Analysis of the Λ-Hypernuclear Three-Body Systems*

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An analysis of the Λ-hypernuclear three-body systems is made by a variational method with a six-parameter trial function. An upper bound for the strength of the Λ-nucleon interaction required to account for the observed binding of the $T=0$ hypertriton ${}_{\Lambda}H^3$ is obtained. This upper bound is 10–20% (depending upon the value taken for the binding energy) lower than the previous estimate by the same authors. An improved quantitative estimate of the spin dependence of the Λ-nucleon interaction (neglecting the possible influence of three-body forces) is obtained from the results of this analysis and those of a previous analysis of ${}_{\Lambda}He^5$; from this estimate it follows that the existence of a bound Λ-nucleon system is strongly excluded. The analysis of the $T=1$ triplet ${}_{\Lambda}He^3$, ${}_{\Lambda}H^3$, ${}_{\Lambda}n^3$ indicates that these systems are not expected to form bound states. It appears that the essential conclusions of this work would not be seriously affected if there exist moderately strong three-body forces arising from pion exchange processes.

1. INTRODUCTION[1]

THE hypernucleus ${}_{\Lambda}H^3$, the hypertriton, is a system of particular interest in the study of the Λ-nucleon interaction. On account of the absence of any bound state for the Λ-nucleon system and on account of the difficulty of obtaining experimental information on the scattering of Λ particles by nucleons, the hypertriton is the simplest system at hand in which a Λ particle interacts with nucleons at low energies. The small total binding energy (≈ 2.3 Mev) of ${}_{\Lambda}H^3$ implies that it is a very loose structure. When any pair of particles is close together, the third particle is, on the average, relatively far away from them. For this reason the Λ-nucleon interactions in the hypertriton can be expected to take place under conditions closely resembling those of free-particle collisions at low relative energy. In this respect the hypertriton plays a role here similar to that played by the deuteron in the study of the ${}^3S$ nucleon-nucleon interaction. Under these circumstances it is reasonable to expect that the properties of the hypertriton depend primarily on certain over-all features of the Λ-nucleon interactions, such as the well-depth parameters and the zero-energy scattering lengths, and that they are insensitive to the details of the interactions.

The short ranges of the nucleon-nucleon and Λ-nucleon interactions, as well as the small total binding energy of the system, imply that the wave function of the hypertriton consists predominantly of S states of relative motion between the particles.[2] The presence of a tensor force in the Λ-nucleon interaction will not give rise to an appreciable D-state component in the hypertriton wave function. The centrifugal barriers effective in such a D-state motion will strongly suppress the corresponding component of the wave function (except possibly in the regions of small separation between the particles) in the outer regions of their short-range interactions. This does not imply, however, that the tensor-force part of the Λ-nucleon interaction can simply be omitted; it is well known that a strong tensor force will contribute significantly to the S-wave scattering interaction between Λ particle and nucleon even at the lowest relative momenta. For this reason the central potentials which are used in the present work are to be understood as equivalent potentials whose low-energy scattering characteristics are the same as those of the actual Λ-nucleon interactions including their noncentral parts. Since we are not concerned with the calculation of the small D-state component of the hypertriton wave function, tensor forces have not been considered explicitly in the present work.

In these introductory remarks the possibility of three-body potentials between the Λ particle and the two nucleons has been neglected although Weitzner[3] and Spitzer[4] have recently pointed out that the transfer of two pions from the Λ particle, one to each of the two available nucleons, might give rise to an appreciable three-body potential. Of all the hypernuclei, however, the hypertriton is the system for which three-body potentials are expected *a priori* to be the least important; this is both because there is only one pair of nucleons available [compared with $(A-1)(A-2)/2$ pairs in a hypernucleus of mass number A] and also because the small total binding energy of ${}_{\Lambda}H^3$ implies a corresponding small probability for the three particles all to be found within the range of this three-body potential. For these reasons we believe that the structure of the hypertriton is determined primarily by the two-body Λ-nucleon interactions and that it is reasonable to

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[1] In this paper the units have been chosen such that $\hbar=c=1$.

[2] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958).

[3] H. Weitzner, Phys. Rev. **110**, 593 (1958).

[4] R. Spitzer, Phys. Rev. **110**, 1190 (1958).

discuss the role of three-body interactions subsequently by means of perturbation methods.

In a recent paper[5] the features desirable in a trial function for a system such as the hypertriton were discussed in some detail. It was pointed out that, although the small value of B_Λ (the binding energy of the Λ particle in ${}_\Lambda H^3$) implies that the wave function should have a long tail, the presence of Λ-nucleon potentials of short range ($\lesssim 1/2m_\pi$) requires that this wave function should allow strong correlations in position between the Λ particle and each of the nucleons. It appeared, therefore, that a satisfactory trial function should have considerable flexibility in order to represent adequately both the asymptotic regions in which one particle is separated by a large distance from the other two particles and the regions of close approach.

In order to illustrate some of the qualitatively important features of the hypernuclear three-body system, a preliminary discussion was given on the basis of a simple trial function[5]

$$\psi = Ne^{-\alpha(r_1+r_2)}e^{-\beta r_3}, \tag{1.1}$$

where r_1 and r_2 denote the distance of the Λ particle from each of the nucleons and r_3, the distance between the two nucleons. The symmetrical form ($\alpha=\beta$) of this trial function is known to provide an excellent first approximation for the S-state structure of the nuclear three-body systems, H^3 and He^3.[6] This trial function (1.1) is not sufficiently flexible, however (having only one variation parameter associated with each interparticle distance), to provide a good approximation for the case in which the binding energy of one particle (the Λ particle, in this case) of the system is small compared to those of the other particles or for the case in which the total binding energy of the system is very small. In a variational calculation for a potential strength corresponding to a given binding energy (or *vice versa*), the region in which the particles lie within the range of their mutual interactions is more important for the determination of the optimum parameters of the trial function than is the asymptotic region of large separations, where the form of the correct wave function is governed by the binding energies of each of the particles of the system. The optimum parameters of the trial function (1.1) are, consequently, insensitive to B_Λ when it is small or to the total binding energy B when it is small. For H^3 and He^3 it happens that the optimum trial function of the form (1.1) with $\alpha=\beta$ has an asymptotic form approximately consistent with the binding energies of these systems. In the nuclear three-body systems the binding energies B_n and B_p of a neutron and a proton, respectively, are nearly the same ($B_n \approx 6$ Mev and $B_p \approx 8$ Mev for H^3, and *vice versa* for He^3), while the characteristic length $[(3/4MB_p)^{\frac{1}{2}}$, say] of the asymptotic region is approximately the same as the range of nuclear forces. For the hypertriton, however, this fortunate situation by no means obtains; and it is reasonable to expect that the use of a trial function with sufficient flexibility to account for both close-in and asymptotic regions at the same time (that is, a trial function with more than one variation parameter associated with each interparticle distance) would lead to a much greater proportional improvement in a calculation for the hypertriton than in one for the normal triton.[7]

In the present work, a variational calculation of the strength of the Λ-nucleon interactions required to account for the observed binding energy of the hypertriton was made with a six-parameter trial function

$$\psi = (e^{-ar_1}+xe^{-br_1})(e^{-ar_2}+xe^{-br_2})(e^{-a_3r_3}+ye^{-b_3r_3}). \tag{1.2}$$

This function has considerable flexibility and should be capable of giving a good representation of the principal features of a lightly bound system with short-range forces because, for each interparticle distance, there is a factor containing a short-range and a long-range term in adjustable proportion and with both ranges adjustable. In order to satisfy the Pauli principle for the nucleons with this form of wave function, the factors corresponding to the Λ-n and the Λ-p separations are taken to have the same form so that there are, in all, six variation parameters: a, b, x, a_3, b_3 and y.

