so that the cross section is not large. The situation is quite different for the p-Σ^0K^+ process, primarily because the electric dipole moment term Q_j is positive for this process. Thus the μ_p and Q terms add destructively in the pseudoscalar case and constructively in the scalar case. Hence, the ratio of the Σ^0K^+ and Σ^+K^0 cross sections (A_{2a}/A_1) tends to be large in the scalar case and small in the pseudoscalar case. A similar argument may be made concerning the ratio (A_{2a}/A_4); this ratio also tends to be much larger in the scalar than in the pseudoscalar theory.

Again it should be pointed out that recoil effects associated with the finite photon wavelength are important, even for the qualitative effects discussed above.

The present experimental data are insufficient for any conclusions to be made concerning the K parity or the sign of G_Λ/G_Σ. Only the p-Λ^0K^+ and p-Σ^0K^+ cross sections have been measured,[7,8] and the data are not of sufficient accuracy to show whether or not the cross sections are appreciably nonisotropic. Certainly much would be learned if the measured results could be compared to the corresponding cross sections for any of the other processes discussed here. It is seen from Table I that the S wave cross sections for some of the unmeasured processes may be much larger than the Σ^0K^+ and Λ^0K^+ cross sections, so that measurement of the processes Σ^+K^0, Σ^-K^+, Σ^0K^0, and Λ^0K^0 may not be as difficult as one would at first suppose.

Of course the effects of pion interactions and resonance states may seriously alter the perturbation theory conclusions. Nevertheless, it is believed that the perturbation theory is a valuable guide to experiment, and it is not unlikely that many of the qualitative conclusions of the theory will prove to be correct.

ACKNOWLEDGMENT

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Fixation of Coordinates in the Hamiltonian Theory of Gravitation

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The theory of gravitation is usually expressed in terms of an arbitrary system of coordinates. This results in the appearance of weak equations connecting the Hamiltonian dynamical variables that describe a state at a certain time, leading to supplementary conditions on the wave function after quantization. It is then difficult to specify the initial state in any practical problem.

To remove the difficulty one must eliminate the weak equations by fixing the coordinate system. The general procedure for this elimination is here described. A particular way of fixing the coordinate system is then proposed and its effect on the Poisson bracket relations is worked out.

INTRODUCTION AND NOTATION

THE problem of putting Einstein's equations for the gravitational field into the Hamiltonian form, as a preliminary to quantization, has recently received a good deal of attention, because of the development of mathematical methods sufficiently powerful to make it tractable.

The Hamiltonian form involves the concept of a physical state "at a certain time," which means in a relativistic theory a state on a certain three-dimensional space-like surface in space-time. At first people[1,2] chose the space-like surface independent of the coordinates x^μ, which enabled them to preserve the four-dimensional symmetry of the equations. Later it was realized[3,4] that one could effect a substantial simplification, at the expense of giving up four-dimensional symmetry, by choosing a system of coordinates such that the three-dimensional surfaces $x^0=\text{constant}$ are all space-like and dealing with the physical states on these surfaces.

The main features of the Hamiltonian formalism will be recapitulated here. The notation will be that used by the author,[4] with the exception that the sign of the $g_{\mu\nu}$ will be changed throughout, to make g_{00} negative. Greek suffixes take on the values 0, 1, 2, 3, lower-case Roman suffixes take on the values 1, 2, 3, the determinant of the $g_{\mu\nu}$ is $-J^2$, the determinant of the g_{rs} is K^2, and the reciprocal matrix to g_{rs} is e^{rs}. A lower suffix added to a field quantity denotes an ordinary derivative, while $_{|\mu}$ added to it denotes the covariant derivative.

We shall deal with the gravitational field in interaction with other fields, or possibly particles. Spinor fields are excluded, as they require a special treatment.

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We have an action density of the form

$$\mathfrak{L}=\mathfrak{L}_G+\mathfrak{L}_M,$$

where $\mathfrak{L}_G$ is the action density of the gravitational field alone, involving the $g_{\mu\nu}$ and their first derivatives, and $\mathfrak{L}_M$ is the action density of the other fields, involving the other field quantities, q_M say, and their first derivatives and involving also the $g_{\mu\nu}$, but not derivatives of the $g_{\mu\nu}$.

The gravitational action density is

$$\mathfrak{L}_G=(16\pi\gamma)^{-1}Jg^{\mu\nu}(\Gamma_{\mu\rho}{}^{\sigma}\Gamma_{\nu\sigma}{}^{\rho}-\Gamma_{\mu\nu}{}^{\rho}\Gamma_{\rho\sigma}{}^{\sigma}), \tag{1}$$

where γ is the gravitational constant, occurring in the numerator of Newton's law of force. To save writing, we shall take

$$16\pi\gamma=1. \tag{2}$$

HAMILTONIAN FORM OF GRAVITATIONAL THEORY

We shall deal with the physical state on the surface $x^0=t$ and shall set up Hamiltonian equations of motion to determine how the state varies as t varies. The Hamiltonian is, by the usual definition

$$H=\int\left(g_{\mu\nu0}\frac{\partial\mathfrak{L}_G}{\partial g_{\mu\nu0}}+\sum q_{M0}\frac{\partial\mathfrak{L}_M}{\partial q_{M0}}-\mathfrak{L}\right)d^3x, \tag{3}$$

where the sum is over all the nongravitational dynamical coordinates q_M.

