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Deuteron-Pickup Reaction in an Optical-Model Approximation*

KENNETH R. GREIDER
Lawrence Radiation Laboratory, University of California, Berkeley, California
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A theory of the (p,d) pickup reaction is described in which the nuclear interactions of the incoming and outgoing particles are considered. Two different formal expressions that give the transition amplitude are derived, and the wave functions in this amplitude are approximated by an optical-model procedure in which it is assumed that the initial- and final-state particles scatter elastically in the nucleus. Several closed forms for these optical-model wave functions are derived on the basis of a WKB approximation for a complex square-well scattering potential. The use of these wave functions, along with an approximation that gives the form of the transition amplitude in terms of Gaussian functions, allows a closed-form solution for the differential cross section.

It is found that the elastic-scattering processes are not negligible, since they affect considerably the magnitude and the shape of the differential cross section. By comparing the theory with recent pickup experiments on C^{12} at 95 and 145 Mev, one obtains a nuclear-momentum distribution that, unlike the Born approximation analysis, is in good agreement with the results of other determinations of momentum distributions. It is found that a neutron number of from 4 to 6 neutrons and a momentum distribution of $\exp(-E/14)$ are required to fit the data.

I. INTRODUCTION

THE theory of nuclear-rearrangement collisions has been developed and refined to a considerable degree over the past ten years due to both the "direct"-interaction picture given by Serber[1] and the scattering formalism introduced by Lippmann and Schwinger.[2] The two particular processes that have received perhaps the greatest amount of attention during this period are that of deuteron stripping, and its time reverse, deuteron pickup. The theoretical treatments of both these processes have enjoyed remarkable success with the models of Serber,[3] Butler,[4] and Chew and Goldberger,[5] and there have recently been several investigations,[6–9] based on scattering formalism, which obtain more exact results than these earlier theories. Comprehensive reviews of the current status of stripping reactions have been written by Huby[10] and by Butler.[11]

In extending the earlier theories such as the Butler theory for stripping or the Chew-Goldberger theory for pickup, one might wonder whether the nucleus has any appreciable effect on the incoming or the outgoing particles. Certainly one would expect these particles to scatter from the nucleus and change the observed angular distributions. The Butler and Chew-Goldberger theories assume that the nucleus is completely transparent both to incoming and to outgoing nucleons. This presumably accounts for the fact that the Butler cross sections are larger than the experimental results, since the nucleon mean free path at Butler's energies is small compared with nuclear dimensions, and absorption effects should not be negligible. At high energies, where the nucleon mean free path is long, it would be expected that the nucleon-nucleus scattering contributions would not only affect the magnitude of the cross section, but also alter the angular distribution of the observed stripped neutrons.

To include the aforementioned scattering contributions and give an exact treatment of the pickup reaction is by no means easy. The first step, to derive an exact formal expression that is believed to represent the process, is made possible with the results of the Lippmann-Schwinger formalism. However, the exact calculation of such an expression is usually impossible, since the simple Born-approximation description is no longer applicable, and one must either use higher orders in Born approximation, or, better, describe the distorted states by means of the results of multiple-scattering theory.[12] In either case, there is the additional mathematical problem of evaluating the matrix elements, since the integrals that appear are usually quite formidable. Thus it would be desirable to find a formal description of the scattering of particles that is reasonably exact and at the same time not too difficult to handle mathematically.

Among the various methods that aim for this goal, the optical-model approach of Fernbach, Serber, and Taylor[13] appears to be a good approximation to use in rearrangement collisions. (If this method is used, one

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[1] R. Serber, Phys. Rev. **72**, 1114 (1947).
[2] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[3] R. Serber, Phys. Rev. **72**, 1008 (1947).
[4] S. T. Butler, Proc. Roy. Soc. (London) **208**, 559 (1951); and Bhatia, Huang, Huby, and Newns, Phil. Mag. **43**, 485 (1952).
[5] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).
[6] E. Gerjouy, Phys. Rev. **91**, 645 (1953).
[7] N. C. Francis and K. M. Watson, Phys. Rev. **93**, 313 (1954).
[8] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953). It should be noted that in their Eq. (4.4) the first term, $\chi_a^{(+)}$, should read $\psi_a^{(+)}$.
[9] W. Tobocman and M. H. Kalos, Phys. Rev. **97**, 132 (1955).
[10] R. Huby, Progr. in Nuclear Phys. **3**, 177 (1953).
[11] S. T. Butler and O. H. Hittmair, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957).
[12] K. M. Watson, Phys. Rev. **105**, 1388 (1957).
[13] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).

must show that the inelastic processes comprise a small part of the total scattering contributions to the matrix element.) This approach to the stripping problem has been suggested by Francis and Watson[7] and calculations for low-energy processes have been made by Cheston,[14] who also considered polarization effects.

For the sake of clarity, the specific process under consideration here is that in which a proton picks up a bound neutron to form a deuteron, leaving the final nucleus in the ground state or in a low excited state. Then, following Chew and Goldberger, the differential cross section depends on two factors: (a) the probability that the neutron to be picked up has a momentum $\mathbf{k}_n$ in the initial nucleus, and (b) the probability that this momentum, $\mathbf{k}_n$, combined with the incident proton momentum through the mutual interaction, V_{np}, can be found in a deuteron. By describing the nuclear scattering of the incident proton and outgoing deuteron, the optical model referred to above adds another factor to the Chew-Goldberger expression for the differential cross section—the probability that the incident proton (outgoing deuteron) has elastically scattered into a different momentum state before (after) the neutron is picked up. Implicit in this factor is the probability that the particle may be absorbed by the nuclear medium. It is assumed in the following discussion that the energy of the incident proton is high, so that the surface reflection and refraction, as well as the effects of the nuclear Coulomb potential, can be neglected.

Part II shows the development of the matrix element that describes the pickup process. The optical-model restriction that the incoming and outgoing particles scatter coherently is imposed, and it will be shown that the Chew-Goldberger expression can be obtained as a special case of the more general matrix element presented.

General forms for the wave functions that can be used to describe the elastic scattering in a central nuclear potential are developed in Part III. The derivation is based on a WKB approximation to obtain results that are as easy to handle mathematically as the Born approximation's plane waves. Finally the differential cross section is obtained in closed form by using Gaussian expressions to approximate the functions in the matrix element.

