# Interaction of Charged Particles with a Degenerate Fermi-Dirac Electron Gas

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Expressions for the self-energy and interaction probability of charged particles in a degenerate Fermi-Dirac electron gas are given, generalizing from the dielectric theory of Lindhard and Hubbard and treating interactions between electrons in the gas by first order perturbation theory. Numerical results for the interaction probability in a particular case are presented. A more general Feynman diagrammatic analysis of interaction in the gas is carried out, yielding results in agreement with the more elementary approach.

## 1. INTRODUCTION

To treat the complex many-body problem of the interaction of charged particles with a degenerate electron gas, Lindhard[1] and Hubbard[2] have developed independently a quantum theory of the dielectric constant of such a system. In their treatments, the electric field is assumed to be classically prescribed, although the electronic motion in the gas is treated by quantum perturbation theory. This paper is concerned with a generalization of the work of these authors in which a prescribed field is unnecessary and the interaction of charged particles with the electron gas is made consistent in first order perturbation theory.[3]

The derivation of the expression for the interaction probability and the self-energy of the charged particle interacting with the plasma proceeds in the following way: First the Hamiltonian of the electron system and the impinging charged particle will be written and the Hartree equation for the $i$th electron will be found, assuming a simple product wave function for the system. Then, following Lindhard,[1] the development of the wave function of the $i$th electron under the influence of a self-consistent scalar potential $\phi(\mathbf{r},t)$ will be calculated by first order time-dependent perturbation theory. Next, the total charge density arising from the time development of the wave functions of the system electrons under the influence of the potential $\phi(\mathbf{r},t)$ is written as a source of the same field by means of Poisson's equation. In Sec. 2 the quantum dielectric constant of the electron gas is defined and derived in direct analogy with classical ideas.

In Sec. 3, expressions for the interaction probability and self-energy of an incident charged particle are given, neglecting spin effects and the antisymmetry of the system. The scattering cross section obtained reduces to the standard Born approximation for the scattering of charged particles on free electrons in the limit of large momentum transfers, but does not contain exchange terms. The cross section reduces to that obtained by a semiclassical treatment[2] of the interaction of charged particles with plasma when momentum transfers are small.

The procedure which will be used in deriving the interaction probability and self-energy of the incident particle in the electron gas is based on some arguments of Feynman in quantum electrodynamics. In the Appendix the dielectric constant is obtained by summing over certain Feynman diagrams, using the $S$-matrix formulation of Hubbard.

## 2. QUANTUM DIELECTRIC CONSTANT

In the following derivation it will be assumed that all particle velocities are small compared with the velocity of light so that radiative effects may be neglected.

The wave equation of the system of electrons and incident particle may be written

$$\left\{-\frac{\hbar^2}{2m}\sum_i \nabla_i^2-\frac{\hbar^2}{2m}\nabla_R^2+e^2\sum_{i>j}\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}-Ze^2\sum_i\frac{1}{|\mathbf{R}-\mathbf{r}_i|}\right\}\Psi=i\hbar\frac{\partial}{\partial t}\Psi, \quad (2.1)$$

where $\mathbf{r}_i$, $m$ and $\mathbf{R}$, $M$ are, respectively, the position vectors and mass of the $i$th plasma electron and the incident particle of charge $Ze$. It is assumed here that the total electric charge of the electron gas is just neutralized by a uniform positive charge background. If one takes a product wave function for the system, $\Psi=\psi(\mathbf{R},t)\prod_i\psi_i(\mathbf{r}_i,t)$ one may write the Hartree equations for the one-particle wave functions in the usual way:

$$\left\{-\frac{\hbar^2}{2m}\nabla_i^2+e^2\sum_{i\neq j}\int d\mathbf{r}_j\frac{|\psi_j|^2}{|\mathbf{r}_i-\mathbf{r}_j|}-Ze^2\int d\mathbf{R}\frac{|\psi|^2}{|\mathbf{R}-\mathbf{r}_i|}\right\}\psi_i=i\hbar\frac{\partial}{\partial t}\psi_i, \quad (2.2)$$

which may be written

$$\left\{-\frac{\hbar^2}{2m}\nabla_i^2-e\phi_i(\mathbf{r},t)\right\}\psi_i(\mathbf{r},t)=i\hbar\frac{\partial}{\partial t}\psi_i(\mathbf{r},t), \quad (2.3)$$

* Operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

[1] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 28, No. 8 (1954).

[2] J. Hubbard, Proc. Phys. Soc. (London) A68, 978 (1955).

[3] R. H. Ritchie, Phys. Rev. 106, 874 (1957). See note added in proof.

where

$$\nabla^2\phi_i(\mathbf{r},t)=4\pi e\sum_{j\neq i}|\psi_j|^2-4\pi Ze|\psi|^2. \qquad (2.4)$$

In these equations, $\phi_i(\mathbf{r},t)$ is the self-consistent potential acting upon the $i$th electron and is expressed as that solution of Poisson's equation which arises from the charge densities of all particles in the medium excepting that of the $i$th electron itself. This is not a satisfactory formulation, since one desires to deal with a single potential in which all particles move and to which they give rise. One may obtain such a field by including the effect of the $i$th electron on itself. This approximation is justified if one is dealing with a system such as the conduction electrons in metals in which the electronic density is $\sim 10^{23}$ electrons per $\text{cm}^3$. Clearly the addition of one electron to such a system should not change appreciably the self-consistent potential. Accordingly, we will hereafter drop the subscript on $\phi$ and let the sum in Eq. (2.4) be unrestricted. We define the Fourier expansion of an arbitrary function of space and time, $f(\mathbf{r},t)$ as

$$f(\mathbf{r},t)=\frac{1}{L^3T}\sum_{\mathbf{k}}\sum_{\omega} f_{\mathbf{k},\omega}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \qquad (2.5a)$$

where

$$f_{\mathbf{k},\omega}=\int d\mathbf{r}\int dt\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}f(\mathbf{r},t). \qquad (2.5b)$$

$f(\mathbf{r},t)$ is assumed to be defined in a large cube of side $L$ and in a time interval $T$ which will be taken large compared with any of the electronic periods to be considered. Equation (2.4) then becomes, in Fourier representation,

$$k^2\phi_{\mathbf{k},\omega}=-4\pi e\sum_j|\psi_j|_{\mathbf{k},\omega}{}^2+4\pi Ze|\psi|_{\mathbf{k},\omega}{}^2. \qquad (2.6)$$

In direct analogy with classical ideas we assume that the polarization charge density $-e\sum_j|\psi_j|_{\mathbf{k},\omega}{}^2$, is proportional to the perturbing electric field potential $\phi$, so that one may define a dielectric constant $\epsilon_{\mathbf{k},\omega}$ which is the solution of

$$\epsilon_{\mathbf{k},\omega}k^2\phi_{\mathbf{k},\omega}=4\pi Ze|\psi|_{\mathbf{k},\omega}{}^2. \qquad (2.7)$$

Then from Eqs. (2.6) and (2.7)

$$(\epsilon_{\mathbf{k},\omega}-1)k^2\phi_{\mathbf{k},\omega}=4\pi e\sum_j|\psi_j|_{\mathbf{k},\omega}{}^2. \qquad (2.8)$$

If one expands the solutions of the Hartree one-particle Eqs. (2.3) in terms of box eigenfunctions, $\epsilon_{\mathbf{k},\omega}$ has an analytical representation to order $e^2$.[4] The result, which was first found by Lindhard,[1] is

$$\epsilon_{k,\omega}=1-\frac{m\omega_p{}^2}{\hbar k^2}\sum_i f(E_i)\left\{\left[\frac{1}{\omega(\mathbf{k}_i)-\omega(\mathbf{k}_i-\mathbf{k})-(\omega+i\sigma)}+\frac{1}{\omega(\mathbf{k}_i)-\omega(\mathbf{k}_i+\mathbf{k})+(\omega+i\sigma)}\right]\right\}, \qquad (2.9)$$

where $\sigma$ is a small positive constant, $f(E_i)$ is the normalized density of states with energy $E_i$ in the undisturbed electron gas and $\omega_p=[4\pi ne^2/m]^{\frac{1}{2}}$ is the "plasma frequency" in a free electron gas of density $n$. The free electron relation between energy and momentum is assumed, i.e., $\omega(\mathbf{k})=\hbar\mathbf{k}^2/2m$. The sum over initial states $i$ is unrestricted, since the operation of the exclusion principle in final states is automatically included; this may be seen by noticing that the term in Eq. (2.9) corresponding to a transition like $\mathbf{k}_i\rightarrow\mathbf{k}_f=\mathbf{k}_i+\mathbf{k}$ is exactly cancelled by a transition in the opposite direction (involving momentum transfer $-\mathbf{k}$) if the state $\mathbf{k}_f$ lies inside the Fermi sphere.