It is evident that there must be a good deal of arbitrariness in the equations of motion on account of the arbitrariness in the system of coordinates x^μ. In the first place we see that the $g_{\mu0}$ can vary with t in an arbitrary way. To describe the geometry of the surface $x^0=t$ and also the system of coordinates x^r in it, we need the g_{rs} at all points on the surface, but we do not need the $g_{\mu0}$, which refer only to intervals going outside the surface. Different values for the $g_{\mu0}$ correspond to different choices of a neighboring surface $x^0=t+\epsilon$ and to different systems of coordinates x^r in the neighboring surface, and these are completely arbitrary with a given initial surface $x^0=t$.

We get the simplest form for the equations of motion if we describe the physical state on the surface $x^0=t$ entirely in terms of dynamical variables that are independent of the $g_{\mu0}$. Let us consider the kind of quantities that can enter into such a description.

Suppose there is a vector field A_μ. The three covariant components A_r on the surface remain invariant under a change of coordinates which leaves the coordinates of each point on the surface invariant. So these A_r will enter into the description. We cannot have A_0, but we have instead the normal component of A, namely $A_\mu l^\mu$, where l^μ is the unit normal. Similarly for a tensor $B_{\mu\nu}$, which may be the covariant derivative $A_{\mu|\nu}$ of A_μ, we have the quantities B_{rs}, $B_{r\nu}l^\nu$, $B_{\mu s}l^\mu$, $B_{\mu\nu}l^\mu l^\nu$. Each of these quantities is unaffected by a change of coordinates which leaves the points on the surface invariant and is thus independent of the $g_{\mu0}$.

It should be noted that, for a vector A_μ, the ordinary and covariant derivatives A_{rs} and $A_{r|s}$ are both independent of the $g_{\mu0}$. Their difference, namely $\Gamma_{rs}{}^{\mu}A_\mu$, is thus independent of the $g_{\mu0}$. We may take A_μ here to be the unit normal, namely

$$l_\mu=g_\mu{}^0/(-g^{00})^{\frac{1}{2}}, \tag{4}$$

and we find that the quantity

$$\Gamma_{rs}{}^0/(-g^{00})^{\frac{1}{2}} \tag{5}$$

is independent of the $g_{\mu0}$. This quantity may be called the "invariant velocity" of g_{rs}, as it consists of the ordinary velocity g_{rs0} multiplied by a certain factor and with certain terms added on, so as to produce a quantity independent of the choice of coordinate system outside the surface $x^0=t$.

With the physical state described in this way, one easily finds[4] that for a dynamical variable η not involving the $g_{\mu0}$, $d\eta/dx^0$ is of the form

$$d\eta/dx^0=\int\{(-g^{00})^{-\frac{1}{2}}\xi_L+g_{r0}e^{rs}\xi_s\}d^3x, \tag{6}$$

with ξ_L and ξ_s independent of the $g_{\mu0}$. We need equations of motion to determine ξ_L, ξ_s for any η. The coefficients $(-g^{00})^{-\frac{1}{2}}$, g_{r0} in (6) are arbitrary and not restricted by the equations of motion.

One gets equations of motion of the form (6) from a Hamiltonian of the form

$$H=\int\{(-g^{00})^{-\frac{1}{2}}\mathcal{H}_L+g_{r0}e^{rs}\mathcal{H}_s\}d^3x, \tag{7}$$

with $\mathcal{H}_L$ and $\mathcal{H}_s$ independent of the $g_{\mu0}$ and vanishing weakly. It has been shown[4] that the Hamiltonian (3) takes the form (7) provided the dynamical coordinates describing the nongravitational fields are chosen to be independent of the $g_{\mu0}$, in the way discussed above, and provided one takes for $\mathfrak{L}_G$, instead of (1), an expression which differs from (1) by a perfect differential and which does not contain the velocities $g_{\mu00}$, namely

$$\mathfrak{L}_G=Jg^{\mu\nu}(\Gamma_{\mu\rho}{}^{\sigma}\Gamma_{\nu\sigma}{}^{\rho}-\Gamma_{\mu\nu}{}^{\rho}\Gamma_{\rho\sigma}{}^{\sigma})+(Jg^{00})_0(g^{r0}/g^{00})_r-(Jg^{00})_r(g^{r0}/g^{00})_0. \tag{8}$$

With this $\mathfrak{L}_G$, the momenta $p^{\mu0}$ conjugate to $g_{\mu0}$ vanish weakly, which results in the degrees of freedom described by $g_{\mu0}$, $p^{\mu0}$ dropping out from the Hamiltonian formalism. The weak equations $p^{\mu0}\approx 0$ give, when one passes to the quantum theory, the conditions $p^{\mu0}\psi=0$, which show that the wave function ψ does not involve the $g_{\mu0}$.

