Evidence for Three-Body Vector Forces in Light Nuclei*

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The splittings of the P-doublet states of Li^7 and N^{15}, and the D-doublet states of O^{17}, are calculated with phenomenological two-body vector forces of both Gaussian and Yukawa shapes with arbitrary ranges and exchange character. It is found to be impossible to fit the experimental data with a two-body force of fixed strength, in agreement with Pearse's calculations based on a relativistic two-body vector force. The splittings are then calculated with a phenomenological three-body vector force of the type expected to arise from higher order effects of the tensor force. The predicted ratio of the P doublet splitting in N^{15} to that in Li^7 is found to be quite insensitive to the range or shape of the three-body force, and in excellent agreement with experiment, provided that the exchange character of the force is not chosen close to the Serber mixture. The O^{17} D-doublet splitting, while somewhat more sensitive to the choice of range and exchange mixture, can also be fitted simultaneously, with a wide variety of three-body vector potentials. An attempt is then made to derive the parameters of the three-body vector force from the Gammel-Thaler tensor force parameters, but the resultant force has the wrong exchange character, and is also much too weak, to fit the experimental data. This is mainly due to the Serber-like exchange character of the Gammel-Thaler tensor force. A moderate increase in the strength or range of the odd-state tensor force would give a satisfactory three-body vector force. The splitting of the ^3D states of Li^6 is also examined. Here one-, two-, and three-body vector forces all predict too small a splitting, when the strength of the interaction is normalized to the Li^7 P-doublet splitting. For the three-body vector force, however, there exists the possibility that the strength parameter is greater for Li^6 than for Li^7, which would improve the fitting.

I. INTRODUCTION AND SUMMARY

THE original nuclear shell model[1] used, mainly for simplicity, a one-body spin-orbit force, analogous to the atomic spin-orbit force. It was soon recognized that to obtain agreement with the observed level separations, particularly in the p-shell nuclei,[2] one had to assume that the strength of the one-body spin-orbit force increased steadily with A, the mass number. Two-body vector forces,[3] as has been emphasized by Elliott and Lane,[4] do lead to an effective one-body force whose strength does increase with A, and for large A in the manner required by the experimental data.[5] A two-body vector force is indeed necessary to explain the high-energy nucleon-nucleon scattering and polarization data.[6] However, both the Gammel-Thaler and Signell-Marshak forces seem too weak to account for the shell model.[7]

In a careful analysis of the nuclei Li^7, N^{15}, and O^{17}, Pearse[8] found that it was impossible to fit the doublet splittings of these three nuclei with a two-body vector force of fixed strength. The experimental ratio of the P-doublet splitting in N^{15} to that in Li^7 is 13.3, and the ratio of the D-doublet splitting in O^{17} to P-doublet splitting in N^{15} is 0.80.[9] In Sec. II of this paper the theoretical predictions for these ratios for two-body vector forces of various shapes, ranges, and exchange character is reviewed. The optimum values predicted by a two-body force for the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios are 4.5 and 1.4, respectively, in rough agreement with Pearse's calculations. A one-body force of fixed strength predicts the values 3 and 5/3, respectively. While the two-body force predictions are an improvement over the one-body predictions, they are not significantly better.

It is well known that the tensor force will lead to doublet splittings in second and higher order,[10] i.e., to an effective vector spin-orbit force that might be responsible for the shell model.[11–15] Such an effective vector force contains both two-body and three-body terms, as well as terms of higher order. Recent calculations[16] based on the Brueckner formulation of the many-body problem,[17] indicate that the two-body vector force obtained from the tensor force is much too small, or has the wrong sign, to account for the shell model. A crude perturbation-variational calculation by the author[12] indicated, however, that the effective three-body vector

* A preliminary report of this work was presented by the author at the 1957 Stanford meeting of the American Physical Society [Bull. Am. Phys. Soc. Ser. II, 2, 392 (1957)].

[1] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1776 (1949).

[2] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953); D. Kurath, Phys. Rev. **101**, 216 (1956).

[3] C. H. Blanchard and R. Avery, Phys. Rev. **81**, 35 (1951); J. Hughes and K. T. LeCouteur, Proc. Phys. Soc. (London) **A63**, 1219 (1950).

[4] J. P. Elliott and A. M. Lane, Phys. Rev. **96**, 1160 (1954).