The asymptotic form of the correct hypertriton wave function for large separations R of the Λ particle from the center of mass of the two nucleons is

$$\psi \sim \psi_D(r_3)e^{-\alpha R}/R, \tag{1.3}$$

where ψ_D is the wave function of the free deuteron and $\alpha=[4MM_\Lambda B_\Lambda/(2M+M_\Lambda)]^{\frac{1}{2}}$. The smaller B_Λ is, the more dominant the asymptotic form (1.3) is in the normalization integral or in the expression for the probability distribution for the n-p separation (averaged over all positions of the Λ particle). As we have emphasized in reference 5, however, this does not justify the use of a wave function of the form (1.3) for all regions of space for ${}_\Lambda H^3$ in any situation (such as the calculation of the potential strength for a given binding energy or the calculation of matrix elements for the decay processes of ${}_\Lambda H^3$) in which the regions of close Λ-n and Λ-p approach play an important role. The asymptotic form of (1.2) for large separation of the Λ particle from the nucleons is

$$\psi \sim (e^{-a_3r_3}+ye^{-b_3r_3})e^{-2\alpha R}. \tag{1.4}$$

If the wave function (1.2) is to give a good representation of the hypertriton wave function in the asymptotic region, then it is clear that the first factor of (1.4)

[5] R. H. Dalitz and B. W. Downs, Phys. Rev. **110**, 958 (1958).
[6] H. Feshbach and S. I. Rubinov, Phys. Rev. **98**, 188 (1955).
[7] It is of interest to note that increase in the flexibility of the trial function for the normal triton [see, for example, F. W. Brown, Phys. Rev. **56**, 1107 (1939)] gives an improvement of only about 1.5% in the estimate of the nucleon-nucleon potential strengths corresponding to the observed triton binding energy over the value obtained with the simple trial function $\exp[-\alpha(r_1+r_2+r_3)]$.

should not be very different from the wave function for a free deuteron. The wave function

$$\varphi=(e^{-a_3r_3}+ye^{-b_3r_3}), \tag{1.5}$$

does, in fact, give a good representation of the deuteron for suitable values of the parameters a_3, b_3, and y (see Sec. 3), leading to a value for the binding energy accurate to about 0.2%. It therefore seems reasonable to expect that the optimum values of the parameters a_3, b_3, and y for the trial function (1.2) should be quite close to those obtained for the free deuteron with the trial function (1.5). The R-dependence of the asymptotic form (1.4) does not have the correct form (1.3); this, however, is relatively unimportant for the calculations of the present work. The effect of this inadequacy on the calculation of other quantities of interest for the ${}_\Lambda H^3$ system is discussed briefly in the concluding section.

There has been no conclusive experimental evidence requiring the existence of any bound state for the ${}_\Lambda He^3$ or ${}_\Lambda n^3$ systems, and arguments have been given[8] which make it appear unlikely that such bound states should exist. These arguments concern the question of whether the existence of the $T=0$ hypertriton with a certain binding energy B_Λ implies or excludes the existence of bound states for ${}_\Lambda He^3$, ${}_\Lambda n^3$ and the $T=1$ state of ${}_\Lambda H^3$. In order to answer this question (and to substantiate the arguments previously given) a variational calculation has been made with the trial function (1.2) to determine the strength of the Λ-nucleon interactions which would be required to bind the $T=1$ systems with zero total energy.

The details of the variational calculation procedure are discussed in Sec. 2. For the $T=0$ state of ${}_\Lambda H^3$, the results for various values of B_Λ and for two ranges of the Λ-nucleon interaction are given in Sec. 3; these results are a significant improvement (up to 19%) over the estimate previously obtained[5] for the strength of the Λ-nucleon interactions. The results for the $T=1$ hypernuclear triplet ${}_\Lambda He^3$, ${}_\Lambda H^3$, and ${}_\Lambda n^3$ are given in Sec. 4; these results confirm that the absence of these states is reasonable. In the concluding Sec. 5 the results of these calculations are combined with those of earlier calculations on the Λ-nucleon interactions in ${}_\Lambda He^5$ to obtain an estimate of the Λ-nucleon interactions necessary in the ${}^1S$ and ${}^3S$ states to account for the binding energies of both ${}_\Lambda H^3$ and ${}_\Lambda He^5$ when the effect of three-body interactions is completely neglected. Some remarks are also made in Sec. 5 on the reliability of these calculations and on the possible use of the wave function (1.2) for the calculation of other properties of the hypertriton.

2. FORMULATION OF THE VARIATIONAL PROBLEM

In the triangular coordinate system (r_1,r_2,r_3) appropriate to a trial function of the form (1.2), the variation principle for determination of the wave function ψ for a three-particle system consisting of two nucleons (mass M) and a Λ hyperon (mass M_Λ) with total binding energy B can be written in the following way (see reference 5):

$$T(\psi,\psi)-V_0v(\psi,\psi)-Uu(\psi,\psi)+BN(\psi,\psi)\geqq 0. \tag{2.1}$$

The functions which occur in (2.1) are defined as

$$T(\varphi,\chi)=\int\left\{\frac{1}{M}\frac{\partial\varphi^*}{\partial r_3}\frac{\partial\chi}{\partial r_3}+\left(\frac{1}{2M}+\frac{1}{2M_\Lambda}\right)\left(\frac{\partial\varphi^*}{\partial r_1}\frac{\partial\chi}{\partial r_1}+\frac{\partial\varphi^*}{\partial r_2}\frac{\partial\chi}{\partial r_2}\right)+\frac{1}{M}[t(231)+t(312)]+\frac{1}{M_\Lambda}t(123)\right\}r_1r_2r_3dr_1dr_2dr_3, \tag{2.2a}$$

$$v(\varphi,\chi)=\int\varphi^*f(\kappa_3r_3)\chi r_1r_2r_3dr_1dr_2dr_3, \tag{2.2b}$$

$$u(\varphi,\chi)=\int\varphi^*[g(\kappa r_1)+g(\kappa r_2)]\chi r_1r_2r_3dr_1dr_2dr_3, \tag{2.2c}$$

$$N(\varphi,\chi)=\int\varphi^*\chi r_1r_2r_3dr_1dr_2dr_3, \tag{2.2d}$$

with

$$t(ijk)=\frac{r_i^2+r_j^2-r_k^2}{4r_ir_j}\left(\frac{\partial\varphi^*}{\partial r_i}\frac{\partial\chi}{\partial r_j}+\frac{\partial\varphi^*}{\partial r_j}\frac{\partial\chi}{\partial r_i}\right). \tag{2.3}$$

The integrals (2.2) are to be taken over the domain consistent with the triangular inequalities $r_1+r_2\geqq r_3$, $r_2+r_3\geqq r_1$ and $r_3+r_1\geqq r_2$. The depth of the nucleon-nucleon potential is V_0; this is $V_{0,\text{ triplet}}$ for the $T=0$ hypertriton states and $V_{0,\text{ singlet}}$ for the $T=1$ hypernuclear triplet. Charge symmetry requires the Λ-p and Λ-n interactions to be the same; the depth of the mean Λ-nucleon potential in the spin state considered for the hypernuclear system is denoted by U. The mean depth U depends upon the relative orientation of the spins of the Λ particle and the nucleons. This spin dependence has been discussed previously[5]; explicit expressions for U in terms of the depths of the triplet and singlet Λ-nucleon potentials are given in Secs. 3 and 4. The functions f and g denote the forms assumed for the nucleon-nucleon and Λ-nucleon potentials, respectively; κ_3 and κ are the corresponding range parameters.