The surviving gravitational momenta are

$$p^{rs}=K(e^{ra}e^{sb}-e^{rs}e^{ab})\Gamma_{ab}{}^0/(-g^{00})^{\frac{1}{2}}. \tag{9}$$

They are built up from the invariant velocities (5). The fundamental Poisson bracket (P.b.) relations for them are

$$[g_{ab},p'^{rs}]=\tfrac{1}{2}(g_a{}^r g_b{}^s+g_b{}^r g_a{}^s)\delta(x-x'). \tag{10}$$

The expressions for $\mathcal{H}_L$ and $\mathcal{H}_s$ in (7) are found to be

$$\mathcal{H}_L=K^{-1}(p^{rs}p_{rs}-\tfrac{1}{2}p_r{}^rp_s{}^s)+B+\{K^{-1}(K^2e^{rs})_r\}_s+\mathcal{H}_{ML}, \quad (11)$$

$$\mathcal{H}_s=p^{ab}g_{abs}-2(p^{ab}g_{as})_b+\mathcal{H}_{Ms}, \quad (12)$$

where

$$B=\tfrac{1}{4}Kg_{rsu}g_{abv}\{(e^{ra}e^{sb}-e^{rs}e^{ab})e^{uv}+2(e^{ru}e^{ab}-e^{ra}e^{bu})e^{sv}\}, \quad (13)$$

and $\mathcal{H}_{ML}$, $\mathcal{H}_{Ms}$ are the contributions arising from the nongravitational fields. It should be noted that the terms $B+\{K^{-1}(K^2e^{rs})_r\}_s$ are equal to the density of the three-dimensional scalar curvature of the surface $x^0=t$.

We have the weak equations

$$\mathcal{H}_L\approx 0, \quad \mathcal{H}_s\approx 0. \quad (14)$$

They are χ equations or secondary constraints. To see where they come from, we note that Einstein's field equations are

$$R_\mu{}^\nu-\tfrac{1}{2}g_\mu{}^\nu R_\sigma{}^\sigma=\tfrac{1}{2}T_\mu{}^\nu, \quad (15)$$

where $T_\mu{}^\nu$ is the stress tensor produced by the nongravitational fields. The left-hand side of (15) contains second derivatives of the $g_{\alpha\beta}$ and thus in general contains accelerations $g_{\alpha\beta 00}$. The right-hand side of (15) contains no derivatives of the $g_{\alpha\beta}$. Now the well-known identities

$$(R_\mu{}^\nu-\tfrac{1}{2}g_\mu{}^\nu R_\sigma{}^\sigma)_{|\nu}\equiv 0$$

may be written

$$(R_\mu{}^0-\tfrac{1}{2}g_\mu{}^0R_\sigma{}^\sigma)_0\equiv-(R_\mu{}^r-\tfrac{1}{2}g_\mu{}^rR_\sigma{}^\sigma)_r+, \quad (16)$$

where the $+$ at the end indicates that some further terms, not involving third derivatives of the $g_{\mu\nu}$, must be added on. The right-hand side of (16) evidently does not contain any third time derivatives $g_{\alpha\beta 000}$. Thus the left-hand side cannot involve third time derivatives, so $R_\mu{}^0-\frac{1}{2}g_\mu{}^0R_\sigma{}^\sigma$ cannot involve accelerations $g_{\alpha\beta 00}$. Thus if we take $\nu=0$ in (15), we get equations involving only dynamical coordinates and velocities. By substituting for the velocities here in terms of the momenta, we get four equations between dynamical coordinates and momenta only, which yield (14).

The main part of the Hamiltonian is obtained by putting $g_{\mu 0}=-\delta_{\mu 0}$ in (7) and is thus

$$H_{\text{main}}=\int\mathcal{H}_Ld^3x=\int\{K^{-1}(p^{rs}p_{rs}-\tfrac{1}{2}p_r{}^rp_s{}^s)+B+\mathcal{H}_{ML}\}d^3x, \quad (17)$$

after removal of a surface integral at infinity. The removal of this surface integral does not disturb the validity of H_{main} for giving equations of motion, but it results in H_{main} not vanishing weakly.