[5] R. J. Blin-Stoyle, Phil. Mag. **46**, 973 (1955); A. Schroder, Nuovo cimento **7**, 461 (1958).

[6] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957); P. S. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958).

[7] M. Moshinsky, Phys. Rev. **109**, 933 (1958); B. P. Nigam and M. K. Sundaresan, Phys. Rev. **111**, 284 (1958).

[8] C. A. Pearse, Phys. Rev. **106**, 545 (1957).

[9] Experimental data are taken from the compilation of F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

[10] S. M. Dancoff, Phys. Rev. **58**, 326 (1940).

[11] J. Keilson, Phys. Rev. **82**, 759 (1951).

[12] A. M. Feingold, Phys. Rev. **101**, 258 (1956).

[13] D. H. Lyons, Phys. Rev. **105**, 936 (1957).

[14] A. M. Feingold, Phys. Rev. **105**, 944 (1957).

[15] L. S. Kisslinger, Phys. Rev. **104**, 1077 (1956).

[16] B. Jancovici, Phys. Rev. **107**, 631 (1957); Nuovo cimento **7**, 290 (1958); B. P. Nigam and M. K. Sundaresan, Can. J. Phys. **36**, 571 (1958); Brueckner, Gammel, and Weitzner, Phys. Rev. **110**, 431 (1958).

[17] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955); K. A. Brueckner, Phys. Rev. **100**, 36 (1955); Brueckner, Eden, and Francis, Phys. Rev. **99**, 76 (1955); H. A. Bethe, Phys. Rev. **103**, 1353 (1956); K. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

force derived from the tensor force may have the correct strength, and sign, to give the observed shell-model splittings.

It therefore seemed worthwhile to see what a three-body vector force would predict for the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios. For this purpose phenomenological three-body vector forces of the type given by the theory of references 12 and 14 were used, with Yukawa or Gaussian radial shapes, and arbitrary ranges and exchange character. Harmonic oscillator shell model wave functions are used. Section III includes some general theory of the three-body vector force, and the results of the calculations are given in Sec. IV. Excellent agreement with the experimental N^{15}/Li^7 ratio is found, practically independent of the shape or range of the vector force, provided the exchange character does not approach the Serber mixture. The predicted O^{17}/N^{15} ratio is somewhat more dependent on the shape and range of the force, but for both Gaussian and Yukawa shapes, a wide choice of ranges and exchange character exists where agreement is found with the experimental value.

An attempt to derive the characteristics of the three-body vector force from the Gammel-Thaler[6] tensor parameters, using the theory of references 12 and 14, is given in Sec. V. The resultant three-body force has the wrong exchange character, and is also much too weak, to fit the data. This is mainly due to the Serber-like exchange character of the Gammel-Thaler tensor force. Increasing the strength or range of their odd-state tensor force would lead to a satisfactory three-body vector force. Section VI contains a general discussion of the results and also a discussion of the ^3D splittings in Li^6.

II. ONE- AND TWO-BODY VECTOR FORCES

A one-body vector force of the form $\xi \mathbf{l}_i \cdot \mathbf{s}_i$, where ξ is a constant, predicts that the P-doublet splitting of N^{15} should be equal (but inverted) to that of He^5. Experimentally the N^{15} splitting is about twice that of He^5 (6.33 Mev compared to ~ 3 Mev), and it was mainly for this reason that Elliott and Lane[4] preferred the two-body vector force which, for a reasonable choice of range and shape, and assuming the same single-particle radius parameter for He^5 and N^{15}, does predict a ratio of 2 for the splittings. However, the use of He^5 in determining the nature of the vector force is undesirable due to its heavy-particle instability and the consequent uncertain "radius" to be used for it, the predicted level splitting for a two- or three-body force being quite sensitive to the choice of nuclear radius (see Fig. 1).