In Part IV, calculations are made to fit the theory with recent pickup experiments on C^{12}.[15,16] It is shown that by the inclusion of particle interactions with the nucleus, a fit can be obtained without using the high neutron momenta that are required in the Chew-Goldberger analysis. Furthermore, since the optical model accounts for particle absorption, Part IV shows that a reasonable value is obtained for the magnitude of the cross section.

An alternative expression for the matrix element that describes the pickup process is given in the Appendix. A form is obtained that duplicates Francis and Watson's results, which were derived by applying time reversal to the stripping process. This form is compared with that of Part II.

II. FORMALISM

In order to obtain information on the cross section, energy, spin, etc. of the deuterons produced from nuclei bombarded with protons, it is most convenient to use the scattering formalism introduced by Lippmann and Schwinger.[2] This formalism provides a method for obtaining the transition-matrix elements that contain the desired information about the final-state deuterons in terms of the initial conditions (momentum, spin, etc.) on the incoming proton-nucleus system. It is necessary first to define explicitly the noninteracting initial and final states in order to describe the operations that later lead to the transition matrix.

Consider an initial state consisting of a proton, whose momentum is $\hbar\mathbf{k}_0$ in the center-of-mass system, incident on a nucleus which is described by a wave function $\chi_0(\boldsymbol{\tau}_A)$. Here $\boldsymbol{\tau}_A$ represents the coordinates of the A nuclear particles, and the subscript 0 designates a nucleus in its ground state. For clarity of notation, all spin indices have been suppressed for the present. The wave function for the proton-nucleus system,

$$\phi_i(\boldsymbol{\tau}_A,\mathbf{r}_p)=\chi_0(\boldsymbol{\tau}_A)e^{i\mathbf{k}_0\cdot\mathbf{r}_p}, \tag{1}$$

is the solution of the Schrödinger wave equation

$$(H_0+T_n+T_p+V_n)\phi_i(\boldsymbol{\tau}_A,\mathbf{r}_p)=E_i\phi_i(\boldsymbol{\tau}_A,\mathbf{r}_p), \tag{2}$$

where H_0 is the total Hamiltonian for $A-1$ nucleons, and T_n and T_p are the kinetic energy operators for the neutron and proton respectively. The neutron to be picked up is bound to the core of $A-1$ nucleons by the potential V_n.

The final state, which consists of a free deuteron of momentum $\hbar\mathbf{K}$, and $A-1$ nucleons in some nuclear state n, is described by the wave function

$$\phi_f(\boldsymbol{\tau}_A,\mathbf{r}_p)=\chi_n(\boldsymbol{\tau}_{A-1})\Phi_d(\mathbf{r})e^{i\mathbf{K}\cdot\mathbf{R}}, \tag{3}$$

which is the solution of

$$(H_0+T_n+T_p+V_{np})\phi_f(\boldsymbol{\tau}_A,\mathbf{r}_p)=E_f\phi_f(\boldsymbol{\tau}_A,\mathbf{r}_p), \tag{4}$$

where $\Phi_d(\mathbf{r})$ is the wave function of the bound deuteron, and V_{np} is the neutron-proton potential.

The total wave function, ψ, which describes the complete interacting proton-nucleus system is the solution of the wave equation involving the total Hamiltonian, H:

$$H\psi=(H_0+T_n+T_p+V_n+V_p+V_{np})\psi=E_i\psi. \tag{5}$$

[14] W. B. Cheston, Phys. Rev. **96**, 1590 (1954), and J. Sawicki, Phys. Rev. **106**, 172 (1957).
[15] W. Selove, Phys. Rev. **101**, 231 (1956).
[16] P. F. Cooper, thesis, Harvard University, 1958 (unpublished).

The potential V_p of Eq. (5) describes the interaction of the proton with the $A-1$ core nucleons.

Following Lippmann and Schwinger, we can write the expression for the solution of Eq. (5) corresponding to incoming plane waves and outgoing spherical waves:

$$\psi^{(+)}=\Omega^{(+)}\phi_i$$

$$=\left[1+\frac{1}{a^{(+)}-V_n-V_p-V_{np}}(V_p+V_{np})\right]\phi_i, \quad (6)$$

$$=\phi_i+\frac{1}{a^{(+)}-V_n}(V_p+V_{np})\psi^{(+)}, \quad (6a)$$

where the notation has been condensed by setting

$$a^{(+)}=E_i-H_0-T_n-T_p+i\epsilon. \quad (7)$$

The transition amplitude is then

$$T_{fi}=(\psi^{(-)}|V_p+V_{np}|\phi_i). \quad (8)$$

We may use Eq. (8) and proceed to derive the same form for the transition amplitude that was obtained in a different way by Francis and Watson. This derivation can be found in the Appendix. Although the resultant transition matrix can now be used to obtain the desired information about the outgoing deuteron states, there are two objections that may be raised concerning its form:

(a) Within the limits of the optical-model approximations, Francis and Watson's result is still not an exact expression, owing to the additional approximation that neglects the last V_{np} interaction in the scattering of the incoming proton from the A nucleons.

(b) The final-state wave function defined in their Eq. (27) and in Eq. (56) here (see Appendix) appears rather unsymmetric in that it represents an outgoing deuteron in which only the neutron interacts with the residual nucleus. It would be desirable, then, to find a new form for the transition matrix which requires no approximations beyond the optical-model assumptions and represents the outgoing-deuteron state so that both the neutron and proton interact symmetrically with the residual nucleus.[17] An expression that fulfills both of these requirements has been partially developed by Gell-Mann and Goldberger[8] and is derived below.