We can write the total Hamiltonian (7) in the form

$$H=H_{\text{main}}+\int\{(-g^{00})^{-\frac{1}{2}}-1\}\mathcal{H}_Ld^3x+\int g_{r0}e^{rs}\mathcal{H}_sd^3x, \quad (18)$$

when it appears as H_{main} with arbitrary linear combinations of $\mathcal{H}_L$ and of $\mathcal{H}_s$, for various values of x^r, added on. These additional terms in the Hamiltonian produce terms in the equations of motion in addition to those produced by H_{main}, corresponding to the surface $x^0=t$ undergoing arbitrary deformations and having arbitrary changes of its coordinate system x^r as t varies.

NEED FOR FIXATION OF THE COORDINATES

To specify a physical state at a particular time in the classical theory, we must choose numerical values for all the dynamical coordinates and momenta so as to satisfy the constraints (14). This involves solving some differential equations, so it is not such a straightforward matter as specifying a state in particle dynamics.

In the quantum theory the situation is more complicated. The constraints (14) go over into the conditions on the wave function

$$\mathcal{H}_L\psi=0, \quad (19)$$

$$\mathcal{H}_s\psi=0. \quad (20)$$

To specify a state at a particular time involves obtaining a solution of Eqs. (19), (20), which are functional equations.

Equation (20) expresses merely that ψ must be invariant under changes of the coordinate system x^r in the surface $x^0=t$. To get ψ to satisfy this equation is thus not difficult. Equation (19) expresses the requirement that the state shall be specified in a way that is independent of deformations of the surface $x^0=t$. The treatment of such deformations is essentially as complicated as the treatment of the passage from the surface $x^0=t$ to a neighboring surface $x^0=t+\epsilon$, so to get ψ to satisfy (19) is essentially as complicated as solving the equations of motion. Thus we have the situation that we cannot specify the initial state for a problem without solving the equations of motion. The formalism is thus not suitable for dealing with practical problems.

The difficulty does not arise in the weak-field approximation, because then many of the terms in (19) get neglected and the remaining ones, if expressed in terms of Fourier components, are easy to handle.

To obtain a practical formalism of greater accuracy than the weak-field approximation, it is necessary to introduce into the theory some new constraint that fixes the surface $x^0=t$, so that we no longer have the possibility of making arbitrary deformations in it. Then the supplementary condition (19) gets eliminated. We may also introduce some further constraints that fix the coordinate system x^r in the surface. While not essential for getting a practical formalism, such further constraints serve to simplify the formalism by eliminating the conditions (20), and so making the task of specifying the initial state a trivial one.

The fixation of coordinates is advantageous also in the weak-field approximation, because it leads to some degrees of freedom dropping out from the formalism, the procedure being similar to the elimination of the longitudinal waves in electrodynamics.

When dealing with gravitational waves, people usually restrict the coordinate system by introducing the harmonic conditions

$$(Jg^{\mu\nu})_\nu=0.$$

These conditions would be quite unsuitable in the present formalism because they involve the $g_{\mu0}$, which the present formalism allows to be completely arbitrary. Any restriction imposed on the $g_{\mu0}$ would not help one in dealing with Eqs. (14) or (19) and (20). We need some restrictions which affect only the variables involved in (14), namely g_{rs} and p^{rs}, and possibly also the nongravitational variables.

GENERAL METHOD

Let us examine the general principles which come into play when we introduce some new restrictions or constraints on the dynamical variables in a Hamiltonian theory. Suppose we have a number of weak equations $\chi_n\approx 0$ $(n=1, 2,\cdots N)$, which may be either primary or secondary constraints. We are taking N to be finite for definiteness, but the same principles apply with N infinite. Suppose further that these weak equations are all first-class, so that

$$[\chi_n,\chi_{n'}]=0.$$

Now introduce some new restrictions, say the M independent equations

$$Y_m\approx 0, \quad m=1, 2,\cdots, M$$

with $M\leqslant N$. They are, of course, weak equations. Suppose that none of them (and no linear combination of them) has zero P.b. with all the χ's, so that they are all second-class constraints. They will cause M of the χ's to become second-class, while $N-M$ of the χ's (or linear combination of them) remain first-class.

Suppose χ_1, $\chi_2,\cdots\chi_M$ become second-class, while $\chi_{M+1},\cdots, \chi_N$ remain first-class. We now have the $2M$ second-class constraints $\chi_m\approx 0$, $Y_m\approx 0$ $(m=1, 2,\cdots M)$. Let us write $\chi_m=Y_{M+m}$, so that the $2M$ second-class constraints become $Y_s\approx 0$ $(s=1, 2,\cdots, 2M)$.

There is no place for second-class weak equations in the quantum theory, so we have to transform them in some way. We shall see that we can change them into strong equations (holding as equations between operators in the quantum theory) provided we adopt a new definition of P.b.'s, which corresponds to the number of effective degrees of freedom being reduced by M.