The inequality (2.1) has been used as a variational principle to obtain an upper bound on the strength U of the Λ-nucleon interaction potential required to account for the binding energy B of the system. The potential forms f and g were taken to be Yukawa functions.[9] The range parameter appropriate to the

[8] See reference 5 and also Sec. 4 in the following.

[9] In reference 5 it was found that Yukawa and exponential forms for g which have the same intrinsic range lead to essentially the same values for those quantities which are a measure of the strength of the Λ-nucleon potential, such as the well-depth parameter, scattering length and the volume integral of the potential.

Λ-nucleon interaction has not yet been established empirically; the value of this parameter will depend upon which of the conceivable mechanisms[2,5] contribute most to this interaction. In order to span the reasonable range parameters, calculations have been made here for the Yukawa shape $e^{-\kappa r}/\kappa r$ with values of the parameter $1/\kappa=1/m_K\approx 0.4\times 10^{-13}$ cm and $1/\kappa=1/2m_\pi \approx 0.7\times 10^{-13}$ cm, the Compton wavelength of the K meson and one-half that of the pion, respectively.

It is convenient to write the trial function (1.2) in the form

$$\psi=\prod_{i=1}^{3}\left(\sum_{\alpha=1}^{2}\lambda_{i\alpha}e^{-a_{i\alpha}r_i}\right), \tag{2.4a}$$

where the coefficients $\lambda_{i\alpha}$ and $a_{i\alpha}$ are given by

$$\begin{aligned}&\lambda_{i1}=1 \quad \text{for all } i,\\ &\lambda_{12}=\lambda_{22}=x, \quad \lambda_{32}=y;\\ &a_{11}=a_{21}=a, \quad a_{31}=a_3,\\ &a_{12}=a_{22}=b, \quad a_{32}=b_3.\end{aligned} \tag{2.4b}$$

The integral $T(\psi,\psi)$ of Eq. (2.1) then becomes

$$T(\psi,\psi)=\sum_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}\lambda_{1\alpha}\lambda_{1\alpha'}\lambda_{2\beta}\lambda_{2\beta'}\lambda_{3\gamma}\lambda_{3\gamma'}T_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}. \tag{2.5}$$

The coefficient $T_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}$ denotes the integral (2.2a); that is,

$$T_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}=T(\exp(-a_{1\alpha}r_1-a_{2\beta}r_2-a_{3\gamma}r_3),\ \exp(-a_{1\alpha'}r_1-a_{2\beta'}r_2-a_{3\gamma'}r_3)). \tag{2.6}$$

The integrals $u(\psi,\psi)$, $v(\psi,\psi)$, and $N(\psi,\psi)$ can be similarly expanded in terms of corresponding coefficients $u_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}$, $v_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}$, and $N_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}$. Explicit expressions for all these coefficients are given in the Appendix.

For the actual computations, the variation principle (2.1) was put into the form

$$\frac{\{T(\psi,\psi)-Vv(\psi,\psi)+BN(\psi,\psi)\}}{u(\psi,\psi)}=\Phi(\psi)\geqq U. \tag{2.7}$$

The function $\Phi(\psi)$ was obtained as an explicit algebraic function of a, b, a_3, b_3, x, and y in the manner described above. Both numerator and denominator of $\Phi(\psi)$ are quadratic functions of y, so that

$$\Phi(\psi)=(K_1y^2+L_1y+M_1)/(K_2y^2+L_2y+M_2); \tag{2.8}$$

the K, L, and M are functions of a, b, a_3, b_3, and x. For each set of values (a, b, a_3, b_3, x) considered, the coefficients K, L, and M were evaluated, and the expression (2.8) was minimized analytically with respect to y.[10] These calculations were done by electronic computer.[11] For each set of values (a, b, a_3, b_3), the computer was programmed to search for the optimum value of x and then to provide values of Φ for 8 values of x in the immediate neighborhood of the optimum value. The optimum x and the corresponding Φ were then obtained graphically and were checked, at least for the sets (a, b, a_3, b_3) near the optimum set, by fitting an appropriate polynomial to the calculated values. The optimum values of a_3 and b_3 turned out to be relatively insensitive to changes in the parameters a and b in the region of their optimum values[12]; this property was used to fix values of a_3 and b_3 from a preliminary calculation with a fairly wide grid of parameters. The values of a_3 and b_3 having been fixed, values of Φ minimized with respect to x and y were then obtained for each set (a,b). For each value of a, Φ was minimized with respect to b, and then these lowest values of Φ were finally minimized with respect to a. This last step yielded both the minimum value of Φ with respect to the four parameters (a, b, x, y) for the fixed set (a_3,b_3) and the corresponding optimum value of a. With this value of a, it was possible to go back and obtain the corresponding optimum value of b by interpolation. The values of a and b were fixed at these optimum values, and the function Φ was then minimized with respect to (a_3,b_3) in the same manner as it had previously been minimized with respect to (a,b) for fixed (a_3,b_3). In every case the optimum values of (a_3,b_3) obtained in this way were sufficiently close to the values originally selected from the wide grid to justify concluding the iteration process at this point. With the optimum set (a,b,a_3,b_3) determined in this way, Φ was minimized with respect to x. With this optimum x and the optimum set (a,b,a_3,b_3), the optimum y was calculated by the computer, leading to a direct check on the minimum value obtained for Φ by interpolation from its values at the calculated points.

3. THE $T=0$ HYPERTRITON STATES

The $T=0$ three-particle hypernuclear states consist of a Λ particle interacting with a neutron and a proton whose total isotopic spin is zero and whose total spin is $S=1$. Assuming the Λ particle to have spin $\frac{1}{2}$,[13] the $T=0$ hypertriton state of lowest energy will have spin $J=\frac{1}{2}$ if the singlet Λ-nucleon interaction is more attractive than the triplet and $J=\frac{3}{2}$ if the triplet interaction is the more attractive. Denoting the volume integral of the Λ-nucleon interaction in the triplet state by $\bar{V}_p$ and that in the singlet state by $\bar{V}_a$, the volume integral of the total Λ-nucleon interaction in the hypertriton ground state is

$$U_2=2\bar{V}_p \quad \text{if} \quad \bar{V}_p>\bar{V}_a \quad (J=\tfrac{3}{2}), \tag{3.1a}$$

$$U_2=3\bar{V}_a/2+\bar{V}_p/2 \quad \text{if} \quad \bar{V}_a>\bar{V}_p \quad (J=\tfrac{1}{2}). \tag{3.1b}$$

[10] The condition $d\Phi/dy=0$ leads to a quadratic equation in y. In every case computed, the roots of this equation were of opposite sign, the positive root being the optimum value of y.

[11] These calculations were performed at the Cornell Computing Center. The authors are grateful to Miss V. A. Walbran for having programmed and run the computer for the calculations reported here.