We define a wave function $\chi^{(+)}$ representing an incoming proton wave in which the proton interacts *with the core only* through the potential V_p:

$$\chi^{(+)}=\phi_i+\frac{1}{a^{(+)}-V_n}V_p\chi^{(+)}. \quad (9)$$

Here $\chi^{(+)}$ is related to the total wave function $\psi^{(+)}$ by the integral equation

$$\psi^{(+)}=\chi^{(+)}+\frac{1}{a^{(+)}-V_n-V_{np}-V_p}V_{np}\chi^{(+)}. \quad (10)$$

By substitution of Eq. (9) into Eq. (8), we obtain

$$T_{fi}=(\psi^{(-)}|V_{np}+V_p|\chi^{(+)}) -\left(\psi^{(-)}\middle|(V_{np}+V_p)\frac{1}{a^{(+)}-V_n}V_p\middle|\chi^{(+)}\right). \quad (11)$$

Rearranging terms, we have

$$T_{fi}=(\psi^{(-)}|V_{np}|\chi^{(+)}) +\left(\psi^{(-)}\middle|\left[1-(V_{np}+V_p)\frac{1}{a^{(+)}-V_n}\right]V_p\middle|\chi^{(+)}\right). \quad (12)$$

Using Eq. (6a), we finally obtain

$$T_{fi}=(\psi^{(-)}|V_{np}|\chi^{(+)})+(\phi_i|V_p|\chi^{(+)}),$$

which is equal to

$$T_{fi}=(\psi^{(-)}|V_{np}|\chi^{(+)})+(\chi^{(-)}|V_p|\phi_i). \quad (13)$$

Equation (13) is the same as Eq. (4.4) of Gell-Mann and Goldberger.[8] The first term in Eq. (13) gives the transition amplitude for the pickup process occurring via the V_{np} interaction. The incident proton interacts with all nucleons except the neutron to be picked up, and both the neutron and proton in the emerging deuteron interact with the residual core nucleons. The second term in Eq. (13) gives the amplitude for proton scattering from the nucleus via the interaction V_p with just the $A-1$ nucleons of the core. This term, therefore, should not lead to final-state deuterons, since the deuteron binding potential, V_{np}, does not appear in the final-state wave function, $\chi^{(-)}$. A formal proof that this term is indeed zero has been given by Lippmann.[18] Thus, we obtain

$$T_{fi}=(\psi^{(-)}|V_{np}|\chi^{(+)}). \quad (14)$$

We may now make our optical-model approximations:

$$\chi^{(+)}\simeq\chi_C^{(+)}=\phi_i+\frac{1}{a^{(+)}-V_n}\mathcal{V}_{Cp}\chi_C^{(+)}, \quad (15)$$

and

$$\psi^{(-)}\simeq\psi_C^{(-)}=\phi_f+\frac{1}{a^{(-)}-V_n-\mathcal{V}_{Cp}^{\dagger}-\mathcal{V}_{Cn}^{\dagger}} \times(\mathcal{V}_{Cp}^{\dagger}+\mathcal{V}_{Cn}^{\dagger})\phi_f, \quad (16)$$

which yield our desired symmetric expression,

$$T_{fi}=(\psi_C^{(-)}|V_{np}|\chi_C^{(+)}). \quad (17)$$

[17] The author would like to express his appreciation to Professor Geoffrey Chew for bringing up this point, and to Dr. Leonard Rodberg for his help and interest in the derivation.

[18] B. Lippmann, Phys. Rev. 102, 254 (1956).

The optical model operators $\mathcal{V}_{Cp}$ and $\mathcal{V}_{Cn}$ are defined to be diagonal with respect to the energy of the core. They describe only elastic proton- and neutron-nucleus interactions, respectively, and can be deduced from multiple-scattering theory, or may be replaced by the phenomenological optical potential of Fernbach, Serber, and Taylor.[13]

There are several qualitative arguments that support this approximation. First, the Pauli exclusion principle forbids "hard" nucleon-nucleon scatterings in a nucleus except for those that leave the particle in unoccupied momentum states beyond the Fermi sphere. Consequently, the inelastic collisions that do occur must be associated with large momentum transfers at large scattering angles, and can thus be experimentally separated from elastic effects which describe the scattering at small angles. Secondly, the experimental fact that the pickup process usually leaves the final nucleus in its ground state gives a strong argument for neglecting inelastic scatterings. For a particle incident on a nucleus in its ground state, there are many channels open for inelastic scatterings to excited states where the density of states is large, but there are few channels back to the ground state. Thus it is improbable that an inelastically scattered particle will find its way back to a state of low or zero excitation. Finally, one may argue that the inelastic scattering of the outgoing deuteron will not contribute to the pickup cross section, since it tends to break up the deuteron. This is because such inelastic effects arise from "hard" scatterings of the individual neutron and proton in the deuteron. Unless these two particles scatter coherently in "hard" interactions, the small deuteron binding energy should not be expected to keep them together. An approximate estimate[19] of the inelastic corrections to the cross section gives a value on the order of 1/100 the magnitude of the elastic terms. It will be assumed in the rest of this paper that, for the purposes of describing the pickup reaction, the elastic-scattering approximations give an adequate and reasonably accurate description of the true scattering.

The differences between Eq. (56) [or Eq. (27) of Francis and Watson] and Eq. (17) are now apparent. The initial state of the former describes a wave of incoming protons interacting with all A nucleons of the initial nucleus, while the latter represents interactions with only the $A-1$ nucleons of the core. But, to compensate for this difference, the final-state wave function of Eq. (56) has only neutron interactions, whereas the corresponding wave function of Eq. (17) has both neutron and proton interactions. Therefore some of the proton interactions that seem to be missing in the initial state of Eq. (17) are included in its final state. Thus it is evident that, within the optical-model approximations, Eq. (17) is more exact and more aesthetically appealing than Eq. (56).

[19] This was carried out following the method of reference 12.

On the other hand, for practical reasons, Eq. (56) may in many cases offer a more suitable form for numerical calculations. If we approximate the deuteron as a single coherent particle, the optical parameter $\mathcal{V}_{Cn}$ used in Eq. (56) is much better known than $\mathcal{V}_{Cd}$, the deuteron optical potential, which would be required in Eq. (17). One would then conclude that Eq. (17) is formally a more desirable form for the transition amplitude, but practical necessities may often favor the use of Eq. (56).