In simple cases we can pick out directly the degrees of freedom that have to be dropped and those that survive. Let us take the special case when M of the equation $Y_s\approx 0$ are

$$p_m\approx 0, \quad m=1, 2,\cdots, M. \qquad (21)$$

The remaining M of them must then contain all the variables q_m independently, (otherwise the p_m would not all be second-class) and so it must be possible to solve them for the q_m and write them as

$$q_m\approx f_m(q_{M+1}, q_{M+2}\cdots p_{M+1}, p_{M+2}\cdots). \qquad (22)$$

We now see that the degrees of freedom associated with q_m, p_m $(m=1, 2,\cdots M)$ cease to play an effective role in the dynamics. We can use Eqs. (21) and (22) to eliminate the variables p_m and q_m from the theory, which implies using these equations as definitions or as strong equations. We then work with P.b.'s that refer only to the other degrees of freedom.

In the general case one retains all the dynamical variables and merely changes their P.b.'s to correspond to the reduction in the number of degrees of freedom. To do this one first sets up the matrix of all the P.b.'s $[Y_s,Y_{s'}]$. It can be shown[5] that this matrix has a nonvanishing determinant, provided there is no linear combination of the Y_s that is first-class. One must then obtain the reciprocal matrix $C_{ss'}$, satisfying

$$C_{ss'}[Y_{s'},Y_{s''}]=\delta_{ss''}. \qquad (23)$$

Note that $C_{ss'}$ is a skew matrix, like $[Y_s,Y_{s'}]$. One then defines new P.b.'s by the formula

$$[\xi,\eta]^*=[\xi,\eta]-[\xi,Y_s]C_{ss'}[Y_{s'},\eta]. \qquad (24)$$

It can be checked[5] that the new P.b.'s satisfy all the fundamental relations that P.b.'s ought to satisfy.

From (23) and (24) we see at once that $[\xi,Y_s]^*=0$ for any ξ. Thus the Y_s now have zero P.b. with everything, so that we can consider the equations $Y_s=0$ as strong equations and use them before working out P.b.'s.

In applying this method to the gravitational case we desire, of course, that the change in the P.b.'s shall not be too complicated. In particular, we would like to have no change at all in the P.b. of two quantities, neither of which involves the gravitational variables g_{rs}, p^{rs}. This result is ensured provided the two conditions hold: (i) The Y_m $(m=1, 2,\cdots M)$ involve only the gravitational variables; (ii) The P.b.'s $[Y_m,Y_{m'}]$ all vanish. The proof is as follows.

We have already $(\chi_m,\chi_{m'}]\approx 0$ from the assumption that the χ's were originally first-class. With the further condition $[Y_m,Y_{m'}]\approx 0$ we have $[Y_s,Y_{s'}]\approx 0$ except when $1\leqslant s\leqslant M$ and $M+1\leqslant s'\leqslant 2M$ or vice versa. This

[5] P. A. M. Dirac, Can. J. Math. 2, 129 (1950).

leads to $C_{ss'}\approx 0$ except when $1\leqslant s\leqslant M$ and $M+1\leqslant s'\leqslant 2M$ or vice versa. The surviving elements of C are thus $C_{m,\ M+m'}=-C_{M+m',\ m}$. The elements $C_{m,\ M+m'}$ form a matrix of M rows and columns, which is the reciprocal of the matrix $[\chi_{m'},Y_m]$.

The formula (24) now reduces to

$$[\xi,\eta]^*-[\xi,\eta]=-C_{m,\ M+m'}\{[\xi,Y_m][\chi_{m'},\eta]-[\xi,\chi_{m'}][Y_m,\eta]\}. \quad (25)$$

If ξ and η do not involve the gravitational variables, the condition (i) above leads to $[\xi,Y_m]=0$ and $[Y_m,\eta]=0$, so the right-hand side of (25) vanishes.

The introduction of the new constraints into the theory, when combined with the appropriate change in the P.b.'s, leaves the Hamiltonian first-class. It follows that the Hamiltonian equations of motion preserve all the constraints.

FIXATION OF THE SURFACE

To fix the surface $x^0=t$, the natural conditions to take are

$$p_r{}^r=g_{rs}p^{rs}\approx 0. \quad (26)$$

This involves bringing into the theory one Y equation for each point of the surface.

One easily checks that

$$[\mathcal{H}_s,g'_{uv}p'^{uv}]=g_{uv}p^{uv}\delta_s(x-x')\approx 0,$$

so the conditions (26) do not disturb the first-class character of the equations $\mathcal{H}_s\approx 0$. This means that the conditions (26) do not restrict the coordinate system x^r in the surface, a result which is evident from the tensor character of (26). The conditions (26) mean geometrically that the surface shall have a maximum three-dimensional "area." The equations (26) and $\mathcal{H}_L\approx 0$ are now second-class and we can use them to eliminate one degree of freedom at each point of space.