Due to this sensitivity of the splitting on the value of the nuclear radius, it is important to have accurate values (or rather accurate relative values) of the radii of Li^7, N^{15}, and O^{17}. According to the Coulomb energy analysis of the mirror nuclei by Carlson and Talmi,[18] the harmonic oscillator parameter is essentially identical for all three nuclei. Using harmonic oscillator single-particle wave functions with radial dependence $r^l \exp(-\alpha^2 r^2)$, and assuming that α is the same for the s, p, and d particles in a given nucleus, the Carlson-Talmi analysis, using the most recent Coulomb energies given by Kofoed-Hansen,[19] gives $\alpha^{-1}(Li^7)=2.31$, $\alpha^{-1}(N^{15})=2.34$, and $\alpha^{-1}(O^{17})=2.37$ fermis. Use of the exact harmonic oscillator Coulomb energy formulas for the p shell,[20] and the corresponding formula for O^{17} given by Pearse,[8] yields identical results.[21] High-energy electron scattering data,[22] when interpreted in terms of harmonic oscillator wave functions, yields a somewhat larger value for $\alpha^{-1}(Li^7)$.[23] We adopt the value $\alpha^{-1}=2.34$ fermis for all three nuclei, Li^7, N^{15}, and O^{17}.

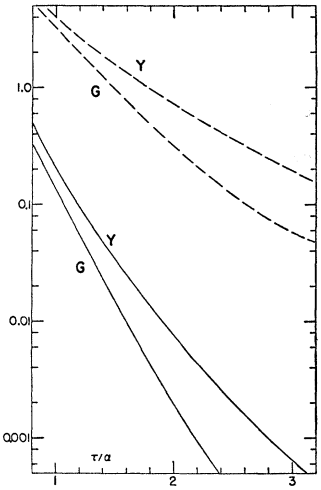

FIG. 1. The P doublet splitting in He^5, in units of V_0, as a function of τ/α for two-body vector forces (dashed lines) and three-body vector forces (solid lines). Curves labeled G and Y are for Gaussian- and Yukawa-shaped forces, respectively. The curves are shown for nonexchange forces. For two-body vector forces characterized by the exchange parameter a [Eq. (1)], the ordinate should be multiplied by the factor $(1-a)$. For three-body vector forces characterized by the exchange parameter g, the ordinate should be multiplied by the factor $[1+(18/5)g(g-1)]$.

The most general two-body charge-independent vector force linear in the momentum of the particles is[24]

$$[V(r_{12})+V'(r_{12})P_{12}][(\mathbf{r}_{12}\times\mathbf{p}_{12})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)]/\hbar,$$

where V and V' are arbitrary scalar functions of the nucleon separation, r_{12}, P_{12} is the Majorana space exchange operator, and $\mathbf{r}_{12}$ and $\mathbf{p}_{12}$ are the relative position and momentum vectors for the two nucleons. We shall consider the more restricted form

$$V(r_{12})[1+aP_{12}][(\mathbf{r}_{12}\times\mathbf{p}_{12})\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)]/\hbar, \qquad (1)$$

a being an adjustable parameter, and $V(r_{12})$ having a

[18] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).

[19] O. Kofoed-Hansen, Revs. Modern Phys. **30**, 449 (1958).

[20] E. Feenberg and E. P. Wigner, Phys. Rev. **51**, 95 (1937).

[21] Due apparently to a numerical error, Pearse obtains a considerably smaller value for $\alpha^{-1}(N^{15})$, and thus concludes that the d particle in O^{17} has a much larger value of α^{-1}. This error does not affect any of his other conclusions.

[22] J. F. Streib, Phys. Rev. **100**, 1797 (1955); R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

[23] A serious discrepancy is the fact that the Coulomb energies indicate that $\alpha^{-1}(Li^6)$ is some 20% larger than $\alpha^{-1}(Li^7)$, while the electron scattering data implies only a 3% difference. See W. M. Visscher and R. A. Ferrell, Phys. Rev. **107**, 781 (1957), for a discussion of this point.

[24] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **27**, 281 (1941).

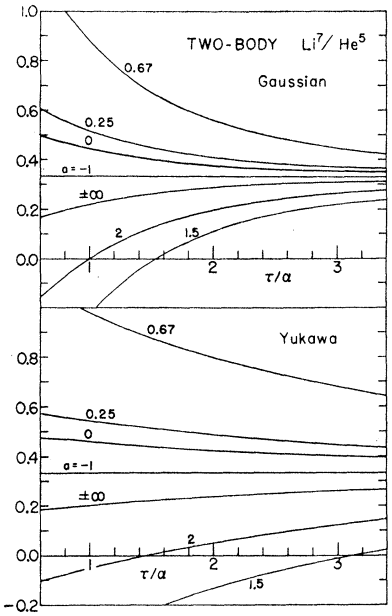

FIG. 2. Ratio of the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in Li^7 to the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in He^5 as a function of τ/α and the exchange parameter a, for the two-body vector force of Eq. (1). The upper set of curves is for a Gaussian-shape force, while the lower set is for a Yukawa shape. The curves are used with Fig. 1 to fix the scale of the Li^7 P doublet splitting.