III. EVALUATION OF THE MATRIX ELEMENT FOR CENTRAL POTENTIAL INTERACTIONS

It is now necessary to find manageable forms of the optical-model wave functions of Part II so that the required integrations may be performed without excessive difficulty. In this section some explicit forms for the wave functions $\chi_C^{(+)}$ and $\psi_C^{(-)}$ are derived. Separating the nuclear and proton coordinates in Eq. (2), we obtain an equation for the proton wave function (with $V_p=\mathcal{V}_{Cp}$):

$$\left(-\frac{\hbar^2}{2m}\nabla_{r_p}{}^2+\mathcal{V}_{Cp}(\mathbf{r}_p)-E_{p0}\right)\psi_0(\mathbf{r}_p)=0, \tag{18}$$

where we have

$$E_{p0}=\hbar^2k_0^2/2m.$$

Solving Eq. (18) in a one-dimensional WKB approximation, we obtain

$$\psi_0(x)\sim\exp\left\{\frac{i}{\hbar}\int^x\{2m[E_{p0}-\mathcal{V}_{Cp}(x)]\}^{\frac{1}{2}}dx\right\}. \tag{19}$$

We make the substitutions

$$k_0=(2mE_{p0})^{\frac{1}{2}}/\hbar,$$

and

$$n_1(x)k_0=[2m(E_{p0}-\mathcal{V}_{Cp}(x))]^{\frac{1}{2}}/\hbar, \tag{20}$$

where

$$n_1^2(x)=1-\mathcal{V}_{Cp}(x)/E_{p0}, \tag{20a}$$

defines the index of refraction in the nuclear medium. Equation (19) becomes

$$\psi_0(x)\sim\exp\left\{ik_0\int^x n_1(x)dx\right\}. \tag{21}$$

To find the three-dimensional form of $\psi(\mathbf{r}_p)$, we need to assume some distribution and shape for $\mathcal{V}_{Cp}(\mathbf{r}_p)$. For the case of a hard spherical nucleus, we may write for $r<R_0$:

$$\mathcal{V}_{Cp}(\mathbf{r})=\mathcal{V}_0,\quad\text{and}\quad n_1(\mathbf{r})=n_1,$$

and for $r>R_0$: (22)

$$\mathcal{V}_{Cp}(\mathbf{r})=0,\quad\text{and}\quad n_1(\mathbf{r})=1,$$

which, for an incoming wave parallel to the x axis, become, respectively,

$$\text{for} \quad x < -(R_0^2 - r^2 \sin^2\theta)^{\frac{1}{2}}, \quad n_1(x) = 1,$$
and
$$\text{for} \quad x > -(R_0^2 - r^2 \sin^2\theta)^{\frac{1}{2}}, \quad n_1(x) = n_1. \tag{22a}$$

Substituting $x = r\cos\theta$, we can evaluate Eq. (21) to obtain a closed form solution for $\psi_0(\mathbf{r})$:

$$\psi_0(\mathbf{r}) = \exp\{in_1\mathbf{k}_0\cdot\mathbf{r} + i(n_1-1)k_0(R_0^2 - r^2\sin^2\theta)^{\frac{1}{2}}\}. \tag{23}$$

We note that Eq. (23) obtains the classical result for a particle following a trajectory through a spherical region that is characterized by an index of refraction n. The fact that the classical expression is obtained is not surprising, since we have used the WKB solution of zero order in $\hbar$.

By approximating $r^2\sin^2\theta \ll R_0^2$, we arrive at an easily integrable expression,

$$\psi_0(\mathbf{r}) \simeq \exp\{i(n_1-1)k_0R_0 + in_1\mathbf{k}_0\cdot\mathbf{r}\}, \tag{24}$$

which has the same form as the scattering wave functions of Hart and Montroll[20] if one neglects the internal reflected wave. For $n \lesssim 0.10$, one obtains results from Eq. (24) that compare favorably to more exact numerical calculations that use Eq. (23). For larger values of n, Eq. (23) may be approximated by

$$\psi_0(\mathbf{r}) \simeq \exp\{in_1\mathbf{k}_0\cdot\mathbf{r} + i(n_1-1)k_0 \times[R_0 - r(1-|\cos\theta|)]\}, \tag{25}$$

which is, however, a less manageable function than that of Eq. (24).

In the outside region, $r > R_0$, the wave function is merely $\exp(i\mathbf{k}_0\cdot\mathbf{r})$. The small contribution of the transmitted wave has been neglected here.

We now turn to the problem of obtaining a form for the final-state wave function of Eq. (17) or Eq. (56) of the Appendix. If we use Eq. (56) we write the wave equation for $\Omega_{Cn}^{(-)}\phi_f$, specifying the coordinates explicitly:

$$[H_0(\boldsymbol{\tau}_{A-1}) + T_R(\mathbf{R}) + T_r(\mathbf{r}) + V_{np}(\mathbf{r}) + \mathcal{V}_{Cn}^\dagger(\mathbf{r}_n) - E_{if}] \times\Omega_{Cn}^{(-)}\phi_f(\mathbf{r},\mathbf{R},\boldsymbol{\tau}_{A-1}) = 0, \tag{26}$$

where $E_{if} = (\hbar^2K^2/4M) - B_d$, and B_d is the magnitude of the deuteron binding energy. In order to solve Eq. (26), we assume $R \gg r$, so that we have

$$\mathcal{V}_{Cn}^\dagger(\mathbf{r}_n) = \mathcal{V}_{Cn}^\dagger(\mathbf{R} - \mathbf{r}/2) \simeq \mathcal{V}_{Cn}^\dagger(\mathbf{R}). \tag{27}$$

This means that we approximate the deuteron as behaving like a single coherent particle in the nucleus, or, more exactly, we assume that the average neutron-proton separation in the deuteron is small compared with the nuclear dimensions. If we use the transition matrix of Eq. (17), and assume that the deuteron propagates as a single particle, then Eq. (26) is changed by replacing $\mathcal{V}_{Cn}^\dagger(\mathbf{r}_n)$ by $\mathcal{V}_{Cd}^\dagger(\mathbf{R})$, and the approximation of Eq. (27) is no longer necessary. We may simplify the notation then by letting $\mathcal{V}_C^\dagger(\mathbf{R})$ stand for either $\mathcal{V}_{Cn}^\dagger(\mathbf{R})$ or $\mathcal{V}_{Cd}^\dagger(\mathbf{R})$.