We have

$$[g_{rs},p'_u{}^u]=g_{rs}\delta(x-x').$$

It follows that the ratios of the g_{rs} at any point have zero P.b.'s with $p_u{}^u$ at all points of the surface. Let us put

$$K=\kappa^3,\quad \tilde{g}_{rs}=g_{rs}\kappa^{-2},\quad \tilde{e}^{rs}=e^{rs}\kappa^2. \quad (27)$$

Then $\tilde{g}_{rs}$ involves only such ratios and has zero P.b. with $p_u{}^u$ at all points. There are five independent $\tilde{g}_{rs}$, as their determinant is unity. The $\tilde{e}^{rs}$ form the reciprocal matrix to the matrix $\tilde{g}_{rs}$, and also have the determinant unity.

We have

$$[K^2,p'_u{}^u]=3K^2\delta(x-x'),$$

and so

$$[\ln\kappa,p'_u{}^u]=\tfrac{1}{2}\delta(x-x'). \quad (28)$$

Put

$$\tilde{p}^{rs}=(p^{rs}-\tfrac{1}{3}e^{rs}g_{ab}p^{ab})\kappa^2, \quad (29)$$

$$\tilde{p}_{rs}=\tilde{g}_{ra}\tilde{g}_{sb}\tilde{p}^{ab}.$$

We find that $\tilde{p}^{rs}$ and $\tilde{p}_{rs}$ have zero P.b. with $p_u{}^u$ and κ at all points.

Let us change our basic dynamical coordinates from the six g_{rs} to the five independent $\tilde{g}_{rs}$ and $\ln\kappa$. The momentum conjugate to $\ln\kappa$ is now, from (28), just $2p_u{}^u$, and the momenta conjugate to the $\tilde{g}_{rs}$ are certain functions of the $\tilde{p}^{rs}$ and $\tilde{g}_{rs}$.

The conditions (26) now take the form (21) and we have the equations $\mathcal{H}_L\approx 0$ playing the role of (22). To put them into the form of (22) we must solve them, with the help of (26), to get κ expressed in terms of quantities having zero P.b. with $p_u{}^u$ and κ. Such quantities are the $\tilde{g}_{rs}$, $\tilde{e}^{rs}$, $\tilde{p}^{rs}$, $\tilde{p}_{rs}$, and the nongravitational variables.

From (11), the equation $\mathcal{H}_L\approx 0$ gives,

$$-\{\kappa^{-3}(\kappa^4\tilde{e}^{rs})_r\}_s\approx\kappa^{-3}\tilde{p}^{rs}\tilde{p}_{rs}+B+\mathcal{H}_{ML}, \quad (30)$$

in which we look upon the g_{rs} in B and $\mathcal{H}_{ML}$ as expressed in terms of the $\tilde{g}_{rs}$ and κ. This is a difficult equation to solve generally for κ. However, for gravitational fields that are not too strong, the important terms are those that involve second derivatives of κ, i.e., those on the left-hand side. We can therefore obtain the solution by a method of successive approximation, first putting $\kappa=1$ on the right and solving the resulting simplified equation, then substituting the first approximation for κ on the right and solving to get the second approximation, and so on. We shall consider this equation further in the next section, with reference to a particular system of coordinates, and for the present we shall assume that the solution has been obtained.

Following the method of the preceding section for dealing with the second-class equations (21) and (22), we express H_{main} and $\mathcal{H}_s$ in terms of the variables $\tilde{g}_{rs}$, $\tilde{e}^{rs}$, $\tilde{p}^{rs}$, $\tilde{p}_{rs}$, $p_u{}^u$, and κ, and then eliminate $p_u{}^u$ and κ from them by means of (26) and the solution of (30), which we may now use as strong equations. The elimination from $\mathcal{H}_s$ is trivial, as we get from (12), using (26),

$$\mathcal{H}_s=\tilde{p}^{ab}\tilde{g}_{abs}-2(\tilde{p}^{ab}\tilde{g}_{as})_b+\mathcal{H}_{Ms}. \quad (31)$$

If the nongravitational field variables are suitably chosen, $\mathcal{H}_{Ms}$ will not contain κ. The elimination from H_{main} leads to an expression

$$H^*{}_{\text{main}}=\int(\kappa^{-3}\tilde{p}^{rs}\tilde{p}_{rs}+B+\mathcal{H}_{ML})d^3x, \quad (32)$$

in which κ is understood to have the appropriate value. The integrand here may be considered as the energy density or mass density. The complete Hamiltonian is now

$$H^*{}_{\text{main}}+\int g_{r0}e^{rs}\mathcal{H}_s d^3x. \quad (33)$$

The term corresponding to the freedom of deformation of the surface, i.e., the middle term of (18), has disappeared.