[12] It is clear from the form of the function $(e^{-a_3r_3}+ye^{-b_3r_3})$ that a change in a_3 or b_3 can be compensated, to some extent, by a corresponding change in y.

[13] See references 2 and 5 for a summary of the evidence supporting the assignment of spin $\frac{1}{2}$ to the Λ particle.

TABLE I. The Λ-nucleon interaction in the hypertriton.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)	(xi)	(xii)[a]
B_Λ(Mev)	a(f^{-1})	b(f^{-1})	a_3(f^{-1})	b_3(f^{-1})	x	y	s	a_0(f)	r_0(f)	U_2(Mev f^3)	U_2(Mev f^3) (Gaussian)	U_2(Mev f^3)
					(a) Intrinsic range 0.8411 f ($\kappa=m_K$)							
0	0.093	1.13_5	0.382	1.13	2.29	2.22	0.635_9	−1.03	1.45	406.9	405	489
0.25	0.158	1.38	0.382	1.13_5	1.97	2.24	0.659_3	−1.13	1.40	421.8	420	
1.00	0.239	1.66	0.393	1.15	1.74	2.16	0.694_8	−1.33	1.32	444.5	444	501
					(b) Intrinsic range 1.4843 f ($\kappa=2m_\pi$)							
0	0.047	0.59	0.380	1.13	2.13	2.21	0.550_0	−1.30	3.05	621.0	615	766
0.25	0.111	0.80	0.380	1.14	1.67_5	2.14	0.595_2	−1.55	2.75	672.1	667	
1.00	0.184	0.98	0.393	1.17	1.42	1.94	0.653_8	−1.96	2.49	738.1	736	813

[a] See reference 5.

The total binding energy B of the hypertriton ground state is the binding energy of the deuteron $B_D=2.226$ Mev plus the binding energy B_Λ of the Λ particle. There is still considerable uncertainty in the value of B_Λ for the hypertriton; it is almost certain that B_Λ is less than 1 Mev, and it is likely that it is close to zero.[14]

The neutron-proton potential V_{triplet} was taken to be that Yukawa potential whose range is consistent with the low-energy proton-proton scattering data[15] and whose depth is determined by the known binding energy of the deuteron. This potential has an intrinsic range of 2.4995 fermi [1 fermi (f)$=1\times10^{-13}$ cm] and a depth of 68.104 Mev; the volume integral of this potential is 1403.4 Mev f^3.[16]

The results of the variational calculations are given in Table I for the two ranges mentioned above for the Λ-nucleon potential and for three values of B_Λ. The values of the variation parameters corresponding to the minima are listed in columns (i) through (vi). The optimum values of a_3 and b_3 were found to be quite close to those found in a variational calculation of the binding energy of the deuteron with the trial function (1.5) and the n-p potential described above. These values, $a_3=0.38_0$ f^{-1} and $b_3=1.12_0$ f^{-1}, lead to $B_D=2.221$ Mev with $y=2.27_1$.[17] For the hypertriton the optimum value of y depends somewhat on the value of B_Λ and on the range parameter of the Λ-nucleon potential, but it does not deviate much from its value for the free deuteron.[18]

Several measures of the strength of the Λ-nucleon interactions are given in Table I as functions of B_Λ and the range parameter κ:

[14] Levi Setti, Ammar, Slater, Limentani, Roberts, Schlein, and Steinberg, Nuovo cimento (to be published) obtain a most probable value of B_Λ which is less than zero, namely, $B_\Lambda=-0.23\pm0.35$ Mev. It should be noted here that the Q value for the decay of the free Λ particle is not known precisely. The value used by Levi Setti *et al.* was $Q_\Lambda=37.22\pm0.2$ Mev; an increase in Q_Λ by an amount ΔQ_Λ would lead to an increase in the value of B_Λ given by them by an amount ΔQ_Λ.

[15] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).

[16] The depth of this potential was determined by a variational calculation using a Hulthèn trial function, one parameter being fixed by the known value of B_D and the other being the variation parameter. The value of the depth of the potential determined in this way is about 0.03% less than that given by the relevant power series of Blatt and Jackson [J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949)]. The particular value of the intrinsic range quoted above resulted from a choice of the range parameter to be exactly $\kappa_3=0.848$ f^{-1}. The intrinsic range taken for V_{triplet} is somewhat shorter than the value $b_t=2.96$ f appropriate to the Yukawa potential which fits the deuteron binding energy and the triplet scattering length. Since the potential used was adjusted to give the correct value of B_D, it is clear that small changes in the intrinsic range will not appreciably affect our estimates of U_2. Values of U_2 were calculated with the appropriate V_{triplet} having an intrinsic range of 2.96 f for $B_\Lambda=0.25$ Mev and the two intrinsic ranges of the Λ-nucleon potential. For these calculations the optimum values of a, b, a_3, and b_3 given in Table I were used, while x and y were treated as variation parameters. The values of U_2 thus obtained are 430 Mev f^3 for $\kappa=m_K$ and 684 Mev f^3 for $\kappa=2m_\pi$; these values are about 2% larger than the corresponding values in Table I. This seems to be a reasonable confirmation of the foregoing statement that U_2 is insensitive to small changes in the intrinsic range of V_{triplet}, particularly since the values of a, b, a_3, and b_3 used in this check are not necessarily the optimum values for the V_{triplet} in question.

[17] Since the depth of V_{triplet} was determined with a Hulthèn trial function, this value of B_D represents a comparison between the description of the deuteron given by the function $[\exp(-a_3r)+y\exp(-b_3r)]$ and that given by the Hulthèn function. With $y=0$, the simple exponential trial function leads to the value $B_D=1.621$ Mev with $a_3=0.62$ f^{-1}.

[18] It should be emphasized that this does not mean that the probability distribution for the n-p separation in the hypertriton is the same as that in the free deuteron. The probability distribution is actually given by $\rho(r)r^2dr$, where

$$\rho(r)=C\rho_D(r)\Big\{\phi(2a,0)+4x\phi\Big(\frac{3a+b}{2},\frac{a-b}{2}\Big)+2x^2[2\phi(a+b,0)+\phi(a+b,a-b)]+4x^3\phi\Big(\frac{a+3b}{2},\frac{a-b}{2}\Big)+x^4\phi(2b,0)\Big\},\quad\text{(i)}$$

in which

$$\rho_D(r)=(e^{-a_3r}+ye^{-b_3r})^2,$$

$$\phi(\alpha,\beta)=r^3\frac{e^{-\alpha r}}{\alpha r}\Big\{\frac{\sinh\beta r}{\beta r}\Big[\frac{1}{\alpha r}+\frac{1}{(\alpha r)^2}\Big]+\frac{1}{(\beta r)^2}\Big[\cosh\beta r-\frac{\sinh\beta r}{\beta r}\Big]\Big\},$$

and C is a normalization constant. This distribution is obtained by averaging the square of the wave function (1.2) over the positions of the Λ particle consistent with the triangular inequalities for the interparticle distances. The factor in the braces of expression (i) is a monotonically decreasing function of r, so that the distribution $\rho(r)$ generally appears compressed relative to the deuteron distribution $\rho_D(r)$, as one would expect. As a approaches zero, however, the term $\phi(2a,0)$ becomes dominant in this factor, which then varies only slightly over the region in which $\rho_D(r)$ is appreciably different from zero. According to the discussion in the Introduction, the distribution $\rho(r)$ should be identical with $\rho_D(r)$ when $B_\Lambda=0$. With the approximate wave function (1.2), $\rho(r)$ is compressed relative to $\rho_D(r)$ even at zero B_Λ because a does not reach the value zero when B_Λ goes to zero; this deviation is related to the incorrect asymptotic form of the wave function (1.2) (see the discussion in Sec. 5). As B_Λ increases, the value of a increases, and the compression of $\rho(r)$ relative to $\rho_D(r)$ becomes increasingly more severe.