Gaussian or Yukawa radial dependence:

$$V^G(r_{12}) = V_0{}^G \exp(-\tau^2 r_{12}{}^2),$$
$$V^Y(r_{12}) = V_0{}^Y e^{-\tau r_{12}}/\tau r_{12}.$$

The calculation of the doublet splittings in He^5, Li^7, N^{15}, and O^{17} due to such a potential is straightforward,[25] and the results are shown graphically in Figs. 1–4. The splittings were calculated in first order only. While this is exact for He^5, N^{15}, and O^{17}, it involves the assumption of pure LS coupling for Li^7, which presumably is close to the truth.[2]

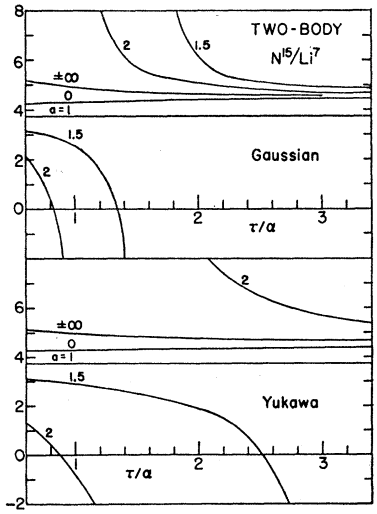

FIG. 3. Ratio of the $^2P_{\frac{1}{2}}$-$^2P_{\frac{3}{2}}$ splitting in N^{15} to the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in Li^7 as a function of τ/α and the exchange parameter a, for two-body vector forces of Gaussian and Yukawa shape. The experimental value of the ratio is 13.3.

The dashed curves in Fig. 1 give the He^5 P-doublet splitting as a function of the parameter τ/α for the two-body vector force. Due to the unstable nature of He^5, Fig. 1 is intended only to show the sensitivity of the splitting to the choice of τ/α, and, together with Figs. 2–7, to determine the absolute magnitudes of the splittings in Li^7, N^{15} and O^{17}. The curve for the two-body Yukawa potential in Fig. 1 was originally given by Elliott and Lane.[4] For large values of τ/α (δ-function potential) the splitting is inversely proportional to the fifth power of τ/α. The operator P_{12} is, for He^5, equivalent to multiplication by the factor -1. Hence for fixed V_0, τ, and α, the He^5 splitting is proportional to $(1-a)$, the parameter a determining the exchange character of the potential (1). The He^5 splitting thus vanishes for a Serber mixture ($a=+1$).[26] The Li^7, N^{15}, and O^{17} doublet splittings vanish for different τ/α values for a near $+1$, and this is the cause of the wild fluctuations in the curves of Figs. 2–4 for such a values. The value $a=2$ corresponds to the symmetric mixture, $\tau_1 \cdot \tau_2$, discussed by Elliott and Lane.[4]

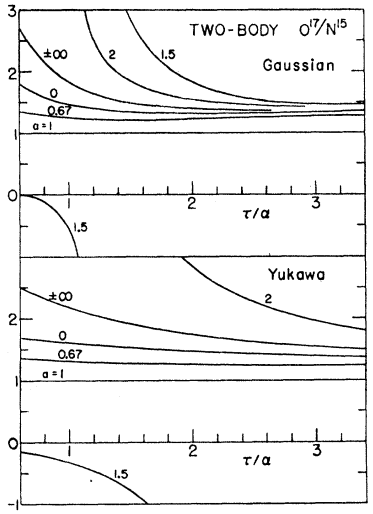

FIG. 4. Ratio of the $^2D_{5/2}$-$^2D_{\frac{3}{2}}$ splitting in O^{17} to the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in N^{15} as a function of τ/α and a for two-body vector forces of Gaussian and Yukawa shape. The experimental value of the ratio is 0.80.