This dielectric constant has the very interesting property that it may be employed for any value of the wave vector $\mathbf{k}$ and is capable of describing both the collective properties of the electron gas and the properties of individual electrons in the gas.

We may write for the analytic representation of $\epsilon_{k,\omega}$ for a degenerate Fermi-Dirac gas, employing variables similar to Lindhard's,

$$\epsilon_{k,\omega}=1+\chi^2f(x,z)/z^2, \qquad (2.10)$$

where

$$f(x,z)=\frac{1}{2}+\frac{1}{8z}\{1-(z-x/4z)^2\}\ln\left(\frac{z-x/4z+1}{z-x/4z-1}\right)+\frac{1}{8z}\{1-(z+x/4z)^2\}\ln\left(\frac{z+x/4z+1}{z-x/4z-1}\right), \qquad (2.11)$$

and $\chi^2=e^2/\pi\hbar v_F$, $z=k/2k_F$, $x=\hbar(\omega+i\sigma)/E_F$, and $k_F$, $v_F$, and $E_F$ are, respectively, the Fermi momentum, velocity and energy in the electron gas.

The assumption implicit in the perturbation solution of the Hartree equations is that the polarization field shall be small compared with the external field, i.e., that $\epsilon_{\mathbf{k},\omega}-1\ll 1$. This will be more nearly true the smaller is $\chi^2$. Since $v_F\sim n^{\frac{1}{3}}$, the accuracy of the dielectric treatment should increase with electronic density and the results will be exact at the limit $n\rightarrow\infty$. One recognizes that $\chi^2$ is proportional to the well-known parameter $e^2/\hbar v$, the smallness of which is a measure of the validity of perturbation theory in the treatment of electron-electron scatter at relative velocity $v$. Further, one might expect the results to be reasonably accurate even at low electronic density, since it is known that the Born treatment of Coulomb scattering is exact even in the low-velocity limit.

[4] It is interesting to consider cases other than those in which one takes momentum eigenfunctions for the perturbation expansion. One may easily obtain an approximate solution for $\epsilon_{\mathbf{k},\omega}$ in another case, *viz.*, a randomly-spaced assembly of atoms coupled only through the longitudinal electromagnetic field, and far enough apart so that overlap of wave functions between different atoms may be neglected. The dielectric constant found in this way has a wave vector dependence and is capable of describing both the collective motion of the assembly and the individual motion of electrons in the assembly to the order $e^2$.

## 3. PLASMA PROPAGATION FUNCTION

Let us now consider Eq. (2.7) which we write as

$$\phi_{\mathbf{k},\omega}=4\pi\rho_{\mathbf{k},\omega}/k^2\epsilon_{\mathbf{k},\omega}, \tag{3.1}$$

where $\rho_{\mathbf{k},\omega}=Ze|\psi|_{\mathbf{k},\omega}{}^2$ is the $\mathbf{k}$, $\omega$ Fourier component of the charge density of the perturbing particle. To the first order of perturbation we have eliminated the motion of individual electrons and have codified the behavior of the assembly in the single quantity $\epsilon_{\mathbf{k},\omega}$. Thus if one has given the space-time behavior of the charge density $\rho(\mathbf{r},t)$, the variation of the self-consistent electric field at any point in space and time may be written, from the properties of the Fourier integrals

$$\phi(\mathbf{r},t)=\int d\mathbf{r}'\int dt'\, \mathcal{V}(\mathbf{r}-\mathbf{r}', t-t')\rho(\mathbf{r}',t), \tag{3.2}$$

where

$$\mathcal{V}(\mathbf{r},t)=\frac{4\pi}{L^3T}\sum_{\mathbf{k}}\sum_{\omega}\frac{1}{k^2\epsilon_{\mathbf{k},\omega}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{3.3}$$

One sees that $[k^2\epsilon_{\mathbf{k},\omega}]^{-1}$ is an operator[5] which yields the potential at a point in space-time if the charge density at other space-time points is known. The reason that $\epsilon$ must be regarded as an operator resides in the fact that polarization in the medium does not follow instantaneously the variation of a perturbing charge, but requires a certain time to establish itself, e.g., if a charge is created suddenly in the free electron gas, oscillations are engendered which persist indefinitely in the absence of damping. Since the collective motion of the plasma system has been codified in the form of the function $\epsilon$, one may regard equation (3.1) as the basic potential variation in the free electron gas which replaces Coulomb's law.

Then, in direct analogy with field theory methods, we may define $\mathcal{V}(2,1)\equiv\mathcal{V}(\mathbf{r}_2-\mathbf{r}_1, t_2-t_1)$, which is the potential at the point $\mathbf{r}_2$, $t_2$ due to an instantaneous unit charge appearing at $\mathbf{r}_1$, $t_1$, as the "plasma propagation function" which replaces the instantaneous Coulomb interaction in the free electron gas. When $\epsilon\rightarrow 1$, $\mathcal{V}(2,1)$ reduces to $\delta(t_2-t_1)/|\mathbf{r}_2-\mathbf{r}_1|$. One may easily show that $\mathcal{V}(2,1)=0$ when $t_2<t_1$ since the zeros and branch points of the $\omega$ integral lie below the real $\omega$ axis.

## 4. SELF-ENERGY AND TRANSITION RATE OF AN INCIDENT CHARGED PARTICLE IN THE FREE ELECTRON GAS

One may now consider the calculation of the self-energy of an incident charged particle due to its interaction with the free electron gas by the methods developed by Feynman[6] in quantum electrodynamics. The imaginary part of the self-energy will be proportional to the real transition rate of the incident particle from its original state. It should be noted that one is not employing rigorous quantum field theory in this case, since one does not quantize the "plasma" field itself. However, the use of quantized longitudinal fields in plasma problems by Bohm and Pines[7] is limited by certain of the mathematical techniques employed, since separate treatments are necessary for collective interactions and for individual interactions.[8] The present approach is not limited in this way, since both collective and individual effects are contained in the dielectric description of plasma. The use of the arguments of Feynman in this connection is analogous to the use of transition charge densities in the treatment of radiation problems.[9]

Consider the interaction of the incident electron with itself via the plasma. Figure 1 shows a Feynman graph of this interaction to order $e^2$. We can consider that the incident electron is nonrelativistic and that the zero order amplitude $K^{(0)}(2,1)$ for an electron to propagate from $\mathbf{r}_1$, $t_1$ to $\mathbf{r}_2$, $t_2$ is given by

$$\begin{aligned}K^{(0)}(2,1)&=\frac{1}{L^3}\sum_{\mathbf{k}}\exp\left(i\left[\mathbf{k}\cdot\mathbf{r}_{21}-\frac{\hbar k^2}{2m}t_{12}\right]\right), && (t_{21}>0)\\ &=0, && (t_{21}<0)\end{aligned} \tag{4.1}$$

where $\mathbf{r}_{21}=\mathbf{r}_2-\mathbf{r}_1$ and $t_{21}=t_2-t_1$. We could, following Feynman, define a propagation function which would be nonzero for $t_{21}<0$ in order to take into account the possibility of exchange processes in the free electron gas, i.e.,

$$K_+^{(0)}(2,1)=\frac{i}{L^3T}\sum_{\mathbf{k}}\sum_{\omega}\frac{\exp(i[\mathbf{k}\cdot\mathbf{r}_{22}-\omega t_{21}])}{\omega-\hbar k^2/2m+i\epsilon\,\mathrm{Sgn}(k^2-k_F{}^2)}, \tag{4.2}$$

where $\epsilon$ is a small positive constant and

$$\mathrm{Sgn}x=x/|x|.$$

The function $\mathrm{Sgn}(k^2-k_F{}^2)$ determines how the contour integration in $\omega$ is to be carried out. This propagator has the property that for $t_{21}>0$ only states outside the Fermi sphere can exist, while for $t_{21}<0$ only states for which $|\mathbf{k}|<k_F$ are permitted. Again in analogy with the case of the positron considered by Feynman, we may regard $K_+^0(2,1)$ for $t_{21}<0$ as the propagation amplitude of holes in the Fermi sea, or, alternatively, as the amplitude for the propagation of electrons backward in time. We will not consider this propagator further in this section. Since exchange effects in the interaction between electrons in the medium have been

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[5] U. Fano has discussed the operator properties of $\epsilon$ [Phys. Rev. **103**, 1202 (1956)].

