Times

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A simple derivation is given of the mean recurrence time of a linear chain of harmonically bound particles. The derivation is based on a formula of Smoluchowski.

In a recent paper Hemmer, Maximon, and Wergeland[1] derived a formula for the mean recurrence time of a linear chain of harmonically bound particles.

It should be pointed out that their result follows almost immediately from a formula of Smoluchowski,[2] though the answers are slightly different due to a slight difference in the definitions of the mean recurrence time.

If observations are taken at times 0, τ, 2τ, $\cdots$, Smoluchowski's formula for the mean recurrence time θ is

$$\theta=\tau\frac{1-\mu(A)}{\mu(A)-\mu_\tau(A,A)},$$

where $\mu(A)$ is the (invariant) measure of the region of phase space (assumed to be of total measure 1), from which the system starts and $\mu_\tau(A,A)$ the measure of those points of A which are also in A at time τ.

For the system considered by Hemmer, Maximon, and Wergeland, the phase space can be taken to be the $(N-1)$-dimensional torus $0\leq\varphi_j\leq 2\pi$ ($j=1, 2, \cdots, N-1$), while A is an angular interval $\{\Delta\varphi_1,\cdots,\Delta\varphi_{N-1}\}$. Thus

$$\mu(A)=\prod_{j=1}^{N-1}\frac{\Delta\varphi_j}{2\pi}.$$

Now, $\mu(A)-\mu_\tau(A,A)$ is the measure of those points $(\varphi_1,\cdots,\varphi_{N-1})$ in the angular interval $\{\Delta\varphi_1,\cdots,\Delta\varphi_{N-1}\}$ for which $(\varphi_1+\omega_1\tau, \cdots, \varphi_{N-1}+\omega_{N-1}\tau)$ is outside that angular interval. In other words, the set of those points $(\varphi_1,\cdots,\varphi_{N-1})$ in the angular interval for which either $\varphi_1+\omega_1\tau$ is not in $\Delta\varphi_1$ or $\varphi_1+\omega_2\tau$ is not in $\Delta\varphi_2$ or $\cdots$ or $\varphi_{N-1}+\omega_{N-1}\tau$ is not in $\Delta\varphi_{N-1}$.

Using the familiar "exclusion-inclusion" principle, we see at once that to first order in τ

$$\mu(A)-\mu_\tau(A,A)=\tau\prod_{j=1}^{N-1}\frac{\omega_j}{\Delta\varphi_j}.$$

Thus in the limit $\tau\to 0$ we get

$$\theta=\left(1-\prod_{j=1}^{N-1}\frac{\Delta\varphi_j}{2\pi}\right)\prod_{j=1}^{N-1}\left(\frac{2\pi}{\Delta\varphi_j}\right)\Big/\sum_{j=1}^{N-1}\frac{\omega_j}{\Delta\varphi_j},$$

which except for the first factor is identical with formula (17) of Hemmer, Maximon, and Wergeland.

Had we adapted the definition of Hemmer, Maximon, and Wergeland with an appropriate modification to cover the case of intermittent observations, we would be led to the formula

$$\theta=\frac{\tau}{\mu(A)-\mu(A,A)},$$

which, in the limit $\tau\to 0$, would yield formula (17) referred to above.

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[1] Hemmer, Maximon, and Wergeland, Phys. Rev. **111**, 689 (1958).

[2] For an exposition of the work of Smoluchowski see, e.g., M. Kac, Bull. Am. Math. Soc. **53**, 1002 (1947). A more complete discussion will be found in M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience Publishers, Inc., New York), Chap. 3.