The curves for the Li^7/He^5 doublet splitting ratio, Fig. 2, should not of course be compared directly with the experimental value of ~ 0.2, since the curves are computed on the basis of the same value of α for both nuclei, but serve mainly to fix the scale of the Li^7 splitting when used together with Fig. 1. Correction for the 3% difference in the value of α going from Li^7 to O^{17} would reduce the values of the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios given in Figs. 3 and 4 by at most 8%. The experimental D-doublet splitting in O^{17} of 5.08 Mev should perhaps be increased to allow for a possible Ehrmann-Thomas shift of the upper level.[27] If we take 1 Mev as an arbitrary upper limit for this shift, then the "experimental" O^{17}/N^{15} doublet ratio

[25] See J. P. Elliott and A. M. Lane, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. 39, for the general techniques for calculating the matrix elements involved. Specific calculations using a potential of the form (1) have been given by Elliott and Lane (reference 4); G. Abraham, Nuclear Phys. 1, 415 (1956); Visscher and Ferrell (reference 23).

[26] Compare with K. M. Case and A. Pais, Phys. Rev. 80, 203 (1950).

[27] W. M. Visscher and R. A. Ferrell, reference 23.

would be increased from 0.80 to 0.96. It is seen from an examination of the computed N^{15}/Li^7 and O^{17}/N^{15} ratios of Figs. 3 and 4 that for either Gaussian or Yukawa shape two-body vector potentials, it is impossible, for any choice of τ and a, to fit simultaneously the experimental values of the ratios. This is in agreement with Pearse's conclusion, based on a relativistic two-body vector force,[8] that a two-body vector force of fixed strength cannot explain the Li^7, N^{15}, and O^{17} doublet splittings.

III. THREE-BODY VECTOR FORCE

Consider a tensor force of the form

$$t_{12}=T_0 f(r_{12})S_{12}\chi_{12}, \tag{2}$$

where S_{12} is the tensor operator,

$$S_{12}=3(\mathbf{r}_{12}\cdot\boldsymbol{\sigma}_1)(\mathbf{r}_{12}\cdot\boldsymbol{\sigma}_2)/r_{12}^2-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),$$

T_0 is the strength of the potential, $f(r_{12})$ is the shape, which again we take to be either Gaussian, $\exp(-\tau^2 r_{12}^2)$, or Yukawa, $e^{-\tau r_{12}}/\tau r_{12}$, and χ_{12} describes the exchange character, $\chi_{12}=(1-g)+gP_{12}$, where P_{12} is the Majorana space exchange operator and g is a mixing parameter. The exchange character has been so written that the effect of the tensor force on the deuteron ground state is independent of g. According to the theory of reference 14, such a tensor force will give rise to an effective three-body vector force of the form

$$\begin{aligned}\mathcal{U}&=\sum_{ijk}\chi_{ij}\mathcal{U}(i,jk)\chi_{ik},\\ \mathcal{U}(1,23)&=V_0 f(r_{12})f(r_{13})[(\mathbf{r}_{12}\cdot\mathbf{r}_{13})(\mathbf{r}_{12}\times\mathbf{r}_{13})/r_{12}^2r_{13}^2]\\ &\quad\cdot\{(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)+\tfrac{1}{5}i[4\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)\\ &\qquad-\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_1)-\boldsymbol{\sigma}_3(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]\}.\end{aligned} \tag{3}$$

The sum in (3) is to be extended over all triples of nucleons i, j, k. For the nonexchange case ($g=0$), $\chi_{12}=\chi_{13}=1$ and the spin terms in (3) that are multiplied by the factor $\frac{1}{5}i$ may be omitted.[14]

According to the theory of reference 14, the strength parameter V_0 is given by

$$V_0=-(9/2)T_0^2/\bar{E}, \tag{4}$$

where $\bar{E}$ is the "average" excitation energy of high-lying states connected to the low states via the tensor force.[28] From the work of references 12 and 13, the calculated value of V_0 was too small to account completely for the multiplet splittings in Li^6 and Li^7. However, Tamura[29] has shown that by slightly modifying the wave functions, considerably larger effects can occur.[30] In the present paper we are primarily concerned with the ratios of the doublet splittings in different nuclei, and thus shall consider V_0 as an adjustable parameter so chosen as to fit, say, the N^{15} P-doublet splitting.