[6] R. P. Feynman, Phys. Rev. **76**, 749 (1949); **76**, 769 (1949).

[7] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953); D. Pines, Phys. Rev. **92**, 626 (1953).

[8] However, see P. Nozières and D. Pines, Phys. Rev. **113**, 1268 (1959).

[9] See, e.g., Schweber, Bethe, and de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, New York, 1956), Vol. 1, p. 76.

neglected, exchange between the incident electron and the Fermi gas[10] will also be neglected.

The amplitude for the incident electron to propagate from $(\mathbf{r}_1,t_1)$ to $(\mathbf{r}_2,t_2)$ differs from the free propagation amplitude $K^{(0)}(2,1)$ by the first order correction term

$$K^{(1)}(2,1)=-i\frac{e^2}{\hbar}\int d\mathbf{r}_3dt_3d\mathbf{r}_4dt_4 \times K^{(0)}(2,4)K^0(4,3)\mathcal{V}(4,3)K^{(0)}(3,1). \quad (4.3)$$

This correction term describes free particle propagation from 1 to 3, emission of a "plasmon"[11] of momentum $\mathbf{k}$ at 3, propagation to 4 where absorption of the "plasmon" occurs and finally free propagation to 2. This first order correction to the amplitude is to be interpreted in view of the fact that one expects the amplitude for arrival at 2 to be altered due to the emission and absorption of the virtual "plasmon." If the energy shift due to this process is $\Delta E$, then the amplitude of the initial state is changed by a factor $\exp(-i\Delta Et_{21}/\hbar)$, or to first order by the difference $-i(\Delta ET/\hbar)$ where we take $t_2-t_1$ to be the large time interval $T$. Then we set

$$\Delta E=\frac{i\hbar e^2}{T}\int\psi_0^*(2)K^{(1)}(2,1)\psi_0(1)d\mathbf{r}_1dt_1d\mathbf{r}_2dt_2, \quad (4.4)$$

where $\psi_0(1)$ is the time-dependent wave function of the incident particle in its initial state. We may also put

$$\Delta E=(\Delta E)_r-i\hbar\Gamma/2, \quad (4.5)$$

where $\Gamma$ is the transition rate from the initial state and $(\Delta E)_r$ is the real energy shift due to the interaction.

Employing the expression above for $K^{(1)}(2,1)$ and taking $\psi_0(r,t)=(1/L^{\frac{3}{2}})\exp\{i(\mathbf{k}_0\cdot\mathbf{r}-\omega_0t)$, we find[12]

$$(\Delta E)_r=\frac{4\pi e^2}{L^3}\sum_{k_f}\frac{1}{k^2}\,\mathrm{Re}\left(\frac{1}{\epsilon_{\mathbf{k},\omega}}-1\right), \quad (4.6)$$

$$\Gamma=-\frac{8\pi e^2}{\hbar L^3}\sum_{k_f}\frac{1}{k^2}\,\mathrm{Im}\left(\frac{1}{\epsilon_{\mathbf{k},\omega}}\right), \quad (4.7)$$

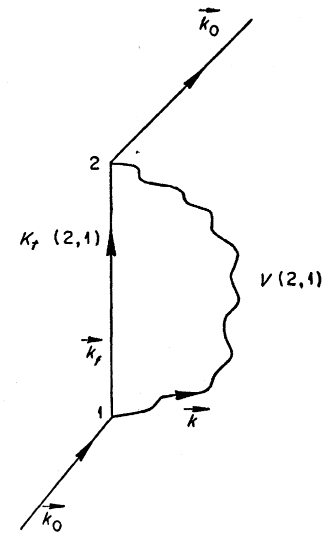


FIG. 1. Feynman graph illustrating the interaction of a charged particle with itself via the plasma field.

where $\omega=\omega_0-\omega_f$, $\mathbf{k}=\mathbf{k}_0-\mathbf{k}_f$, $\omega_f=\hbar k_f^2/2m$, $\omega_0=\hbar k_0^2/2m$, and $\mathbf{k}_f$ is the wave vector of the electron in its intermediate state. The vacuum self-energy of the electron has been subtracted in Eq. (4.6).[12] Note that the expression for $(\Delta E)_r$ does not account for the contribution of holes to the self-energy. To include this contribution, one must employ the electron propagation function (4.2) together with a plasma propagation function which is symmetrical with respect to time (see the Appendix). One finds in this case that $(\Delta E)_r$ must be left in the form of an integral over $\omega$ but that the expression for $\Gamma$ (Eq. 4.7) is unchanged, since it corresponds to real transitions only.

From Eq. (4.7) one may obtain the cross section, or alternately, $\tau$, the interaction probability per unit path length in the free electron gas. Letting the sum in (4.7) go to an integral, we obtain

$$\sum_{\mathbf{k}_f}\rightarrow 2\left(\frac{L}{2\pi}\right)^3\int d\Omega\, k_f^2dk_f=\frac{2m}{\hbar}\left(\frac{L}{2\pi}\right)^3\int k_fd\Omega_fd\omega_f, \quad (4.8)$$

where $d\Omega_f$ is the element of solid angle about the direction of the wave vector $\mathbf{k}_f$. Dividing by the incident flux and by the volume of the medium, we find

$$\tau(\theta,\omega_f)d\Omega_fd\omega_f=\frac{m^2e^2}{\hbar^3\pi^2}\frac{k_f}{k_0k^2}\frac{\mathrm{Im}\epsilon_{\mathbf{k},\omega}}{|\epsilon_{\mathbf{k},\omega}|^2}d\Omega_fd\omega_f, \quad (4.9)$$

where $\theta$ is the angle between $\mathbf{k}_0$ and $\mathbf{k}_f$.

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[10] There is an additional Feynman diagram, which is "of order $e^2$" in the present sense, which contributes to the self-energy. This is an unconnected diagram in which the incident electron remains in its original state while an electron-hole pair is created spontaneously together with a plasmon and subsequently annihilates itself. This corresponds to vacuum fluctuations in electrodynamics. Such processes do not contribute to observable effects as far as the incident particle is concerned.

[11] The term plasmon has been applied by Bohm and Pines to the quantized oscillations of plasma. It is employed in a different sense here to emphasize the similarity with the self-energy calculation in quantum electrodynamics. In the present case the "plasmon" includes the effect of virtual single-electron excitations as well as bound coherent states.

[12] An expression equivalent to Eq. (4.4) for the self-energy was obtained by Lindhard [reference 1, p. 55, Eq. (A-12)] using a different method. It is felt that the derivation given above, using Feynman's approach, is simpler and more transparent. Lindhard did not apply the self-energy expression to the calculation of transition probabilities. After the major portion of the present work had been completed, it was found that an independent formulation of similar ideas using Feynman's techniques had been carried out by Ferrell and co-workers. [J. J. Quinn and R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, 3, 202 (1958), and A. J. Glick and R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, 3, 191 (1958).] However, it is felt that the differences in approach are sufficient to warrant presentation of the present paper. Quinn and Ferrell have employed the equivalent of Eq. (4.4), together with a theorem by Seitz, to calculate the ground-state energy of the free electron gas. Hence we will not consider this question in the present paper.