(a) The effective well-depth parameter s [column (vii)]. This is the value of the well-depth parameter for the Λ-nucleon potential $Ug(\kappa r)$, where U is the depth of the mean Λ-nucleon potential in the hypertriton. Explicitly, U is chosen so that

$$2U\int g(\kappa r)d_3r=U_2,$$

where U_2 is the volume integral of the total Λ-nucleon interaction in the hypertriton.

(b) The scattering length a_0 [column (viii)] for Λ-nucleon scattering at zero energy by the potential $Ug(\kappa r)$.

(c) The effective range r_0 [column (ix)] of the Λ-nucleon potential $Ug(\kappa r)$.

(d) The volume integral U_2 [column (x)] of the total Λ-nucleon interaction in the hypertriton.

(e) The volume integral U_2 [column (xi)] for a Λ-nucleon potential of Gaussian form which has the same intrinsic range and leads to the same zero-energy scattering length a_0 as the Yukawa potential to which the volume integral U_2 of column (x) pertains.

In column (xii) the values of U_2 calculated previously[5] with the simple trial function (1.1) (with Yukawa potentials) are given for comparison. Even neglecting three-body potentials, the quantities a_0 and r_0 have a direct interpretation in terms of the parameters of the Λ-nucleon system only for the case $\bar{V}_p>\bar{V}_a$ and $j=\frac{3}{2}$ for the hypertriton. In this case a_0 and r_0 are the zero-energy scattering length and the effective range, respectively, of the ^3S state. With $\bar{V}_a>\bar{V}_p$ and $j=\frac{1}{2}$ for ${}_\Lambda H^3$, the well-depth W is $(3U_a+U_p)/4$, a combination of the ^1S and ^3S well depths, and the parameters a_0 and r_0 of Table I then have no direct physical interpretation.

From the values of U_2 given in columns (x) and (xii) of Table I, it is apparent that the trial function (1.2) gives a significant improvement over the simpler trial function (1.1). The smaller the binding energy B_Λ, the greater is the relative improvement in the value of U_2; this was expected from the considerations given in the Introduction. As a function of B_Λ, these values of U_2 [column (x) of Table I] now fall on a curve which has the correct dependence on $\sqrt{B_\Lambda}$ as B_Λ approaches zero. This functional dependence can be represented by the expression

$$U_2(B_\Lambda)=U_2(0)\{1+\alpha(B_\Lambda)^{\frac{1}{2}}+\beta B_\Lambda+\cdots\}, \quad (3.2)$$

where (α,β) have the values (0.054, 0.038) for $\kappa=m_K$ and (0.140, 0.049) for $\kappa=2m_\pi$ when B_Λ is expressed in Mev. These values of (α,β) are to be contrasted with those of the earlier work[5] which led to an unacceptable B_Λ dependence for U_2 [column (xii) of Table I], the corresponding coefficient α being negligible compared to the coefficient β there.

4. THE $T=1$ HYPERNUCLEAR TRIPLET

Each member of the $T=1$ hypernuclear triplet ${}_\Lambda He^3$, ${}_\Lambda H^3$, ${}_\Lambda n^3$ consists of a Λ particle in interaction with two nucleons of total isotopic spin 1 and total spin $S=0$. Since the nucleon spins are paired, the volume integral U_2' of the total Λ-nucleon interaction in these states is just twice the volume integral of the spin-average interaction:

$$U_2'=3\bar{V}_p/2+\bar{V}_a/2. \quad (4.1)$$

For either $\bar{V}_a>\bar{V}_p$ or $\bar{V}_p>\bar{V}_a$, this interaction U_2' is necessarily less than U_2 [see Eqs. (3.1)], U_2' and U_2 being equal only if $\bar{V}_p=\bar{V}_a$.

It is the ^1S nucleon-nucleon potential which is effective in the $T=1$ systems. In the present work, this potential V_{singlet} was taken to be that Yukawa potential which is consistent with the low-energy proton-proton scattering data.[15] This potential has an intrinsic range of 2.4995 f, a depth of 46.17 Mev and a volume integral of 951 Mev f^3.

At present there is no experimental evidence which requires the existence of bound states for the $T=1$ systems. Since the total binding energy of the $T=0$ hypertriton is small ($\approx$2.3 Mev) and since both the nucleon-nucleon interaction and the mean Λ-nucleon interaction are weaker in the $T=1$ systems than are the corresponding interactions in the ground state of the $T=0$ system, the absence of such bound states does not seem unreasonable. The likelihood of a bound state for the ${}_\Lambda He^3$ system is further reduced by the Coulomb repulsion between the protons.

For the analysis of the $T=1$ systems, we confined attention to the state with total binding energy $B=0$. The variational principle (2.7) was used to calculate the strength $U_2'^*$ necessary for the total Λ-nucleon interaction U_2' in order that these systems should have a bound state at zero energy.[19] If the value calculated for $U_2'^*$ were smaller than U_2' given by (4.1), then the existence of bound $T=1$ systems with $B>0$ would be indicated; on the other hand, if $U_2'^*>U_2'$, no bound states would be expected for these systems. On account of the considerable variation of U_2 with B for the $T=0$ hypertriton shown in Table I, it was not *a priori* certain that a decrease in the required value of B, to zero, for the $T=1$ systems would not more than offset the weaker interactions in those systems and imply the existence of bound states.[20]

The calculational procedure described in Sec. 2 was carried through with the nucleon-nucleon potential

[19] The Coulomb energy term appropriate to ${}_\Lambda He^3$ was not included; since these calculations show that no bound state is to be expected for the $T=1$ systems even in the absence of the Coulomb repulsion, this omission has no effect on the essential conclusion.

[20] The corresponding calculations of reference 5 indicated that bound states were not to be expected for the $T=1$ systems. That evidence could not be considered conclusive, however, because of the manifestly incorrect B_Λ dependence of the values of U_2 calculated there with the simple trial function (1.1) (see the discussion at the end of Sec. 3).

V_{singlet} and with $B=0$. In the case of the $T=0$ hypertriton, an initial estimate of the parameters a_3 and b_3 could be obtained from a study of the deuteron. In the case of the $T=1$ systems a corresponding estimate was obtained from the ^1S nucleon-nucleon system by using the trial wave function (1.5) to calculate the strength of the interaction required to give a resonance at zero energy. The values of the parameters a_3 and b_3 obtained in this way were not so close (and were not expected to be) to the optimum values for the three-body system as were those in the case of the $T=0$ system; they did, however, provide a rational starting point in the search for the optimum parameters. The minimum value of the function Φ was determined with a grid of parameters (a,b,a_3,b_3) which was considerably more open than that used for the $T=0$ hypertriton. The precise location of the minimum with respect to the six parameters is not of prime interest in this case because there is no apparent application for the wave function itself. It is the minimum value of Φ which is important here, and this could be obtained sufficiently accurately with a little refinement in the grid in the region of the minimum.