A serious difficulty, however, is that in principle $\bar{E}$, and hence V_0, are functions of the mass number, A. This dependence is difficult to estimate, but from a review of earlier calculations on the nuclei from He^4 through Li^7,[12] it appears to be small. This possible dependence will be unimportant for the O^{17}/N^{15} doublet splitting ratio but might be considerable for the N^{15}/Li^7 ratio. $\bar{E}$, and hence V_0, can also be expected to depend on the type of nucleus, i.e., whether $A=4n$, $4n\pm1$, or $4n+2$, since the supermultiplet structure[31] will almost certainly affect $\bar{E}$. Since He^5, Li^7, N^{15}, and O^{17} are all of the same type, this effect will not occur here. Some possible evidence for such a structure effect will be discussed below in Sec. VI in connection with Li^6. $\bar{E}$ can also be expected to depend on the parameters α and τ. Earlier calculations[12] indicate that this dependence is much smaller than the α and τ dependence of $\mathcal{U}$ itself (with fixed V_0). We can therefore safely neglect this source of variation in V_0. For most of what follows, we shall use (3) as the form of the three-body vector force, with V_0 constant.

For the calculations on He^5, N^{15}, and O^{17} we shall use the $nlm_lm_sm_\tau$ representation, as this makes the application of hole theory particularly simple. Consider such a state in a nucleus consisting of closed shells (total number of particles m) plus ϵ particles in an incomplete shell. A properly antisymmetrized wave function for the state A is then

$$\psi_A=\frac{1}{(A!)^{\frac{1}{2}}}\sum_P(-1)^p P u_1(1)\cdots u_{m+\epsilon}(m+\epsilon),$$

where the u's are single-particle wave functions, the subscripts being particle labels, the numbers in parentheses the state labels, P represents a permutation of the particle labels, and p is the signature of the permutation. The diagonal matrix element of a three-body operator, $\mathcal{O}=\sum_{ijk}\mathcal{O}(i,jk)$, is then

$$\begin{aligned}\mathcal{O}_{AA}&=\langle u_1(1)\cdots u_{m+\epsilon}(m+\epsilon)\,|\,\mathcal{O}\,|\\ &\qquad\times\sum_P(-1)^p P u_1(1)\cdots u_{m+\epsilon}(m+\epsilon)\rangle\\ &=\sum_{ijk}\langle u_1(i)u_2(j)u_3(k)\,|\,\mathcal{O}(1,23)\,|\\ &\qquad\times\sum_P(-1)^p P u_1(i)u_2(j)u_3(k)\rangle\\ &=\tfrac{1}{6}\sum_{ijk}\langle u_1(i)u_2(j)u_3(k)\,|\,\mathcal{O}'\,|\\ &\qquad\times\sum_P(-1)^p P u_1(i)u_2(j)u_3(k)\rangle\\ &=\sum_{i<j<k}\langle u_1(i)u_2(j)u_3(k)\,|\,\mathcal{O}'\,|\\ &\qquad\times\sum_P(-1)^p P u_1(i)u_2(j)u_3(k)\rangle\\ &\equiv\sum_{i<j<k}[ijk]=\tfrac{1}{6}\sum_{ijk}[ijk],\end{aligned}$$

[28] $\bar{E}$, as defined here, is identical with the expression $(\bar{E}-\varepsilon_0)$ of reference 14. The right-hand side of Eq. (25) of reference 14 should be divided by the factor $(\varepsilon_0-\bar{E})$.

[29] T. Tamura, Phys. Rev. **105**, 1808 (1957).

[30] Tamura's calculations specifically refer only to the singlet-triplet separation in Li^6 produced by the tensor force in second order, i.e., the scalar part of the operator t^2 (see references 12–14), and not to the effective vector operator. But it seems reasonable to expect similar effects on the strength of the vector operator.