We now have a Hamiltonian formalism in which the degree of freedom described by $p_u{}^u$ and $\ln\kappa$ has dropped out. The Hamiltonians (32) and (33) are first-class even with the condition (26), so they lead to equations of motion that preserve (26). The procedure of substituting for κ in the derivation of $H^*{}_{\text{main}}$ caused the introduction of the right amount of $\mathcal{H}_L$ into the Hamiltonian to ensure the preservation of (26).

FIXATION OF COORDINATES IN THE SURFACE

To get the theory into a more convenient form, one must also fix the coordinate system x^r in the surface. The most natural conditions to take for this purpose, from the geometrical point of view, are the harmonic conditions in three dimensions:

$$(Ke^{rs})_s \approx 0. \qquad (34)$$

However, (34) does not have zero P.b. with (26), so if we adopt (34) together with (26) we must change the P.b. relationships between the nongravitational variables. To avoid this inconvenience, it is better to replace (34) by

$$\tilde{e}^{rs}{}_s \approx 0, \qquad (35)$$

which does have zero P.b. with (26).

With the coordinates fixed by (35), Eq. (30) reduces to

$$-4\nabla^2\kappa = \kappa^{-3}\tilde{p}^{rs}\tilde{p}_{rs} + B + \mathcal{H}_{ML}. \qquad (36)$$

where ∇^2 denotes the Laplacian operator with respect to the metric $\tilde{g}_{rs}$, namely

$$\nabla^2 = \tilde{e}^{rs}\partial^2/\partial x^r \partial x^s. \qquad (37)$$

The right-hand side in (36) equals the integrand in (32) and is the mass density. To interpret (36), let us restore the gravitational constant into the theory in accordance with (2). It then becomes

$$-(4\pi\gamma)^{-1}\nabla^2\kappa = 16\pi\gamma\kappa^{-3}\tilde{p}^{rs}\tilde{p}_{rs} + (16\pi\gamma)^{-1}B + \mathcal{H}_{ML}. \qquad (38)$$

We now see that $\kappa - 1$ is the Newtonian potential generated by the mass density in a space with the metric $\tilde{g}_{rs}$. The fact that κ occurs in the right-hand side of (38) can be understood as due to the Newtonian potential itself having some influence on the mass density which generates it.

Let us examine the term with B in (38). The expression (13) for B, written in terms of the new variables, is

$$B = \tfrac{1}{4}\kappa^{-1}(\kappa\tilde{g}_{rsu} + 2\kappa_u\tilde{g}_{rs})(\kappa\tilde{g}_{abv} + 2\kappa_v\tilde{g}_{ab}) \times \{(\tilde{e}^{ra}\tilde{e}^{sb} - \tilde{e}^{rs}\tilde{e}^{ab})\tilde{e}^{uv} + 2(\tilde{e}^{ru}\tilde{e}^{ab} - \tilde{e}^{ra}\tilde{e}^{bu})\tilde{e}^{sv}\}.$$

With the help of the equation

$$\tilde{g}_{rsu}\tilde{e}^{rs} = 0,$$

which follows from the determinant of the $\tilde{g}_{rs}$ being unity, and of the equation

$$\tilde{g}_{rsu}\tilde{e}^{ru} = 0,$$

which follows from (35), this reduces to

$$B = \tfrac{1}{4}\kappa\tilde{g}_{rsu}\tilde{g}_{abv}(\tilde{e}^{ra}\tilde{e}^{sb}\tilde{e}^{uv} - 2\tilde{e}^{ra}\tilde{e}^{bu}\tilde{e}^{sv}) - 2\kappa^{-1}\kappa_u\kappa_v\tilde{e}^{uv}. \qquad (39)$$

The last term here, divided by $16\pi\gamma$, can be interpreted as the mass density (or energy density) of the Newtonian field with the potential $\kappa - 1$. It is negative definite, corresponding to the Newtonian force being attractive. The remaining terms of B, together with the first term on the right-hand side of (38), give the energy density of the gravitational waves.

THE NEW POISSON BRACKETS

With the coordinates fixed by (35), the P.b.'s of the gravitational variables with one another and with the nongravitational variables will be altered. The new P.b.'s are given by formula (25) with Y_m replaced by $\tilde{e}^{ru}{}_u$ and $\chi_{m'}$ replaced by $\mathcal{H}'_s$. It thus reads

$$[\xi,\eta]^* - [\xi,\eta] = -\int\int C_r{}^s(x,x')\{[\xi,\tilde{e}^{ru}{}_u][\mathcal{H}'_s,\eta] - [\xi,\mathcal{H}'_s][\tilde{e}^{ru}{}_u,\eta]\}d^3x d^3x'. \qquad (40)$$

The coefficient $C_r{}^s(x,x')$ is the reciprocal of the matrix $[\mathcal{H}'_s,\tilde{e}^{ru}{}_u]$ and thus satisfies

$$\int C_v{}^s(x'',x')[\mathcal{H}'_s,\tilde{e}^{ru}{}_u]d^3x' = g_v{}^r\delta(x - x''). \qquad (41)$$