Following our method for the proton wave function, we separate the nuclear and deuteron coordinates, and subtract the equation for the bound state of the deuteron. We are left with

$$\left[-\frac{\hbar^2}{4m}\nabla_R^2 + \mathcal{V}_C^\dagger - (E_{if} - B_d)\right]\chi^{(-)}(\mathbf{R}) = 0, \tag{28}$$

which can be solved by the WKB method outlined above. However, the form will not be quite the same as the result in Eq. (23), since $\Omega_C^{(-)}\phi_f$ or $\psi^{(-)}$ is a solution that is asymptotic to outgoing plane waves at infinity and to incoming spherical waves.[2] For a spherically symmetric, square-well potential, we find

$$\psi_0^{(-)}(\mathbf{R}) = \exp\{in_2'\mathbf{K}\cdot\mathbf{R} - i(n_2'-1)K \times(R_0^2 - R^2\sin^2\theta)^{\frac{1}{2}}\}, \tag{29}$$

where

$$(n_2')^2 = 1 - \mathcal{V}_C^\dagger/E_d, \tag{30}$$

and

$$E_d = \hbar^2K^2/4M. \tag{31}$$

The approximations of Eqs. (24) and (25) may then be applied to Eq. (29) in order to obtain an easily integrable expression for the outgoing deuteron wave function.

It should be pointed out that, for small n, wave functions of the form of Eq. (24) can be handled with no more difficulty than the plane waves of the Born approximation. These wave functions may now be substituted into Eq. (17) or (56), in order to find the transition amplitude, T_{fi}, for the pickup process.

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \sum_f \frac{3}{4}\frac{m^2}{2\pi^2\hbar^4}\frac{K}{k_0}\frac{A(A-1)}{(A+1)^2}|T_{fi}|^2. \tag{32}$$

The factor of $\frac{3}{4}$ is due to spin statistics, and the sum is over all final states. The sum can be written

$$\sum_f |T_{fi}|^2 = \sum_f |(\chi_f(\boldsymbol{\tau}_{A-1})\Phi_d(\mathbf{r})\psi_0^{(-)}(\mathbf{R}) \times|V_{np}(\mathbf{r})|\chi_i(\boldsymbol{\tau}_{A-1},\mathbf{r}_n)\psi_0^{(+)}(\mathbf{r}_p))|^2,$$

or

$$\sum_f |T_{fi}|^2 = \int d\boldsymbol{\tau}_{A-1}\left|\int d\mathbf{r}_n d\mathbf{r}_p\, \Phi_d^*(\mathbf{r})\psi_0^{(-)*}(\mathbf{R}) \times V_{np}(\mathbf{r})\chi_i(\boldsymbol{\tau}_{A-1},\mathbf{r}_n)\psi_0^{(+)}(\mathbf{r}_p)\right|^2. \tag{33}$$

We need next to evaluate the integral, M, inside the

[20] R. Hart and E. Montroll, J. Appl. Phys. 22, 376 (1951), and E. Montroll and J. M. Greenberg, Phys. Rev. 86, 889 (1952).

absolute value signs. By allowing $V_{np}(\mathbf{r})$ to operate on $\Phi_d(\mathbf{r})$, we obtain $[(\hbar^2/m)\nabla^2-B_d]\Phi_d(\mathbf{r})$, in accordance with the Schrödinger equation for the deuteron ground state. Therefore we denote

$$V_{np}(\mathbf{r})\Phi_d(\mathbf{r})=\Phi(\mathbf{r})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{k}\,\Phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{34}$$

and we use the partial Fourier transform of the initial state nuclear wave function:

$$\int d\boldsymbol{\tau}_{A-1}|\chi_i(\boldsymbol{\tau}_{A-1},\mathbf{r}_n)|^2 =\left|\frac{1}{(2\pi)^{\frac{3}{2}}}\int d\mathbf{k}_n\,\chi_i(\mathbf{k}_n)e^{i\mathbf{k}_n\cdot\mathbf{r}_n}\right|^2, \tag{35}$$

where $\chi_i(\mathbf{k}_n)$ is assumed to be the square root of the neutron momentum density distribution in the nucleus.

Using these expressions in Eq. (33), we have for M,

$$M=\frac{1}{(\pi)^3}\int d\mathbf{k}\,\Phi(\mathbf{k})\int d\mathbf{k}_n\,\chi_i(\mathbf{k}_n) \times\int d\mathbf{R}\,e^{-i(2\mathbf{k}-2\mathbf{k}_n)\cdot\mathbf{R}}\psi_0^{(-)}(\mathbf{R}) \times\int d\mathbf{r}_p\,e^{i(2\mathbf{k}-\mathbf{k}_n)\cdot\mathbf{r}_p}\psi_0^{(+)}(\mathbf{r}_p). \tag{36}$$

It should be pointed out here that Eq. (36) reduces to the Born-approximation expression of the Chew-Goldberger theory for $n_1=n_2=1$. Since the incoming proton is then represented by a plane wave, the integral over $\mathbf{r}_p$ gives a delta function, $\delta(2\mathbf{k}-\mathbf{k}_n+\mathbf{k}_0)$, and the integral over $\mathbf{R}$ gives $\delta(2\mathbf{k}-2\mathbf{k}_n+\mathbf{K})$, which permits immediate evaluation of the integrals.

Since some effort has been spent to obtain integrable wave functions for use in the matrix element, it would obviously be desirable to continue further and use analytic expressions in Eq. (36) that allow its solution in closed form. To accomplish this, we have assumed Gaussian forms for the factors in the integrand,

$$\chi(\mathbf{k}_n)=C\exp\left[-c\left(\mathbf{k}_n+\frac{p-q}{2}\mathbf{k}_0\right)^2\right], \tag{37}$$

where we have accounted for nuclear recoil by defining the parameters

$$q=(A-1)/A;\quad p=(A+1)/A. \tag{38}$$

Assuming a Hulthén wave function for the deuteron, we find

$$\Phi(\mathbf{k})=\frac{4\pi}{(2\pi)^{\frac{3}{2}}}A\frac{\hbar^2}{m}(\alpha^2-\beta^2)\times\frac{1}{k^2+\beta^2}, \tag{39}$$

TABLE I. Optical-model parameters for the process $C^{12}(p,d)C^{11}$.