[31] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

where $\mathcal{O}'=\sum_P P\mathcal{O}(1,23)$. Note that $[ijk]$ is independent of the ordering of i, j, and k. Now

$$\sum_{ijk=1}^{m+\epsilon}[ijk]=\sum_{ijk=1}^{m}+\sum_{ijk=m+1}^{m+\epsilon}+3\sum_{i=1}^{m}\sum_{jk=m+1}^{m+\epsilon}+3\sum_{i=m+1}^{m+\epsilon}\sum_{jk=1}^{m}.$$

For $\mathcal{O}=\mathcal{V}$, a three-body vector force, the sum over closed shells vanishes, leaving

$$\mathcal{V}_{AA}=\tfrac{1}{6}\{\sum_{ijk=m+1}^{m+\epsilon}+3\sum_{i=1}^{m}\sum_{jk=m+1}^{m+\epsilon}+3\sum_{i=m+1}^{m+\epsilon}\sum_{jk=1}^{m}\}. \quad (5)$$

For a single particle outside closed shells, such as He^5 and O^{17}, the first two sums in (5) vanish, leaving

$$\mathcal{V}_{AA}=\sum_{k>j=1}^{m}[m+1, jk]. \quad (6)$$

Hole Theory

Consider now the "complementary" state in the "complementary" nucleus, $A'=m+n-\epsilon$, where n is the total number of possible states in the incomplete shell, and where we have replaced the occupied states in the incomplete shell by holes, and vice versa. The $^2P_{\frac{3}{2}}$ states of N^{15} and He^5 are such a pair of complementary states, for example. The relationship between the diagonal matrix element of a three-body force for the complementary state and the original state in the nucleus, $A=m+\epsilon$, is readily found by an extension of the original method used by Shortley for one- and two-body forces.[32] We have

$$\begin{aligned}\mathcal{O}_{A'A'}&=\tfrac{1}{6}\sum_{ijk=1\cdots m,m+\epsilon+1}^{m+n}[ijk]\\
&=\tfrac{1}{6}\{\sum_{ijk=1}^{m}+\sum_{ijk=m+\epsilon+1}^{m+n}+3\sum_{i=1}^{m}\sum_{jk=m+\epsilon+1}^{m+n}+3\sum_{i=m+\epsilon+1}^{m+n}\sum_{jk=1}^{m}\}\\
&=\tfrac{1}{6}\{\sum_{ijk=1}^{m}+\sum_{ijk=m+1}^{m+n}+3\sum_{i=1}^{m}\sum_{jk=m+1}^{m+n}+3\sum_{ij=1}^{m}\sum_{k=m+1}^{m+n}\\
&\quad-\sum_{ijk=m+1}^{m+\epsilon}+3\sum_{i=1}^{m}\sum_{jk=m+1}^{m+\epsilon}-3\sum_{ij=1}^{m}\sum_{k=m+1}^{m+\epsilon}\\
&\quad-3\sum_{ij=m+1}^{m+n}\sum_{k=m+1}^{m+\epsilon}+3\sum_{i=m+1}^{m+n}\sum_{jk=m+1}^{m+\epsilon}\\
&\quad-6\sum_{i=1}^{m}\sum_{j=m+1}^{m+n}\sum_{k=m+1}^{m+\epsilon}\},\end{aligned}$$

where ϵ is the number of holes in the complementary nucleus, A'. Again for $\mathcal{O}=\mathcal{V}$, a three-body vector operator, the sums over closed shells vanish, leaving

$$\begin{aligned}\mathcal{V}_{A'A'}=-\mathcal{V}_{AA}+\{&\sum_{i=1}^{m}\sum_{jk=m+1}^{m+\epsilon}-\sum_{i=1}^{m}\sum_{j=m+1}^{m+n}\sum_{k=m+1}^{m+\epsilon}\\
&+\sum_{i=m+1}^{m+n}\sum_{j>k=m+1}^{m+\epsilon}-\sum_{i>j=m+1}^{m+n}\sum_{k=m+1}^{m+\epsilon}\}. \quad (7)\end{aligned}$$

For the case of a single hole in a closed shell, as in N^{15}, (7) reduces to

$$\begin{aligned}\mathcal{V}_{A'A'}=-\mathcal{V}_{AA}&-\sum_{k>j=m+2}^{m+n}[m+1, jk]\\
&-\sum_{k=1}^{m}\sum_{j=m+2}^{m+n}[m+1, jk]. \quad (8)\end{aligned}$$

For a one-body force, the two sums in (8) are absent and one has the well-known result that the one-body vector splitting for a hole is the same in magnitude, but opposite in sign to that for a particle. We shall use formula (6) to calculate the P- and D-doublet splittings in He^5 and O^{17}, respectively, and formula (8) to obtain the P-doublet splitting in N^{15}. The Li^7 P-doublet splitting is most easily computed by using the method of reference 12, where similar matrix elements (called V_1 and V_3 there) were computed.