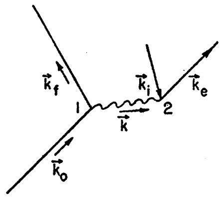


FIG. 2. Interaction of an incident charged particle with the free electron gas.

One may obtain the expression for $\tau$ by a more straightforward approach: The incident particle may be assumed to interact with electrons in the Fermi sea according to the modified potential $\mathcal{V}(2,1)$ above and then the matrix element $M$ for the process indicated in the Feynman graph of Fig. 2 may be obtained immediately. One finds

$$M=\frac{4\pi e^2}{i\hbar L^3 T}\sum_{\mathbf{k}}\sum_{\omega}\frac{1}{k^2\epsilon_{\mathbf{k},\omega}}\times\frac{\delta(\mathbf{k}_f+\mathbf{k}-\mathbf{k}_0)\delta(\mathbf{k}+\mathbf{k}_i-\mathbf{k}_e)}{(\omega+\omega_f-\omega_0-i\sigma)(\omega-\omega_s+\omega_i+i\sigma)}, \quad (4.10)$$

where $\sigma$ is a convergence term which is allowed to approach zero after all integrations are performed. The transition probability per unit time is found by dividing the $M^2$ by the time $T$. The interaction probability per unit path length of the incident particle is found to be

$$\tau=\frac{2}{v_0}\left(\frac{4\pi e^2}{\hbar L^3}\right)^2\sum_{\mathbf{k}_e}\sum_{\mathbf{k}_i}\sum_{\mathbf{k}_f}\frac{1}{k^4|\epsilon_{k,\omega}|^2}\delta(\mathbf{k}_e-\mathbf{k}_i+\mathbf{k}_f-\mathbf{k}_0)\pi\delta(\omega_0-\omega_f-\omega_e+\omega_i), \quad (4.11)$$

where we set $\mathbf{k}=\mathbf{k}_e-\mathbf{k}_i$, $\omega=\omega_e-\omega_i$ in the summand. Note that if we carry out the sum of $\mathbf{k}_e$ first, and then sum on $\mathbf{k}_i$, holding $\mathbf{k}=\mathbf{k}_0-\mathbf{k}_f$ constant we obtain just the result found by the self-energy method, i.e., Eq. (4.9), for the interaction probability of the incident particle with the free electron gas. This is easily seen by noting that

$$\frac{4\pi e^2}{\hbar k^2 L^3}\sum_{\mathbf{k}_i}\pi\delta(\omega-\omega_e+\omega_i)\delta(\mathbf{k}+\mathbf{k}_i-\mathbf{k}_e)=\mathrm{Im}\epsilon_{k,\omega}, \quad (4.12)$$

where the sum over $\mathbf{k}_i$ is restricted to the region $|\mathbf{k}_i+\mathbf{k}|<k_F$, and we need consider only positive energy transfers to the free electron gas.

To obtain a more detailed expression for $\tau_s$ the probability for creation of a secondary electron of wave vector $\mathbf{k}_e$, first carry out the sum over $\mathbf{k}_f$ and let

$$\sum_{\mathbf{k}_f}=\frac{2m}{\hbar}\left(\frac{L}{2\pi}\right)^3\int k_e d\omega_e d\Omega_e, \quad (4.13)$$

where $\hbar\omega_e$ is the energy of the secondary electron and $d\Omega_e$ is the element of solid angle about its direction of emission. Then

$$\tau_s d\omega_e d\Omega_e=\frac{me^4k_e}{\pi^3 v_0\hbar^4}\int d\mathbf{k}_i\frac{\delta(\omega_0-\omega_f-\omega_e+\omega_i)}{k^4|\epsilon_{k,\omega}|^2}d\Omega_e d\omega_e, \quad (4.14)$$

where $\mathbf{k}=\mathbf{k}_e-\mathbf{k}_i=\mathbf{k}_0-\mathbf{k}_f$ and $\omega=\omega_e-\omega_i$.

The integration is carried out over the volume of the Fermi sphere, and $\mathbf{k}_e$ must lie outside of this region.

## 5. THE INTERACTION PROBABILITY IN THE LIMIT $n\to 0$

To aid in understanding the interaction cross section of an incident electron with a realistic Fermi gas, it will be helpful to consider a fictitious assembly of stationary electrons. This corresponds to the limit $n\to 0$ in the dielectric constant of Eq. (2.9). One finds

$$\epsilon\to 1+\omega_p^2\Big/\left\{\frac{\hbar^2k^4}{4m^2}-(\omega+i\sigma)^2\right\}, \quad (5.1)$$

and from Eq. (4.8)

$$\tau(\theta,\omega)=\frac{m^2e^2}{\pi^2\hbar^3}\frac{k_f\omega_p^2}{k_0k^2}\left\{\frac{2\sigma\omega}{(\omega^2-\omega_p^2-\hbar^2k^4/4m^2)^2+4\sigma^2\omega^2}\right\}. \quad (5.2)$$

Since we have assumed $\sigma\to 0$, the resonance term in the curly brackets has negligible value except when $\omega^2=\omega_p^2+\hbar^2k^4/4m^2$. Thus a first order perturbation theory of interparticle action in the electron gas has resulted in a prediction of resonance which agrees exactly with that of classical theory when momentum transfer from the incident particle is negligible. We will see immediately that when the momentum transfer is large the Born cross section is found.

To find the connection between energy loss and scattering angle (see Fig. 3), one must solve the equations

$$\mathbf{k}=\mathbf{k}_0-\mathbf{k}_f,\quad \omega=\hbar(k_0^2-k_f^2)/2m=[\omega_p^2+\hbar^2k^4/4m^2]^{\frac{1}{2}}.$$

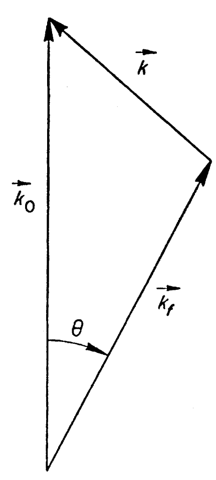


FIG. 3. Momentum diagram of scattering process showing relation between initial and final wave vectors of incident particle.

We shall not deal with these equations directly, but rather some limiting cases will be considered.

(a) $\omega_p \gg \hbar k^2/2m$

In this case

$$\omega \simeq \omega_p = \frac{\hbar}{m}(\mathbf{k}_0\cdot\mathbf{k} - k^2/2).$$

If we integrate Eq. (5.2) over $\omega$, we find

$$\tau(\theta)d\Omega = \frac{me^2\omega_p{}^2 k_f}{2\pi\hbar^2 v_0 k^2}\frac{d\Omega}{[\omega_p{}^2+\hbar^2k^4/4m^2]^{\frac{1}{2}}}, \tag{5.3}$$

which, in the case considered, reduces to

$$\tau(\theta)d\Omega = \frac{e^2\omega_p{}^2}{2\pi\hbar v_0{}^2}\frac{d\Omega}{\theta^2+(\hbar\omega_p/2E_0)^2}, \tag{5.4}$$

where $v_0 = \hbar k_0/m$, $E_0 = mv_0{}^2/2$. This agrees with the result obtained classically.[2]

(b) $\omega_p \ll \hbar k^2/2m$

In this case one finds the simple result

$$k_0{}^2 - k_f{}^2 = (\mathbf{k}_0 - \mathbf{k}_f)^2,$$

or

$$k_f = k_0 \cos\theta, \quad k = k_0 \sin\theta,$$

and from Eq. (5.3)

$$\tau(\theta)d\theta = \frac{2\pi ne^4}{E_0{}^2}\frac{\cos\theta d\theta}{\sin^3\theta}, \tag{5.5}$$

where we have set $d\Omega = 2\pi \sin\theta d\theta$. This is just the Born approximation for the scattering probability of an electron on an assembly of electrons with which it interacts individually. This scattering formula is expressed in the laboratory system of coordinates, which is the natural system to which the electron gas is referred.