E_{lab} (Mev)	$\mathcal{V}_{Cp}$ (Mev)	$\mathcal{V}_{Cd}$ or $\mathcal{V}_{Cn}$ (Mev)	$R_0\times10^{-13}$ cm	E_0 (Mev)	Neutron number
95	$-25\ -15i$	$-20\ -15i$	3.4	14	4 to 6
145	$-13\ -18i$	$-20\ -15i$	3.4	14	4 to 6

where we can approximate

$$\frac{1}{k^2+\beta^2}\simeq\frac{1}{\beta^2}[0.902\exp(-0.729k^2/\beta^2) +0.0971\exp(-0.0313k^2/\beta^2)], \tag{40}$$

and we have chosen $\beta=6.2\alpha$.

Finally, the integrations over $\mathbf{r}_p$ and $\mathbf{R}$ give spherical Bessel functions, which also may be approximated by a sum of Gaussian functions. The first-order spherical Bessel function may be written

$$j_1(x)/x\simeq\tfrac{1}{3}\{\exp(-0.111x^2) -0.121\exp[-0.290(x-5.20)^2] +0.038\exp[-0.460(x-9.00)^2]\}, \tag{41}$$

which is accurate to $x\lesssim10$. It turns out that at the energies under consideration, $j_1(x)/x$ is required only for values of x less than 3; therefore the first term of Eq. (41) is sufficiently accurate.

The integral of Eq. (36) now becomes very simple if we substitute the approximations of Eqs. (37) through (41). Our matrix element will be the sum of several integrals over both $\mathbf{k}$ and $\mathbf{k}_n$ of the form

$$\int d\mathbf{k}\exp(-dk^2+2\mathbf{k}\cdot\mathfrak{R})=\exp(\mathfrak{R}^2/d)\left(\frac{\pi}{d}\right)^{\frac{3}{2}}, \tag{42}$$

where the vector $\mathfrak{R}$ is a linear combination of the incoming proton momentum $\mathbf{k}_0$ and the outgoing deuteron momentum $\mathbf{K}$. The actual values of $\mathfrak{R}$ and d in Eq. (42) depend on the parameters given in Eqs. (37) through (41) for the particular Gaussian considered. Then these Gaussian terms, when summed, constitute the solution of Eq. (36). Finally, Eq. (32) allows the calculation of the differential cross section.

IV. COMPARISON OF THE CENTRAL POTENTIAL THEORY WITH EXPERIMENT

Measurements of the deuteron-pickup cross section have been reported by Selove[15] and by Cooper,[16] who have obtained angular distributions for the process $C^{12}(p,d)C^{11}$, at 95 and 145 Mev, respectively. The results of the optical-model analysis are obtained by using the parameters listed in Table I, and the resulting angular distributions are shown in Curve A in Figs. 1 and 2. The values of the nucleon optical potentials are taken from Glassgold.[21] It should be noticed from Eq. (30) that $n_2'^*=n_2$ if $\mathcal{V}_{Cd}$ is in the form shown in Table I

[21] A. Glassgold, Revs. Modern Phys. 30, 419 (1958).

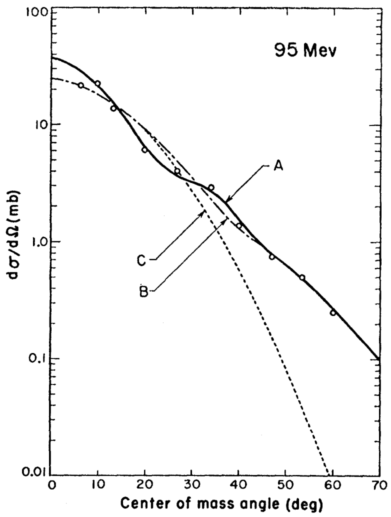

FIG. 1. Comparison of the optical-model treatment for the reaction $C^{12}(p,d)$-C^{11} with Selove's data. Curve A is obtained by using the parameters listed in Table I. Curve B shows the effect for $n_2=1$, and curve C gives the Born approximation results with $n_2=n_1=1$. A nuclear momentum distribution of $\exp(-E/14)$ is used in each case. The statistical errors on the experimental points are reported to be approximately $\pm 6\%$.

above, i.e., a real and an imaginary part. This follows from the fact that n_2' was obtained from $\mathcal{V}_{Cd}^{\dagger}$. Thus if n_2' is merely the complex conjugate of n_2, then $n_2'^*$ is equal to n_2.

To obtain the results shown in Figs. 1 and 2, the nuclear radius was chosen to be $1.5A^{\frac{1}{3}}\times 10^{-13}$ cm, and the nuclear momentum distribution of the initial neutron state was of the form $\exp(-E/E_0)$ [see Eq. (37)]. Table I shows that the fit at both energies was obtained with $E_0=14$ Mev, which is in general agreement with the scattering results of Wilcox and Moyer,[22] and Cladis, Hess, and Moyer.[23] Selove reported that in order to fit the Born approximation (Chew-Goldberger) theory to the results of his pickup experiments he required high-momentum components in the nucleon momentum distribution, which had the form $\exp(-E/7)+0.15\exp(-E/50)$.[15] These high-momentum components ($E_0=50$ Mev) were needed to reproduce the observed wide-angle distribution of deuterons. In the Born-approximation theory, the deuteron momentum is just the sum of the incident-proton momentum and the bound-state neutron momentum. In order to observe deuterons at appreciably large angles, therefore, one must have a nucleus that contains neutrons whose energy is about as large as the incident-proton energy.

Such energetic neutrons are not required in theory presented here, since the wide-angle distribution of deuterons can be obtained by allowing the incident and final particles to scatter in the nuclear field. For example, an incident proton may have scattered through a considerable angle with respect to the initial $\mathbf{k}_0$ direction before it encounters the neutron. Consequently, after the pickup of a low-energy neutron from inside the nucleus, a deuteron may emerge at an even larger angle. By further allowing this deuteron to scatter while leaving the nucleus, one can obtain very broad angular distributions.