The minimum values of Φ are 520 Mev f^3 for $\kappa=m_K$ and 880 Mev f^3 for $\kappa=2m_\pi$. These estimates of U_2'* are 17% and 19% higher than even the largest values (for $B_\Lambda=1$ Mev) of U_2 given in Table I for the cases $\kappa=m_K$ and $2m_\pi$, respectively. This means that, even in the most favorable case possible (that is, assuming $B_\Lambda=1$ Mev for the $T=0$ hypertriton and no spin dependence at all), the existence of bound states for the $T=1$ systems is excluded.

5. DISCUSSION

The use of the trial function (1.2) has led to a substantial improvement in our estimate of U_2 for the hypertriton. The values of U_2 for $B_\Lambda=1$ Mev given in Table I [column (x)] are about 10% lower than the previous estimates[5] [column (xii) of Table I]; and those for $B_\Lambda=0$ are about 20% lower than those previously obtained. It seems reasonable to believe that these present values of U_2 are quite close to the exact ones appropriate to the physical assumptions underlying our calculations. Since a deviation of the trial function from the exact wave function by an amount ϵ leads only to a deviation in the estimate of U_2 by an amount of order ϵ^2, it follows that the hypertriton wave function we have obtained [the function (1.2) with the optimum parameters given in Table I] is not so close to the exact one. This is, of course, a characteristic of the variational method. In the present case, so far as applications to other properties of the hypertriton are concerned, the most important defect in the wave function (1.2) is the incorrectness of its asymptotic form. For example, in the calculation of the matrix element for the decay process

$$ {}_\Lambda H^3 \rightarrow He^3+\pi^-, \tag{5.1}$$

the main contribution to the integral involved comes from the inner regions of the ${}_\Lambda H^3$ wave function where the form of the wave function is probably described adequately by (1.2); however, with the small value of B_Λ, the normalization of the wave function is determined to a large extent by the form of the wave function in the asymptotic region. Consequently, with the incorrect asymptotic form of the wave function (1.2), the matrix element for the process (5.1) calculated with this wave function will differ from the correct one by an undetermined factor exceeding unity. This is apparent for the limiting case of $B_\Lambda=0$, for which the matrix element for this process should be zero, whereas use of the function (1.2) leads to a nonzero value.

It is, of course, possible to construct a trial function which does have the correct asymptotic form in the regions of large separations. For a large separation of the Λ particle from the center of mass of the nucleons (with $r_1\sim r_2\sim R$) such a wave function should be proportional to $\exp(-a_\Lambda R)/R$, with

$$a_\Lambda=\{4MM_\Lambda B_\Lambda/(2M+M_\Lambda)\}^{\frac{1}{2}}. \tag{5.2}$$

For a large separation ρ of either nucleon from the center of mass of the Λ particle and the other nucleon (with $r_1\sim r_3\sim\rho$) the wave function should be proportional to $\exp(-a_n\rho)/\rho$, where

$$a_n=\{2M(M+M_\Lambda)B/(2M+M_\Lambda)\}^{\frac{1}{2}}, \tag{5.3}$$

and B denotes the total binding energy of the system, B_D+B_Λ. An example of such a trial function with the correct asymptotic forms and with the flexibility of the function (1.2) is

$$\psi=\frac{(e^{-\alpha r_1}+xe^{-br_1})(e^{-\alpha r_2}+xe^{-br_2})(e^{-\alpha_3 r_3}+ye^{-b_3 r_3})}{\{(r_1+A)(r_2+A)(r_3+C)\}^{\frac{1}{2}}}. \tag{5.4}$$

The quantities α and α_3 are chosen to give the correct asymptotic forms; that is

$$2\alpha=a_\Lambda, \quad \alpha+\alpha_3=a_n. \tag{5.5}$$

The variation parameters are b, b_3, x, y, A, and C. Unfortunately the amount of labor involved in a variational calculation for U_2 with this function would greatly exceed that of the present work with the trial function (1.2). The integrals occurring with the use of (5.4) are not at all simple although they can all be expressed in terms of derivatives of the basic integral

$$\int\int\int_D \frac{\exp[-(a_1r_1+a_2r_2+a_3r_3)]}{(r_1+A_1)(r_2+A_2)(r_3+A_3)}dr_1dr_2dr_3, \tag{5.6}$$

where D is the usual domain $r_1+r_2\geqq r_3$, $r_2+r_3\geqq r_1$, $r_3+r_1\geqq r_2$.

An *ad hoc* method of normalizing the wave function (1.2) in a manner appropriate for such applications as the calculation of the matrix element for the hypertriton decay (5.1) is the following: Each factor of (1.2) is

TABLE II. Spin dependence of the Λ-nucleon interaction.[a]

Range	U_2(Mev f³)	U_4(Mev f³)	$\bar{V}_p > \bar{V}_a$ $\bar{V}_p$(Mev f³)	$\bar{V}_a$(Mev f³)	$s_>$	$\bar{V}_a > \bar{V}_p$ $\bar{V}$(Mev f³)	$\bar{V}_a$(Mev f³)	$s_>$
$\kappa=m_K$	420±15	695±25	210± 8	65±34	0.66	156±10	228±12	0.72
$\kappa=2m_\pi$	667±52	910±45	334±26	−90±90	0.60	174±21	386±39	0.69

[a] The most probable value given for U_2 in this table corresponds to $B_\Lambda=0.25$ Mev; the uncertainty given for U_2 is such that the lower limit equals the value of U_2 corresponding to $B_\Lambda=0$.

replaced by the corresponding asymptotic form of the function (5.4) beyond the point at which the logarithmic derivatives of these two functions are equal. For the factor containing r_1, for example, $e^{-ar_1}+xe^{-br_1}$ is to be replaced by $e^{-\alpha r_1}/r_1^{\frac{1}{2}}$ for $r_1>s_1$, where s_1 is defined by

$$\frac{a+xbe^{-s_1(b-a)}}{1+xe^{-s_1(b-a)}}=\alpha+\frac{1}{2s_1}. \tag{5.7}$$

The wave function (1.2) having been modified in this way, the normalization constant for the wave function [determined from the normalization integral (2.2d)] will have the correct dependence (that is, proportional to $B_\Lambda^{\frac{1}{2}}$) on B_Λ for sufficiently small B_Λ.

These remarks on the defects of the trial function (1.2) apply more strongly to its use for the $T=1$ three-body hypernuclear systems with $B=0$ than they do to the $T=0$ hypertriton. For the $T=1$ systems, the use of the wave function (5.4) with $\alpha=\alpha_3=0$ should result in some further reduction (perhaps by several percent) in the estimate of the critical value $U_2'^*$. The values obtained for $U_2'^*$ on the basis of the function (1.2), however, are already between 15 and 20% higher than the values obtained for U_2 even with the assumption of a value for B_Λ (1 Mev) above the range at present consistent with the ${}_\Lambda H^3$ data. Even if there were no spin dependence in the two-body Λ-nucleon interaction, this result implies that the existence of a bound state is highly improbable for ${}_\Lambda n^3$; and the additional Coulomb repulsion makes the existence of a bound state for ${}_\Lambda He^3$ even less probable. Any spin dependence in the two-body Λ-nucleon interaction makes this conclusion stronger. Moreover, it appears unlikely that this conclusion would need any modification if three-body potentials due to pion exchange processes were to contribute appreciably to the interactions in the three-body hypernuclear systems.[21]

It is of interest to compare the values of U_2 obtained here with the values recently obtained[2] for U_3 and U_4 from the analyses of the ${}_\Lambda H^4$, ${}_\Lambda He^4$ doublet and of ${}_\Lambda He^5$. On account of the uncertainty of the value of B_Λ for the hypertriton, we can only say that it is unlikely that U_2 lies outside the range 407 to 430 Mev f³ for $\kappa=m_K$ and outside 621 to 700 Mev f³ for $\kappa=2m_\pi$. A further uncertainty in this comparison arises from the fact that the calculations of U_n for the heavier hypernuclei were carried out for a Λ-nucleon potential of Gaussian form, whereas the use of a Yukawa potential was more convenient for the calculations of U_2 reported in this paper. In column (xii) of Table I volume integrals U_2 are given for the Gaussian potentials equivalent to the Yukawa potentials used here, in that they have the same intrinsic range and give rise to the same zero-energy scattering length. It is with the volume integrals U_2 of these equivalent Gaussian potentials that the following comparisons are made.