It is also of interest to integrate over $d\Omega$ first in order to find the distribution of energy losses of the incident particle. Then

$$\tau(\omega)d\omega = \frac{e^2\omega_p{}^2}{2\hbar v_0{}^2}\frac{d\omega}{\omega^2-\omega_p{}^2}, \tag{5.6}$$

if there is at least one root of

$$1-[1-\hbar\omega/E_0]^{\frac{1}{2}} - [4\hbar^2(\omega^2-\omega_p{}^2)/E_0{}^2]^{\frac{1}{4}} = 0, \tag{5.7}$$

and $\tau = 0$, otherwise. This shows again that for energy losses large compared with $\hbar\omega_p$ the cross section behaves like the free electron value. The rise in cross section as the energy loss decreases toward the plasma resonance value $\hbar\omega_p$ is clearly seen. One may show from Eq. (5.7) that $\tau$ is always finite.

The energy loss of the incident electron per unit path length, $-dw/dx$, is given by

$$-dw/dx = \int \hbar\omega\tau(\theta,\omega)d\Omega d\omega = \frac{4\pi ne^4}{mv_0{}^2}\ln\left(\frac{k_{\max}}{k_{\min}}\right), \tag{5.8}$$

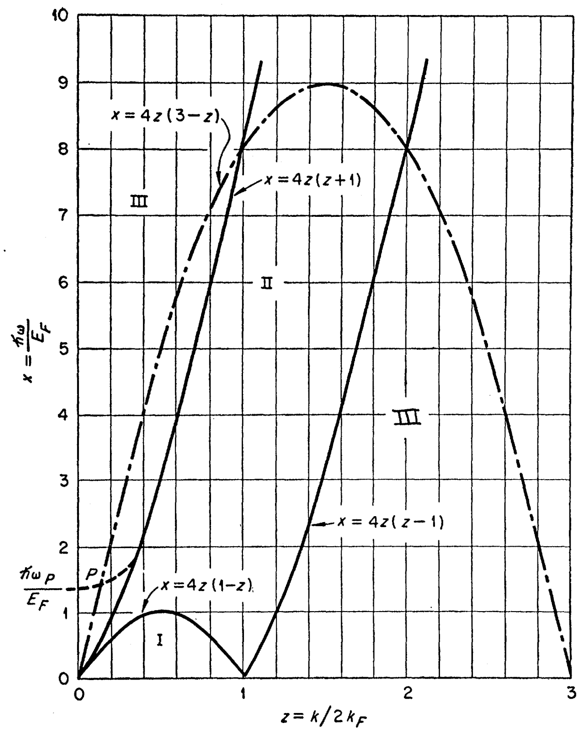


FIG. 4. Regions of the $x$-$z$ plane in which different representations of $f_2(x,z)$ must be used.

where $k_{\max}$ and $k_{\min}$ are the roots of

$$k^3 - k_0k^2 + m^2\omega_p{}^2/\hbar^2k_0 = 0, \tag{5.9}$$

which lie in the range $0 \le k \le k_0$. When $\hbar\omega_p \ll E_0$, we have approximately

$$-dw/dx = \frac{4\pi ne^4}{mv_0{}^2}\ln(mv_0{}^2/\hbar\omega_p). \tag{5.10}$$

Equation (5.9) shows that when $E_0 < (27)^{\frac{1}{2}}\hbar\omega_p/4$, the interaction probability of the incident electron in the zero-density gas vanishes.

The Debye shielding of electrical charges, an important characteristic of the free electron gas, is not present in the limit $n=0$.

## 6. THE INTERACTION PROBABILITY, $n \neq 0$

In this case one must employ the full expressions for both the real and imaginary parts of $\epsilon_{\mathbf{k},\omega}$. Following Lindhard, we write in the limit $\sigma \to 0$

$$\epsilon_{k,\omega} = 1 + (\chi^2/z^2)[f_1(x,z) + if_2(x,z)], \tag{6.1}$$

where again $\chi^2 = e^2/\pi\hbar v_F$, $z = k/2k_F$, and $x = \hbar\omega/E_F$, and

$$f_1(x,z) = \frac{1}{2} + \frac{1}{8z}\{1-(z-x/4z)^2\}\ln\left|\frac{z-x/4z+1}{z-x/4z-1}\right| + \frac{1}{8z}\{1-(z+x/4z)^2\}\ln\left|\frac{z+x/4z+1}{z+x/4z-1}\right|. \tag{6.2}$$

Figure 4 gives a plot of the $x$-$z$ plane showing the boundaries of regions I, II, and III in which different analytical representations of $f_2(x,z)$ must be employed.

Region I is bounded by the lines $x=0$, and $x=4z(1-z)$. In this region

$$f_2^{\mathrm{I}}=\pi x/8z. \tag{6.3a}$$

Region II is bounded by the line $x=4z(1-z)$, $4z(1+z)$, and $4z(z-1)$. In this region

$$f_2^{\mathrm{II}}=\frac{\pi}{8z}\{1-(z-x/4z)^2\}. \tag{6.3b}$$

Region III consists of two separate regions in the $x$-$z$ plane; (1) that between the lines $z=0$ and $x=4z(z+1)$; (2) that bounded by $x=0$ and $x=4z(z-1)$. In this region

$$f_2^{\mathrm{III}}=\sigma g(x,z). \tag{6.3c}$$

Since $\sigma$ is a small positive constant which is allowed to approach zero after all integrations are performed, the exact form of $g$ is not important. From Eqs. (4.8) and (6.1) we may write for the interaction probability of an incident electron

$$\tau(\theta,\omega)d\omega d\Omega=\frac{m^2e^2\chi^2k_f}{4\pi^2\hbar^3k_F{}^2k_0}\frac{f_2(x,z)}{\{(z^2+\chi^2f_1)^2+\chi^4f_2{}^2\}}d\Omega d\omega. \tag{6.4}$$

With the following relation between scattering angle, $z$ and $x$:

$$\cos\theta=(2\alpha^2-4z^2-x)/2\alpha(\alpha^2-x)^{\frac{1}{2}}, \tag{6.5}$$

where $\alpha=k_0/k_F$.

One may see the major features of the behavior of $\tau$ as a function of $x$ and $z$ by referring to Fig. 4. The dashed line represents a schematic plot of the equation

$$F(x,z)=z^2+\chi^2f_1=0. \tag{6.6}$$

The portion of this line lying in region III constitutes the resonance line which reduces to $\omega^2=\omega_p{}^2+\hbar^2k^4/4m^2$ as $n\rightarrow 0$, as discussed in Sec. 5. This is clearly seen by writing

$$\tau_{\mathrm{res}}\sim\frac{\sigma g}{F^2(x,z)+\chi^4\sigma^2g^2}. \tag{6.7}$$

If one expands Eq. (6.6) for small $z$, one finds plasma resonance occurring at

$$\omega_{\mathrm{res}}{}^2=\omega_p{}^2+\tfrac{3}{5}v_F{}^2k^2+\left[\hbar^2/4m^2+\frac{12}{175}v_F{}^4\right]k^4/\omega_p{}^2+\cdots. \tag{6.8}$$

which agrees with the corresponding classical condition for resonance except for the quantum correction term $\hbar^2k^4/4m^2$. This relation was first obtained by Bohm and Pines[13] from their quantum theory of plasma and has been studied by Ferrell.[14] The connection between $\omega$ and $k$ found experimentally by Watanabe[15] for electrons interacting in thin foils of Be, Al, Mg, and Ge has been interpreted by Pines[16] as proof of the plasma-like character of energy losses in these substances.

In region I the response of the electron gas to the incident electron is primarily in producing Debye shielding of the external charge. In region II the response takes on both screening and oscillatory characteristics. For large momentum transfers, i.e., for $\hbar k^2/2m\gg\omega_p$, the cross section reduces to the Born approximation as in Eq. (5.5).

We shall now examine some limiting cases to illustrate the behavior of $\tau$ for different incident charged particles.

### (a) Low-Energy Positron in the Fermi Gas

The Eqs. (6.2), (6.3), and (6.4) completely determine (to the order $e^2$) the interaction cross section for intruder positrons in plasma. The explicit relation between $\theta$, the angular deflection of the electron, and $\mathbf{k}$, its momentum change, is found from the energy-momentum relations and is given by Eq. (6.5).