If one integrates the neutron momentum-density distribution over all momentum space, he obtains the number of neutrons that are effective in the pickup reaction. This number depends on the value of the constant C in Eq. (37), where C is usually chosen so that the magnitude of the theoretical cross section fits the experimental results. In the fit to the experimental data shown in Curves A, a neutron number from four to six is required at both 95 Mev and 145 Mev. These results, which approximately account for the six neutrons in C^{12}, are in striking contrast to the 0.061 neutron obtained by Selove, and the 0.76 neutron obtained by Chew and Goldberger in their Born-approximation analyses. The reason for the large differences is, of course, the inclusion of absorption effects in the optical-model theory.

Curve B in Figs. 1 and 2 shows the result of neglecting the deuteron potential, i.e., of setting $n_2=1$. The general shape of the differential cross section is obtained by considering only the interaction of the proton with the nucleus,[24] but the diffraction dip at about 20° can only be obtained by allowing the deuteron to interact with the nucleus through the potential $\mathcal{V}_{Cd}$ (Curve A). Furthermore, the normalization of Curve B gives a neutron number in the carbon nucleus of about 0.4 neutrons at 95 Mev and about 1.0 neutrons at 145 Mev. These values are smaller than the corresponding numbers for $n_2\neq 1$, since in ignoring the deuteron absorption by the nucleus, we obtain matrix elements whose magnitude is too large.

Curve C shows the result of the Born-approximation analysis[5,15] ($n_2=n_1=1$) for $E_0=14$ Mev. The normalization here requires a neutron number of about 0.25 neutrons at both 95 and 145 Mev. These values are

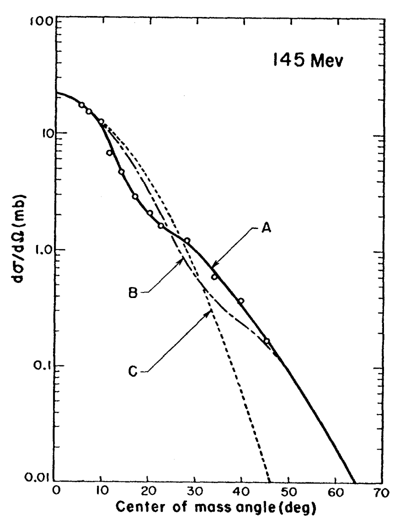

FIG. 2. Comparison of the optical-model treatment for the reaction $C^{12}(p,d)$-C^{11} with Cooper's data. Curve A is obtained by using the parameters listed in Table I. Curve B shows the effect for $n_2=1$, and curve C gives the Born approximation results with $n_2=n_1=1$. A nuclear momentum distribution of $\exp(-E/14)$ is used in each case. The errors on the experimental points are reported to be less than 6% and average to about 3.5%.

[22] J. M. Wilcox and B. J. Moyer, Phys. Rev. **99**, 875 (1955).
[23] Cladis, Hess, and Moyer, Phys. Rev. **87**, 425 (1952).
[24] For a somewhat different calculation using this approximation see Kenneth R. Greider, University of California Radiation Laboratory Report UCRL-8357, July 7, 1958 (unpublished).

smaller than either those for Curves A or B because no nuclear interactions and hence no absorption is considered. It should be pointed out that the Born approximation does predict the gross features of the process, and can obtain reasonable results for small angles.

It is not quite clear what exact value of the deuteron potential, $\mathcal{V}_{Cd}$, is required to fit the experimental curve, because the numbers listed in Table I are only one set of several that obtain a reasonable fit. However, an upper limit may be obtained, since for a good fit, both the real and imaginary parts of $\mathcal{V}_{Cd}$ must fall between 0 and -20 Mev. If the matrix element of Eq. (56) is used to calculate the cross section, then the deuteron potential is just $\mathcal{V}_{Cn}$, the potential felt by the neutron in the deuteron. Values of $\mathcal{V}_{Cn}$ within the limits mentioned above are quite consistent with information on nucleon-nucleus potentials.[21] However, if the matrix element of Eq. (17) is used, then one requires values for $\mathcal{V}_{Cd}$ which are not readily available in the literature.

It appears that in a first approximation, the effective deuteron optical potential should be merely the sum of the neutron and proton potentials. However, in attempts to fit deuteron scattering data,[25] such a simple sum is not adequate for the deuteron potential, as higher-order effects are not negligible. These effects describe the coherent scattering of both neutron and proton by the nucleus, and thus tend to lower considerably the effective deuteron potential from the value given merely by the sum of the single-particle potentials. In this case the value of $\mathcal{V}_{Cd}$ required in the present analysis could be compatible with the actual effective deuteron potential.

It should be mentioned that because of the nature of the approximations used in this treatment of the pickup process, it is unlikely that any exact information can be obtained for the deuteron potential. The general effect of the square-well approximation is small, as can be seen by averaging the scattering amplitudes for several values of the radius parameter, R_0. But still it can lead to a change in the required deuteron potential by about 5 Mev. More important, as one requires a larger deuteron potential on the order of the deuteron kinetic energy, the requirement that $n_2\sim 1$ is no longer met, and the wave function of Eq. (29) is no longer valid. In order to avoid these difficulties, a more exact treatment would be necessary in which one would use wave functions of the form of Eq. (23), and (or) higher orders in WKB approximation. Certainly, a numerical integration of the matrix element would be inevitable.

However, discounting the failure of the theory with regard to the points mentioned above, it is felt that by an optical-model analysis as presented here, one can adequately describe the nuclear scattering effects in rearrangement collisions such as the pickup process. It should also be quite apparent from the results here that the neglect of the initial- and final-state scattering contributions to the matrix element is not justified at these energies, if one is interested in finding accurate information concerning the details of nuclear structure.

[25] H. P. Stapp, Phys. Rev. **107**, 607 (1957).