Evaluating the P.b. here, we get

$$\int C_v{}^s(x'',x')\{g_s{}^r\tilde{e}^{ab}\delta_{ab}(x - x') + \tfrac{1}{3}\tilde{e}^{ra}\delta_{sa}(x - x')\}d^3x' = g_v{}^r\delta(x - x''),$$

which reduces to

$$\nabla^2 C_v{}^r(x',x) + \tfrac{1}{3}\tilde{e}^{ra}C_v{}^s(x',x)_{sa} = g^r{}_v\delta(x - x'), \qquad (42)$$

with ∇^2 defined by (37).

This equation may be considered for fixed x', when it is a differential equation for the unknown functions $C_v{}^r(x',x)$ in the variables x. The important domain for x is now the neighborhood of x', since when x is far from x' the functions $C_v{}^r(x',x)$ are small. We can therefore get an approximate solution by considering the space as flat in this domain, so that the $\tilde{e}^{ab}$ are constants. With this approximation we get, on differentiating (42) with respect to x^r,

$$\nabla^2 C_v{}^s{}_s = \tfrac{3}{4}\delta_v(x - x'). \qquad (43)$$

The solution of this equation is

$$C_v{}^s{}_s = -\frac{1}{4\pi} \times \frac{3}{4}\left(\frac{1}{|x - x'|}\right)_v,$$

where $|x - x'|$ denotes the distance from x to x' with respect to the metric $\tilde{g}_{rs}$,

$$|x - x'| = \{\tilde{g}_{rs}(x^r - x'^r)(x^s - x'^s)\}^{\frac{1}{2}}. \qquad (44)$$

Equation (42) now becomes

$$\nabla^2 C_v{}^r = g^r{}_v \delta(x-x') + \frac{1}{16\pi}\tilde{e}^{ra}\left(\frac{1}{|x-x'|}\right)_{va},$$

whose solution is

$$C_v{}^r(x',x) = -\frac{1}{4\pi}g^r{}_v\frac{1}{|x-x'|} + \frac{1}{32\pi}\tilde{e}^{ra}|x-x'|_{va}. \quad (45)$$

One could get the solution of (42) to a higher accuracy by substituting for the $\tilde{e}^{ab}$ in the left-hand side of (42), (remembering that $\tilde{e}^{ab}$ occurs also in the operator ∇^2,) their Taylor expansions in powers of $x-x'$ and using the first approximation for $C_v{}^r$ in those terms in which it occurs with a factor $x^r - x'^r$. By a process of successive approximation one could get the solution to any desired accuracy.

With the coefficients $C_r{}^s(x,x')$ in (40) determined, the new P.b.'s are determined. It should be noted that the new P.b. of any nongravitational variable with $\tilde{g}_{rs}$ or $\tilde{e}^{rs}$ vanishes. However, its new P.b. with $\tilde{p}^{rs}$ does not vanish.

QUANTIZATION

To pass over to the quantum theory, we must make all our dynamical variables into operators satisfying commutation relations corresponding to the new P.b.'s. We must then pick out a complete set of commuting observables. We may take these to consist of the $\tilde{e}^{rs}$ at all points x^r, together with a complete set of commuting nongravitational observables, ζ say. We can then set up the wave function as a function of these variables,

$$\psi(\tilde{e}^{rs},\zeta).$$

The effective domain of ψ is that for which the $\tilde{e}^{rs}$ are restricted to have the determinant unity and to satisfy $\tilde{e}^{rs}{}_s = 0$. ψ may be considered as undefined outside this domain. When we operate on ψ with $\tilde{p}^{ab}$ or with any dynamical variable in the theory, we get another wave function defined in the same domain, on account of $\tilde{p}^{ab}$ commuting with the determinant of the $\tilde{e}^{rs}$ and with $\tilde{e}^{rs}{}_s$.

There are no supplementary conditions to be imposed on ψ. We can choose it arbitrarily to correspond to the initial state in any problem. There is just one equation for ψ, the Schrödinger equation

$$i\hbar(d\psi/dx^0) = H^*{}_{\text{main}}\psi,$$

which fixes the state at later times.

For the theory to be self-consistent it is necessary that the space-like surface on which the state is defined shall always remain space-like. The condition for this is that K^2, the determinant of the g_{rs}, shall remain always positive. In the present formalism this means $\kappa^6 > 0$, with κ determined by (36). If the mass density is always positive, (36) shows that $\kappa > 1$ and there is no trouble. Difficulties arise only where there is a large negative density. This occurs very close to a point particle, on account of the last term in (39). The gravitational treatment of point particles thus brings in one further difficulty, in addition to the usual ones in the quantum theory.