We shall, however, concern ourselves with the total interaction probability and energy loss per unit length for a slow positron, $v_0\ll v_F$, in plasma. We note that its energy loss may be written

$$\hbar\omega=\frac{\hbar^2}{m}(k_0k\nu-k^2/2), \tag{6.9}$$

where $\nu$ is the cosine of the angle between $\mathbf{k}_0$ and $\mathbf{k}$ (see Fig. 2).

In terms of the reduced variables $x$ and $z$, we have

$$x=4\{\alpha z\nu-z^2\}, \tag{6.10}$$

and since $0\leq\nu\leq 1$, only the area in the $x$-$z$ plane enclosed between the lines

$$x=4(\alpha z-z^2), \tag{6.11}$$

and $x=0$ is to be included. The curve of Eq. (6.11) is plotted in Fig. 4 as a dot-dash line for the special case $\alpha=k_0/k_F=3$. When $\alpha\ll 1$, we may use the approximations

$$f_2\simeq\pi x/8z,\quad f_1\sim 1,$$

and setting

$$d\Omega=8\pi k_F{}^2zdz/k_0k_f,\quad d\omega=E_Fdx/\hbar,$$

and integrating under the parabola of Eq. (6.11), we find

$$\tau=\int\tau(\theta,\omega)d\Omega d\omega=\frac{\pi m}{30\hbar}\frac{v_0{}^3}{v_F{}^2}. \tag{6.12}$$

The energy loss per unit path length, $-dw/dx$, is found to be

$$-\frac{dw}{dx}=\int\hbar\omega\tau(\theta,\omega)d\omega d\Omega=\frac{\pi}{105}\frac{m^2}{\hbar}\frac{v_0{}^5}{v_F{}^2}. \tag{6.13}$$

[13] D. Bohm and D. Pines, Phys. Rev. **92**, 608 (1953).

[14] R. A. Ferrell, Phys. Rev. **107**, 450 (1957).

[15] H. Watanabe, J. Phys. Soc. Japan **11**, 112 (1956).

[16] D. Pines, Revs. Modern Phys. **28**, 184 (1956). See also D. Pines, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1955), Vol. 1.

This may be compared with the result obtained by Lee-Whiting[17] for the slowing down rate of positrons in the free electron gas. He assumed a screened Coulomb interaction of the form $-e^2e^{-qr}/r$ between the positron and the individual electrons in the gas and employed perturbation theory to treat the scattering process, taking explicit account of the fact that a struck plasma electron must receive energy sufficient to lift it into the unoccupied region in momentum space. If one uses his expression for the slowing down rate divided by the positron velocity in the same limit ($v_0 \ll v_F$) and takes $q=(3\omega_p{}^2/v_F{}^2)^{\frac{1}{2}}$, the Debye wave vector, one finds that the resulting formula agrees exactly with Eq. (6.13). The dielectric formulation automatically accounts for the operation of the exclusion principle and yields the Debye shielding result in the low-energy limit. At energies comparable with $E_F$, the dielectric formulation should be considerably more accurate than that of Lee-Whiting, since it takes into account the dynamical properties of the plasma.

### (b) Low-Energy Heavy Charged Particle in the Fermi Gas

If the incident particle has charge $Ze$, velocity $v_0$, and mass $M \gg m$, then the energy-momentum condition, Eq. (6.11), becomes

$$x=4(v_0 z/v_F - mz^2/M), \qquad (6.14)$$

showing immediately that one may neglect altogether the particle recoil for quite small velicities of the incident particle; $1 \gg v_0/v_F \gg m/M$, since $m/M \sim 1/2000$. Clearly, one must multiply the expression for $\tau$, Eq. (4.9), by $Z^2$. Then using the approximations

$$f_2=\pi x/8z, \quad f_1=\frac{1}{2}+\frac{1}{42}(1-z^2)\ln\left|\frac{z+1}{z+1}\right| \simeq 1-z^2/2,$$

one finds

$$\tau=\frac{Z^2}{8a_0}\frac{\mathrm{X}^2}{1-\mathrm{X}^2/2}[\{(1-\mathrm{X}^2/2)^{\frac{1}{2}}/\mathrm{X}\}\tan^{-1}\{(1-\mathrm{X}^2/2)/\mathrm{X}^2\} - (1-\mathrm{X}^2/2)/(1+\mathrm{X}^2/2)],$$

$$-\frac{dw}{dx}=\frac{2}{3\pi}\frac{Z^2e^4m^2v_0}{\hbar^3(1-\mathrm{X}^2/2)}\{\ln[(1+\mathrm{X}^2/2)/\mathrm{X}^2] - (1-\mathrm{X}^2/2)/(1+\mathrm{X}^2/2)\}, \qquad (6.15)$$

where $a_0$ is the Bohr radius.

If $\mathrm{X}^2 \ll 1$, this formula reduces to

$$-\frac{dw}{dx}=\frac{2}{3\pi}\frac{Z^2e^4m^2v_0}{\hbar^3}\ln(\pi\hbar v_F/e^2), \qquad (6.16)$$

which is identical with the formula obtained by Fermi and Teller[18] for a slow heavy-charged particle in a Fermi-Dirac gas, assuming $\chi^2 \ll 1$.

### (c) Low-Energy Incident Electron

In this case we need to take account of the fact that the incident electron cannot make transitions to occupied states in the Fermi sea. As mentioned in Sec. 4, exchange scattering of the incident electron with the electron gas is not included here, since exchange processes between plasma electrons have been neglected. The energy loss of an incident electron of momentum $\hbar k_0$ cannot exceed the value

$$\hbar\omega_{\max}=\frac{\hbar^2}{2m}(k_0{}^2-k_F{}^2),$$

and in terms of the reduced variables

$$x_{\max}=\alpha^2-1.$$

The range of $x$, $z$ values which can be assumed by the incident electron lies between the lines

$$x=\alpha^2-1, \quad x=0,$$

and under the curve

$$x=4(\alpha z-z^2).$$

Then if $\alpha^2-1 \ll 1$ we may again put $f_1 \simeq 1-z^2/2$, $f_2=\pi x/8z$, and proceeding as before, neglecting $f_2$ in the denominator of Eq. (6.4) we obtain

$$\tau=\frac{(\alpha^2-1)^2}{32a_0\alpha^2}\left\{\frac{1}{1+\mathrm{X}^2/2} + \frac{1}{\mathrm{X}[1-\mathrm{X}^2/2]}\tan^{-1}[(1-\mathrm{X}^2/2)^{\frac{1}{2}}/\mathrm{X}]\right\}. \qquad (6.17)$$

Also

$$-\frac{dw}{dx}=\left(\frac{mev_F}{8\pi\hbar v_0}\right)^2(\alpha^2-1)^3\left\{\frac{1}{1+\mathrm{X}^2/2} + \frac{1}{\mathrm{X}[1-\mathrm{X}^2/2]^{\frac{1}{2}}}\tan^{-1}[(1-\mathrm{X}^2/2)^{\frac{1}{2}}/\mathrm{X}]\right\}. \qquad (6.18)$$

The expression for $\tau$ has a slightly different appearance than the corresponding expression obtained by Quinn and Ferrell,[19] but reduces to the same value for $\alpha \to 1$ and $x^2 \ll 1$.

### (d) Stopping Power of the Fermi Gas for a High-Energy Electron ($E_0 \gg E_F$)

An approximate formula for the high-energy stopping power of the free electron gas may be obtained by observing that, as will be shown below, the probability

[17] G. E. Lee-Whiting, Phys. Rev. **97**, 1557 (1955).

[18] E. Fermi and E. Teller, Phys. Rev. **72**, 399 (1947).

[19] J. J. Quinn and R. A. Ferrell, Phys. Rev. **112**, 812 (1958).