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APPENDIX: ALTERNATIVE DERIVATION OF THE TRANSITION MATRIX

We may rewrite Eq. (6) as

$$\psi^{(+)}=\phi_i+\frac{1}{a^{(+)}-V_{np}-V_n}V_{np}\phi_i+\frac{1}{a^{(+)}-V_{np}-V_n}V_p\psi^{(+)}, \quad (43)$$

and verify that it satisfies Eqs. (2) and (5) by direct substitution. The Green's function in Eq. (43) represents outgoing deuterons in which the neutron interacts with the core through the potential V_n. We can therefore define a wave function $\Phi_n^{(-)}$ to describe this:

$$\Phi_n^{(-)}=\Omega_n^{(-)}\phi_f=\phi_f+\frac{1}{a^{(-)}-V_{np}}V_n^\dagger\Phi_n^{(-)}, \quad (44)$$

where

$$\Omega_n^{(-)}=1+\frac{1}{a^{(-)}-V_{np}-V_n^\dagger}V_n^\dagger. \quad (45)$$

Now by expanding $(a^{(+)}-V_{np}-V_n)^{-1}$ of Eq. (43) in the usual way in outgoing deuteron states (in which the neutron interacts with the core), we obtain the exact transition-matrix element for the pickup process:

$$T_{fi}=(\Phi_n^{(-)}|V_{np}|\phi_i)+(\Phi_n^{(-)}|V_p|\psi^{(+)}). \quad (46)$$

It should be noted that no approximations have been used in deriving Eq. (46). This transition amplitude, however, is not a very tractable expression as yet, since only the functional forms of V_{np} and ϕ_i are reasonably well known. We begin with the optical-model approximations by replacing the last term of Eq. (46) by an expression that describes only elastic or coherent scatterings of the proton in the nucleus:

$$V_p|\psi^{(+)})\simeq\mathcal{V}_{Cp}|\Omega_{Cp}^{(+)}\phi_i), \quad (47)$$

where we have

$$\Omega_{Cp}^{(+)}=1+\frac{1}{a^{(+)}-V_n-\mathcal{V}_{Cp}}\mathcal{V}_{Cp}, \quad (48)$$

and $\mathcal{V}_{Cp}$ and $\Omega_{Cp}^{(+)}$ are defined to be diagonal with respect to the energy of the core. [See discussion in Sec. II.]

Since, in Eq. (48), $a^{(+)}-V_n$ is diagonal and represents states of all A nucleons, $\mathcal{V}_{Cp}$ then gives the elastic proton interactions with all A nucleons. It might be argued therefore that to make Eq. (47) more exact, we should have an additional term, $V_{np}\psi^{(+)}$, on the left side, since V_p includes the proton interaction with only $A-1$ nucleons. We can justify Eq. (47) by expanding $\psi^{(+)}$ and seeing that the proton interacts through V_p+V_{np} for all but the last scattering, in which V_{np} is absent (i.e., the neutron does not interact). If A is large we can assume that the omission of the neutron from the A nucleons in the last scattering will not affect the elastic description of the proton scattering. Substituting Eq. (47) into (46), we obtain the transition matrix

$$T_{fi}=(\Phi_n^{(-)}|V_{np}|\phi_i)+(\Phi_n^{(-)}|\mathcal{V}_{Cp}|\Omega_{Cp}^{(+)}\phi_i). \quad (49)$$

We may now condense Eq. (49) into a more compact expression. We note from Eqs. (2), (4), (7), (44), and (48) the relations

$$(a^{(-)}-V_{np}-V_n^{\dagger})\Omega_n^{(-)}\phi_f=-i\epsilon\phi_f, \quad (50a)$$

$$(a^{(+)}-V_n)\phi_i=i\epsilon\phi_i, \quad (50b)$$

$$(a^{(+)}-V_n-\mathcal{V}_{Cp})\Omega_{Cp}^{(+)}\phi_i=i\epsilon\phi_i. \quad (50c)$$

In Eqs. (50), we have used $E_f=E_i$. Using (50a) and (50b) in the first term of Eq. (49), we obtain

$$(\Phi_n^{(-)}|V_{np}|\phi_i)=i\epsilon(\Phi_n^{(-)}|\phi_i)-i\epsilon(\phi_f|\phi_i). \quad (51)$$

Likewise, using (50c) and (50a) in the second term of Eq. (49), we obtain

$$\begin{aligned}(\Phi_n^{(-)}|\mathcal{V}_{Cp}|\Omega_{Cp}^{(+)}\phi_i) &= (\Phi_n^{(-)}|V_{np}|\Omega_{Cp}^{(+)}\phi_i)+i\epsilon(\phi_f|\Omega_{Cp}^{(+)}\phi_i)\\ &\quad -i\epsilon(\Phi_n^{(-)}|\phi_i). \quad (52)\end{aligned}$$

Substituting Eqs. (51) and (52) into (49), we find

$$T_{fi}=(\Omega_n^{(-)}\phi_f|V_{np}|\Omega_{Cp}^{(+)}\phi_i) +i\epsilon[(\phi_f|\Omega_{Cp}^{(+)}\phi_i)-(\phi_f|\phi_i)],$$

or

$$T_{fi}=(\Omega_n^{(-)}\phi_f|V_{np}|\Omega_{Cp}^{(+)}\phi_i), \quad (53)$$

since the term involving the factor $i\epsilon$ is asymptotically zero for final-state deuterons.[18]

Our second elastic-scattering approximation is to replace $\Phi_n^{(-)}$, defined by Eq. (44), by $\Phi_{Cn}^{(-)}$, where

$$\Phi_{Cn}^{(-)}=\Omega_{Cn}^{(-)}\phi_f, \quad (54)$$

describes a wave of deuterons in which the neutron interacts elastically with the core nucleons, and where

$$\Omega_{Cn}^{(-)}=1+\frac{1}{a^{(-)}-V_{np}-\mathcal{V}_{Cn}^{\dagger}}\mathcal{V}_{Cn}^{\dagger}. \quad (55)$$

Here $\Omega_{Cn}^{(-)}$ and $\mathcal{V}_{Cn}$ are optical-model operators similar to $\Omega_{Cp}^{(+)}$ and $\mathcal{V}_{Cp}$ of Eq. (48). We finally obtain the transition matrix,

$$T_{fi}=(\Omega_{Cn}^{(-)}\phi_f|V_{np}|\Omega_{Cp}^{(+)}\phi_i), \quad (56)$$

which is the same as Eq. (27) of Francis and Watson.[7]