Neglecting the possible contribution of three-body potentials, the volume integral U_4 of the total Λ-nucleon interaction in ${}_\Lambda He^5$ is

$$U_4=3\bar{V}_p+\bar{V}_a. \tag{5.8}$$

The expressions for U_2 in terms of $\bar{V}_p$ and $\bar{V}_a$ are given by (3.1a) for $\bar{V}_p>\bar{V}_a$ and by (3.1b) for $\bar{V}_a>\bar{V}_p$. The values of $\bar{V}_p$ and $\bar{V}_a$, deduced from expressions (3.1) and (5.8), are given in Table II. The value $s_>$ of the well-depth parameter of the more attractive potential is also given in the table; the improved estimates of U_2 have led to a reduction in the values of $s_>$ by 10–20% from the values previously obtained.[2] From the present results it appears that the existence of a bound state of the Λ-nucleon system is strongly excluded.[22] The essential features of the spin dependence shown in Table II are the same as those of the previous work, although the spin dependence is now somewhat weaker. The values expected for the volume integral U_3 of the total Λ-nucleon interaction in the ${}_\Lambda H^4$, ${}_\Lambda He^4$ doublet are

$$U_3=(U_4+U_2)/2 \quad \text{for} \quad \bar{V}_p>\bar{V}_a, \tag{5.9a}$$

$$U_3=3(U_4+2U_2)/8 \quad \text{for} \quad \bar{V}_a>\bar{V}_p. \tag{5.9b}$$

[21] Weitzner[3] and Spitzer[4] have found that, in the lowest approximation of meson theory, the central, static three-body potential between a Λ particle and two nucleons has the form $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 D(r_1,r_2)$. This potential may contribute as much as 10 or 15% to the value of U_2. Since $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2=-3$ for both $T=0$ and $T=1$ S states, however, the contribution of this three-body potential to $U_2'^*$ will have the same sign as and magnitude comparable with its contribution to U_2, so that the above discussion would be essentially unaffected by the presence of such a three-body potential. In the next approximation of meson theory, there will arise three-body potentials whose sign will be opposite for $T=0$ and $T=1$ states; but these are potentials of shorter range whose contributions to U_2 and $U_2'^*$ will be an order of magnitude less than that of the simplest three-body potential.

[22] T. Truong, Phys. Rev. (to be published) has found that the presence of hard cores in the Λ-nucleon and nucleon-nucleon interactions in the hypertriton sensibly increases the predicted value of $s_>$. The conclusion that no bound state exists for the Λ-nucleon system would not be affected unless the hard core radii were greater than 0.6 f. See also the previous work of D. B. Lichtenberg, Nuovo cimento **8**, 463 (1958), which is an adaptation to the hypertriton of the work of Kikuta *et al.* [Kikuta, Morita, and Yamada, Progr. Theoret. Phys. (Japan) **15**, 222 (1956)] on the nuclear three-body system with hard cores.

The values predicted by (5.9), based on the present values of U_2, are not in disagreement with the values obtained for U_3 from the earlier analysis[2] for either $\bar{V}_p > \bar{V}_a$ or $\bar{V}_a > \bar{V}_p$.[23] The evidence from the ${}_\Lambda H^4$, ${}_\Lambda He^4$ doublet, in addition to that from ${}_\Lambda H^3$ and ${}_\Lambda He^5$, therefore does not allow a decision to be made as to whether the singlet or triplet Λ-nucleon interaction is the stronger. Such a decision can, however, be made on the basis of the observed mesonic decays of ${}_\Lambda H^4$ and ${}_\Lambda H^3$. From the high proportion of two-body decay events it can be concluded[24] that the ground-state spins of ${}_\Lambda H^4$ and ${}_\Lambda H^3$ are 0 and $\frac{1}{2}$, respectively. This means that $\bar{V}_a > \bar{V}_p$ and that the well-depth parameters $s=0.72$ ($\kappa = m_K$) and 0.69 ($\kappa = 2m_\pi$) of Table II are appropriate to the ${}^1S$ Λ-nucleon state.[22]

It is appropriate to emphasize the uncertainty in these remarks on the Λ-nucleon potential which arises from the possibility of three-body potentials. Although three-body potentials may have a moderately small effect for ${}_\Lambda H^3$, contributing u (say) to U_2, their effect may be appreciable for ${}_\Lambda He^5$, with a contribution of at least $6u$ to U_4. The existence of moderate three-body potentials could, therefore, appreciably modify our conclusions on the amount of the spin dependence of the Λ-nucleon interaction. A repulsive three-body potential would mean that the values of U_4 and U_2 could be accounted for with less spin dependence in the two-body Λ-nucleon potential, and *vice versa*. The essential conclusion of this paper, that the two-body potential is not strong enough to bind the Λ-nucleon system, could seriously be brought into question only in the case $\bar{V}_a > \bar{V}_p$ and then only if the three-body potential were so strongly attractive that the ${}_\Lambda He^5$ binding energy required a very weak or repulsive ${}^3S$ Λ-nucleon potential.

One question which depends sensitively on the existence and character of three-body potentials is that of the existence of an excited bound state ${}_\Lambda H^{4*}$. This question is primarily of interest for the case $\bar{V}_a > \bar{V}_p$, where the spin of the ground state of ${}_\Lambda H^4$ is zero (see reference 2). In this case, the potential appropriate to the interaction between Λ and H^3 in the $J=1$ state is measured by the volume integral[25] U_3':

$$U_3' = U_3 - (\bar{V}_a - \bar{V}_p). \tag{5.10}$$

There will then be a bound state ${}_\Lambda H^{4*}$ with $J=1$ only if U_3' exceeds a certain critical value U_3^* [given in reference 2 as U_3 $(B_\Lambda = 0)$]. The greater the spin dependence of the two-body Λ-nucleon potential, the smaller is the value of U_3' and the less likely is the existence of a bound state ${}_\Lambda H^{4*}$. The spin dependence found here neglecting three-body potentials would mean that the existence of this bound state is almost certain, especially for a triton radius R_3 smaller than the radius of the alpha particle. The presence of a repulsive three-body potential would reduce this spin dependence and require an ${}_\Lambda H^{4*}$ state with an appreciable binding energy. On the other hand, the presence of an attractive three-body potential, such as that computed by Spitzer,[4] would call for a stronger spin dependence in the two-body potential and would make it unlikely that a $J=1$ bound state should exist.