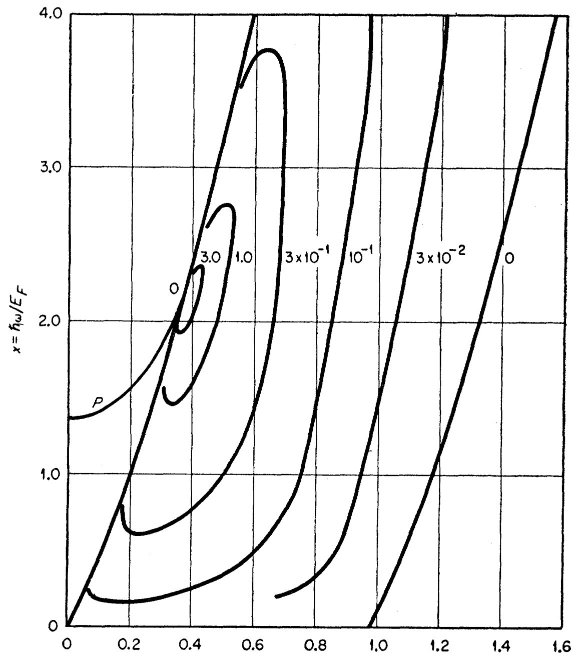


FIG. 5. Contours of equal values of $T(x,z)$ in the $x$-$z$ plane.

of energy absorption varies as the reciprocal of the momentum transfer in two regions of the $x$-$z$ plane: (1) in the neighborhood of the plasma resonance line; and (2) in the range of large energy and large momentum transfers. We designate the energy loss per unit path length in the Fermi gas in these two regions by $-dw_1/dz$, and $-dw_2/dx$, respectively.

As before, we write

$$-dw/dx=\int \hbar\omega\tau d\Omega d\omega, \tag{6.19}$$

where the region of integration lies under the parabola $x=4z(\alpha-z)$. In the neighborhood of the plasma resonance line we put

$$f_2\sim\sigma g, \quad \chi^2 f_1\simeq -z^2 x_p^2/x^2,$$

for $z\ll 1$, and integrate over the resonance at $x=x_p=\hbar\omega_p/E_F$. The $z$ integration will be taken from $z=x_p/4\alpha$, the intersection of the parabola $x=4z(\alpha-z)$ and the plasma resonance line for $\alpha\gg 1$, and an intermediate value of $z$, designated by $z_i$. We find

$$-\frac{dw_1}{dz}=\frac{e^2\omega_p^2}{v_0^2}\ln\left(\frac{4\alpha z_i}{x_p}\right). \tag{6.20}$$

For large momentum transfers we may put $x=4z(z+\delta)$, where $1-\leq\delta\leq 1$ and $f_2=\pi(1-\delta^2)/8z$, and neglect $f_1$ and $f_2$ in comparison with $z^2$ in the denominator of Eq. (6.19). The $z$ integration is carried over from the intermediate value $z_i$ to the intersection of the line $x=4z^2$ with the parabola $x=4z(\alpha-z)$. We find the contribution to the stopping power from this region to be

$$-\frac{dw_2}{dx}=\frac{e^2\omega_p^2}{v_0^2}\ln\left(\frac{\alpha}{2z_i}\right). \tag{6.21}$$

Then

$$-\frac{dw}{dx}=\frac{e^2\omega_p^2}{v_0^2}\ln\left(\frac{mv_0^2}{\hbar\omega_p}\right),$$

in which $z_i$ subtracts out in the sum. The resulting expression agrees exactly with Eq. (5.10). If the incident charged particle had been a heavy particle of charge $Ze$, the result would have been

$$-\frac{dw}{dx}=\frac{Z^2e^2\omega_p^2}{v_0^2}\ln\left(\frac{2mv_0^2}{\hbar\omega_p}\right), \tag{6.22}$$

where the difference of a factor of 2 in the argument of the logarithm is due to the fact that the upper limit in the $z$ integration is taken to be the intersection of the parabola $x=4z^2$ with the line $x=4zv_0/v_F$, since the recoil of the heavy particle may be neglected.

We return now to the more general case of an incident particle with arbitrary (but nonrelativistic) velocity. A numerical evaluation of $\tau(\theta,\omega)$ has been carried out for a plasma density corresponding to $\chi^2=0.344$. This is appropriate to the conduction band in Al, if one assumes that there are three free electrons per atom. The quantity $T(x,z)=(\hbar v_0^2/e^2)\tau$ is plotted in Fig. 5. The interaction probability per unit path is obtained from this plot by setting

$$\tau(\theta,\omega)d\Omega d\omega=\frac{e^2}{\hbar v_0^2}T(x,z)d\Omega d\omega,$$

and by employing the relation expressed by Eq. (6.5).

In Fig. 5 contours of equal intensity in $T$ are plotted on the $x$-$z$ plane. For a high-energy electron, $\alpha^2\gg 1$, which undergoes an energy loss small compared with its original energy, we may write approximately, $\theta\sim 2z/\alpha$ and since $\omega=E_Fx/\hbar$, the intensity plot of Fig.

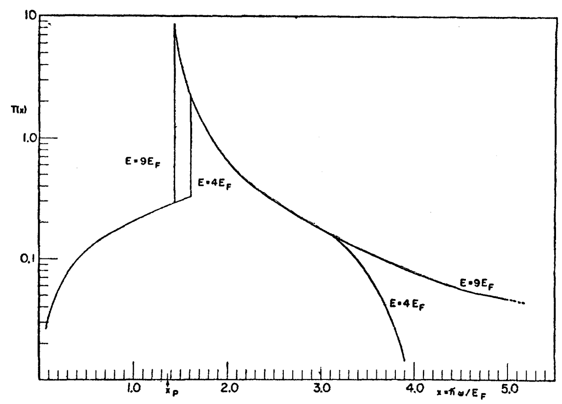


FIG. 6. $T(x)$, interaction probability in a free electron gas as a function loss $\hbar\omega$.

5 represents a prediction of the distribution of energy loss and angular deflection experienced by a fast electron in a Fermi gas. The ingenious experiment of Watanabe[14] showed an approximately parabolic connection between energy loss and angle of deflection of the sort shown in Fig. 5.

The plasma resonance line $P$ beginning at $x_p=\hbar\omega_p/E_F$ has a high intensity and vanishingly small width in the limit of zero damping. There is a quasi-resonance in regions I and II of lower intensity.

Figure 6 shows a plot of $T(x)=\int T(x,z)d\Omega$ for positrons possessing energy $E=4E_F$ and $E=9E_F$.

## 7. SUMMARY

The interaction probability of an incident charged particle with a free electron gas has been calculated in the Born approximation, employing the wave-vector-dependent dielectric constant of Lindhard and Hubbard, and Feynman's treatment of self-energy in the medium. The cross section is shown to reduce to the Born approximation for the Coulomb scattering of charged particles on the electron gas if momentum and energy transfers to the gas are large and to the result obtained by employing a screened interaction if energy transfers are small. It is clear that the methods presented in this paper may be applied to the calculation of many effects in the free electron theory of metals, such as positron slowing down and annihilation, capture and loss of electrons by charged particles, plasma effects in the $K$-edge fine structure in x-ray absorption, etc.


## ACKNOWLEDGMENTS

The author takes pleasure in thanking Dr. R. G. Alsmiller and Dr. F. S. Alsmiller for helpful criticism of the ideas presented here, and Mr. P. R. Flusser for help in some of the numerical work. Thanks are due Dr. R. A. Ferrell for sending the author a preprint of a paper by Quinn and Ferrell on the self-energy approach to correlation energy in the degenerate electron gas.


## 8. APPENDIX. DERIVATION OF THE DIELECTRIC CONSTANT BY MEANS OF A FEYNMAN DIAGRAMMATIC ANALYSIS

The derivation of the self-energy given above is somewhat inconsistent in that the dielectric constant is derived by an ordinary perturbation solution of the approximate Hartree one-particle equations for electrons making up the Fermi gas, while the interaction of an external charged particle with the electron gas was found using Feynman's formulation of perturbation theory.

The following derivation employs methods similar to those used by Hubbard.[20] The plasma propagation function will be obtained from an analysis of Feynman diagrams in the interaction of an external charged particle with the electron gas. By identifying with the result obtained above a partial sum in orders of the perturbation which occurs in this function a consistent definition of the dielectric constant of the free-electron gas will emerge.