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APPENDIX

The coefficients $T_{\alpha\beta\gamma,\alpha'\beta'\gamma'}$ required in the expression (2.5) for $T(\psi,\psi)$ and the corresponding coefficients required in the corresponding expressions for $u(\psi,\psi)$, $v(\psi,\psi)$, and $N(\psi,\psi)$ can all be expressed in terms of the algebraic functions

$$I_{lmn} = (-1)^{l+m+n} \times \left(\frac{\partial}{\partial A}\right)^l \left(\frac{\partial}{\partial B}\right)^m \left(\frac{\partial}{\partial C}\right)^n I_{000}(ABC), \tag{A1}$$

where

$$I_{000}(ABC) = \int e^{-(Ar_1+Br_2+Cr_3)} dr_1 dr_2 dr_3 = \frac{2}{(A+B)(B+C)(C+A)}. \tag{A2}$$

[23] The values of U_3 obtained from ${}_\Lambda H^4$, ${}_\Lambda He^4$ depend considerably on the rms radius R_3 assumed for the H^3, He^3 core of these hypernuclei, a parameter for which there is no direct measurement yet available. With the range $1/m_K$ for the Λ-nucleon potential, values of 600 and 695 Mev f^3 were obtained for U_3 with values of R_3 of 1.38 f and 1.58 f, respectively; with the range of $1/(2m_\pi)$, the corresponding values of U_3 were 820 and 915 Mev f^3. The value of U_3 predicted by (5.9b) for $\bar{V}_a > \bar{V}_p$ (the case which appears to be appropriate to the actual situation) would require a value of R_3 close to the smaller of the two values considered. More adequate allowance for the distortion of the H^3, He^3 core by the presence of the Λ particle than was made in reference 2 would lead to some reduction in the values of U_3 obtained there. It is possible that such improved values would agree satisfactorily with the U_3 predicted for $\bar{V}_a > \bar{V}_p$, even for a value of R_3 the same as or larger than the rms radius of He^4.

[24] A qualitative discussion of the two-body decay of ${}_\Lambda H^4$ was given in reference 2. A quantitative calculation for the two systems has been made by R. H. Dalitz, Phys. Rev. **112**, 605 (1958), and for ${}_\Lambda H^3$ by M. Leon, Phys. Rev. (to be published).

Note added in proof.—Further investigation of the effects of the Pauli principle on the ${}_\Lambda H^4$ decay modes by R. H. Dalitz and L. Liu (report in preparation) shows that this conclusion cannot be based on the ${}_\Lambda H^4$ branching ratio data alone. If further arguments concerning the branching ratio for nonmesonic modes of ${}_\Lambda He$ decay are accepted, it appears probable that this conclusion is still valid.

[25] Note that a three-body Λ-nucleon potential of the form corresponding to the simplest possible pion exchange processes will contribute equally to U_3 and U_3' (see footnote 21).

The required coefficients are

$$T_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}=\frac{1}{2M}\left\{2a_{3\gamma}a_{3\gamma'}+\left(1+\frac{M}{M_\Lambda}\right)(a_{1\alpha}a_{1\alpha'}+a_{2\beta}a_{2\beta'})\right\}I_{111}(A,B,C)$$
$$+\frac{1}{4M}(a_{2\beta}a_{3\gamma'}+a_{3\gamma}a_{2\beta'})\{I_{120}(A,B,C)+I_{102}(A,B,C)-I_{300}(ABC)\}$$
$$+\frac{1}{4M}(a_{3\gamma}a_{1\alpha'}+a_{1\alpha}a_{3\gamma'})\{I_{012}(A,B,C)+I_{210}(A,B,C)-I_{030}(ABC)\}$$
$$+\frac{1}{4M_\Lambda}(a_{1\alpha}a_{2\beta'}+a_{2\beta}a_{1\alpha'})\{I_{201}(ABC)+I_{021}(ABC)-I_{003}(ABC)\};\quad\text{(A3)}$$

$$v_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}=I_{110}(A,B,\,C+\kappa_3)/\kappa_3;\quad\text{(A4)}$$

$$u_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}=\{I_{011}(A+\kappa,\,B,C)+I_{101}(A,\,B+\kappa,\,C)\}/\kappa;\quad\text{(A5)}$$

$$N_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}=I_{111}(A,B,C).\quad\text{(A6)}$$

The arguments A, B, and C are combinations of the parameters $a_{i\alpha}$ given in Eq. (2.4):

$$A=a_{1\alpha}+a_{1\alpha'},\quad B=a_{2\beta}+a_{2\beta'},\quad C=a_{3\gamma}+a_{3\gamma'}.\quad\text{(A7)}$$

Explicit expressions are given here for the various I_{lmn} which appear in (A3)–(A6) and for the combinations of I_{lmn} which appear in (A3).

$$I_{111}(x,y,z)=8\{x(x+y)(x+z)+y(y+z)(y+x)$$
$$+z(z+x)(z+y)+2(x+y)(y+z)(z+x)\}$$
$$\times\{(x+y)^3(y+z)^3(z+x)^3\}^{-1}.\quad\text{(A8)}$$

The functions I_{110}, I_{101}, and I_{011}, which appear in (A4) and (A5), differ from one another only by permutations of their arguments, that is,

$$\begin{aligned}I_{011}(x,y,z)&=I_{110}(y,z,x)=I_{110}(z,y,x),\\ I_{101}(x,y,z)&=I_{110}(z,x,y)=I_{110}(x,z,y),\end{aligned}\quad\text{(A9)}$$

where

$$I_{110}(x,y,z)=4\{(x+y)(x+y+z)+(x+z)(y+z)\}$$
$$\times\{(x+y)^3(y+z)^2(z+x)^2\}^{-1}.\quad\text{(A10)}$$

Each of the combinations of I_{lmn} occurring in the second, third, and fourth terms of (A3) can be expressed as a single function which is no more complicated than any one of the I_{lmn} which it includes. These combinations are

$$\begin{aligned}I_{120}(A,B,C)+I_{102}(A,B,C)-I_{300}(ABC)&=J(B,C,A)=J(C,B,A),\\ I_{012}(A,B,C)+I_{210}(ABC)-I_{030}(A,B,C)&=J(C,A,B)=J(A,C,B),\\ I_{201}(A,B,C)+I_{021}(ABC)-I_{003}(ABC)&=J(A,B,C)=J(B,A,C).\end{aligned}\quad\text{(A11)}$$

where

$$J(x,y,z)=16\{(x^3+y^3)z^2+(x^2+y^2)(4z^3+xyz)$$
$$+(x+y)(4z^4+7xyz)+z^5+3x^2y^2z+10xyz^3\}$$
$$\times\{(x+y)^3(y+z)^4(z+x)^4\}^{-1}.\quad\text{(A12)}$$

The use of the function $J(x,y,z)$ reduces the number of functional forms necessary for the evaluation of all the coefficients to the three of (A8), (A10), and (A12).

A simplification in the explicit expressions for $T(\psi,\psi)$, $v(\psi,\psi)$, $u(\psi,\psi)$, and $N(\psi,\psi)$ can result from the evident symmetry of the functions I_{111}, I_{110}, and J if their arguments have common values. In the problem treated in this paper the set of values of A and B, given in Eqs. (A7) and (2.4) are the same. This reduced from 27 to 18 the number of coefficients $N_{\alpha\beta\gamma,\,\alpha'\beta'\gamma'}$ which had to be evlauated for $N(\psi,\psi)$. Similar reductions were possible for $T(\psi,\psi)$ and $v(\psi,\psi)$.