The nonrelativistic electron propagator in the free-electron gas may be written as in Eq. (4.2). This propagator contains only components outside the Fermi sphere when $t_2>t_1$ and components inside only when $t_2<t_1$, and it is directly analogous to the relativistic electron-positron propagator discussed by Feynman.[6] The Coulomb interaction potential may be written

$$V(2,1)=\frac{4\pi}{L^3T}\sum_{\mathbf{k},\omega}\frac{1}{\mathbf{k}^2}\exp\{i\mathbf{k}\cdot\mathbf{r}_{21}-i\omega t_{21}\}. \tag{8.1}$$

We now proceed to calculate the first order correction to the amplitude of a fast electron ($v_0\gg v_F$) interacting with the plasma, using the prescriptions given by Hubbard.[20] These are essentially identical with the prescriptions of the quantum theory of the electromagnetic field,[9] except that only longitudinal interactions are included. Consider first the Feynman graph shown in Fig. 7(a). The contribution to the matrix element to the order $e^4$ from this diagram is

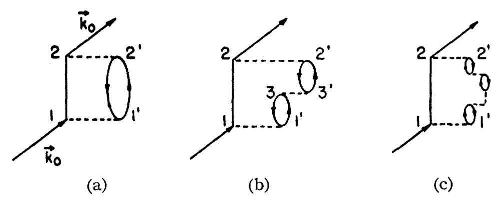


FIG. 7. Feynman graphs showing excitation of virtual electron-hole pairs in the free electron gas.

$$M^{(2)}=-\frac{e^4}{(2i\hbar)^2 2!}\int d^4x_1d^4x_{1'}d^4x_2d^4x_{2'}\,\phi^*(2)K_+(2,1)\times V(2,2')K(1',2')V(1,1')\phi(1), \tag{8.2}$$

where $\phi$ stands for the eigenfunction of the initial state, and $d^4x\equiv d\mathbf{r}dt$. Now the number of diagrams which differ from this only in the naming of the interaction points is exactly $2^2\times 2!$. Hence we may write for the total matrix element

$$M^{(2)}=\frac{e^2}{i\hbar}\int d\mathbf{r}_1\int_{-\infty}^{0}dt_1\int d\mathbf{r}_2\int_{-\infty}^{0}dt_2\times\phi_0^*(2)K_+(2,1)\mathcal{V}^{(2)}(2,1)\phi_0(1), \tag{8.3}$$

where

$$\mathcal{V}^{(2)}(2,1)=-\frac{e^2}{i\hbar}\int d\mathbf{r}_1'\int dt_1'\int d\mathbf{r}_2'\int dt_{2'}\times V(1',1)K_+(2',2)K_+(1',2')V(2_12'). \tag{8.4}$$

[20] J. Hubbard, Proc. Roy. Soc. London **A240**, 539 (1957), and Proc. Roy. Soc. London **243**, 336 (1948).

We find, letting $\omega(\mathbf{k}_0)$, $\omega(\mathbf{k}_f)=(\hbar/2m)(\mathbf{k}_0^2,\mathbf{k}_f^2)$ be the energy of the initial and intermediate states, respectively,

$$\mathcal{V}^{(2)}(2,1)=\frac{4\pi}{L^3T}\sum_{\mathbf{k},\omega}\frac{\alpha}{k^2}\exp\{i\mathbf{k}\cdot\mathbf{r}_{21}-i\omega t_{21}\}, \quad (8.5)$$

where

$$\alpha=\left\{\frac{4\pi e^2}{\hbar L^3}\sum_{\mathbf{k}_0}^{\text{occ}}\left[\frac{1}{\omega(\mathbf{k}_0)-\omega(\mathbf{k}_0-\mathbf{k})-\omega+i\sigma}+\frac{1}{\omega(\mathbf{k}_0)-\omega(\mathbf{k}_0+\mathbf{k})+\omega+i\sigma}\right]\right\}.$$

Now consider the contribution to $M^{(4)}$ from processes of the sort indicated in the graph of Fig. 7(b). Then

$$M^{(4)}=-\frac{e^2}{i\hbar}\int d^4x_1\int d^4x_2 \times\phi_0^*(2)K_+(2,1)\mathcal{V}^{(4)}(2,1)\phi_0(1), \quad (8.6)$$

where

$$\mathcal{V}^{(4)}(2,1)=\frac{4\pi}{L^3T}\sum_{\mathbf{k},\omega}\frac{\alpha^2}{k^2}\exp\{i\mathbf{k}\cdot\mathbf{r}_{21}-i\omega t_{21}\}.$$

In an immediate generalization of the above, we may write

$$\mathcal{V}^{(n)}(2,1)=\frac{4\pi}{L^3T}\sum_{\mathbf{k},\omega}\frac{\alpha^n}{k^2}\exp\{i\mathbf{k}\cdot\mathbf{r}_{21}-i\omega t_{21}\}, \quad (8.7)$$

where the diagram consists of $n$ loops as in Fig. 7(c). If we now sum over all these processes, i.e.,

$$M=\sum_{n=0}^{\infty}M^{(n)}=-\frac{e^2}{i\hbar}\int d^4x_1\int d^4x_2 \times\phi_0^*(2)K_+(2,1)\mathcal{V}(2,1)\phi_0(1), \quad (8.8)$$

in which

$$\mathcal{V}(2,1)=\frac{4\pi}{L^3T}\sum_{\mathbf{k},\omega}\frac{1}{k^2\epsilon_{k,\omega}}\exp\{i\mathbf{k}\cdot\mathbf{r}_{21}-i\omega t_{21}\},$$

and

$$\epsilon_{k,\omega}=1-\frac{4\pi e^2}{\hbar k^2L^3}\sum_{\mathbf{k}_0}^{\text{occ}}\left\{\frac{1}{\omega(\mathbf{k}_0)-\omega(\mathbf{k}_0-\mathbf{k})-\omega+i\sigma}+\frac{}{\omega(\mathbf{k}_0)-\omega(\mathbf{k}_i+\mathbf{k})+\omega+i\sigma}\right\}, \quad (8.9)$$

where it is understood that $\mathbf{k}_f$ must lie outside the Fermi sphere. The function $\epsilon_{k,\omega}$ differs from the Lindhard dielectric constant considered before only in the signs of the imaginary part of the denominators. Recall that the plasma propagation function, Eq. (3.3), involving the Lindhard dielectric constant had the property that

$$\mathcal{V}(2,1)=0 \quad (t_{21}<0).$$

In the present case $\mathcal{V}(2,1)$ is symmetrical in $t_1$ and $t_2$ and hence has only positive energy components. It is thus a proper propagation function for a "field theory" of plasma.

Note that the sum over Feynman graphs which we have performed is not at all exhaustive: It is a very special sum which includes only the simplest processes. For example, in the $\mathcal{V}^{(4)}(2,1)$ term there is a process of the kind depicted in Fig. 8 which has been ignored.

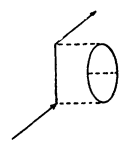

FIG. 8. Feynman graph showing a basic exchange process in the free electron gas.

Let us consider again Eq. (8.8). This is the change in the amplitude of the incident electron of momentum $\hbar\mathbf{k}_0$ due to a "first-order" interaction with the plasma. This amplitude change may clearly be written as $\{\exp(-iT\Delta E/\hbar)-1\}$, where $T$ is the normalization time and $\Delta E$ is the energy shift of the incident electron due to its interaction with plasma. Then one may write for the "self-energy" of the electron

$$\Delta E=\frac{e^2}{T}\int dx_1dx_2\,\phi_0^*(2)K_+(2,1)\mathcal{V}(2,1)\phi_0(1), \quad (8.10)$$

which agrees with Eq. (4.4) which was derived from more intuitive considerations.

The propagation of disturbances in plasma has thus been resolved in terms of processes in which an electron-hole pair interacts with the incident electron after the excitation and subsequent de-excitation of none, one, two, etc., pairs previously. The excitation is passed on from one pair to another in turn. The summation over primitive Feynman graphs gives the result obtained by more elementary methods in Sec. 2. The summation involved in finding $\epsilon_{\mathbf{k},\omega}$ is equivalent to the solution of an integral equation formulation for interactions in the electron gas which has been given by Hubbard,[20] but seems rather closer to the physical processes which occur.