Formulas for a two-body vector force, analogous to (5) and (7), are[33]

$$\mathcal{V}_{AA}=\sum_{i=1}^{m}\sum_{j=m+1}^{m+\epsilon}[ij]+\sum_{i>j=m+1}^{m+\epsilon}[ij], \quad (5')$$

$$\mathcal{V}_{A'A'}=-\sum_{i=1}^{m}\sum_{j=m+1}^{m+\epsilon}+\sum_{i>j=m+1}^{m+\epsilon}-\sum_{i=m+1}^{m+n}\sum_{j=m+1}^{m+\epsilon}, \quad (7')$$

where

$$[ij]=\langle u_1(i)u_2(j)\,|\,\mathcal{V}(12)\,|\,u_1(i)u_2(j)-u_2(i)u_1(j)\rangle.$$

The second sum in (5′) and (7′) vanishes for the case of a single particle (or hole) outside closed shells. The addition term present in (7′), which represents the interaction of the hole with the entire unfilled shell, is the origin of the larger hole splittings compared to single particle splittings for a two-body vector force, as found by Elliott and Lane.[4] For the three-body vector force, there are two additional sums, which have the same origin, in formula (8) for a hole as compared to a particle [formula (6)]. We can therefore expect that the ratio of hole to particle splittings will be larger for the three-body vector force than for a two-body vector force.[34]

[32] G. H. Shortley, Phys. Rev. **40**, 185 (1932); E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1935).

[33] Formulas (5′) and (7′), which follow directly from Shortley's calculations (reference 32) on the electrostatic energy, have also been given by D. M. Brink and G. R. Satchler, Nuovo cimento **4**, 549 (1956) and by Visscher and Ferrell (reference 23) using the second-quantization formalism.

[34] The different hole-particle relationship for a three-body central (scalar) force as compared to that for a two-body force has been used by S. P. Pandya and J. B. French, Ann. Phys. **2**, 166 (1957), as evidence for the existence of three-body scalar forces in the effective nuclear interaction.

IV. APPLICATION TO He^5, Li^7, N^{15}, AND O^{17}

Some details of the calculations of the P-doublet splittings in He^5, Li^7, and N^{15}, and the D-doublet splitting in O^{17}, are given in the Appendix. The results are given in Figs. 1, 5–7 as a function of τ/α and the exchange parameter g for both Yukawa- and Gaussian-shape three-body vector forces. The solid lines in Fig. 1 show the strong dependence of the He^5 P-doublet splitting on the parameter τ/α for the three-body force, inversely proportional to the tenth power of τ/α in the limit of a δ-function force ($\tau/\alpha\rightarrow\infty$), and inversely proportional roughly to the sixth power for the Yukawa shape with $\tau/\alpha=2$–3, which corresponds to the Gammel-Thaler tensor force parameters. This is roughly twice the power dependence on τ/α for the two-body vector force, and is not unexpected in view of the form (3), where $f(r)$ occurs twice. The He^5 splitting is proportional to $[1+(18/5)g(g-1)]$ in its dependence on the exchange parameter g, reaching its minimal value of one-tenth the nonexchange (or pure space-exchange) value for $g=\frac{1}{2}$, the Serber mixture. For $g<0$ or >1, the He^5 splitting increases indefinitely above the nonexchange value, but this of course would imply a tensor force stronger in odd states than in even states in the nucleon-nucleon problem.

The reason for the small splitting for near Serber mixtures can be made more explicit as follows. We can rewrite $\chi_{12}\mathcal{U}(1,23)\chi_{13}$ of Eq. (3) in the form

$$\chi_{12}\mathcal{U}(1,23)\chi_{13}=(1-g)^2\mathcal{U}(1,23)+g(1-g)[P_{12}\mathcal{U}(1,23)+\mathcal{U}(1,23)P_{13}]+g^2P_{12}\mathcal{U}(1,23)P_{13}. \quad (9)$$

Let us call $\mathcal{U}(1,23)$ the nonexchange term, $P_{12}\mathcal{U}(1,23)P_{13}$ the pure-exchange term, and $P_{12}\mathcal{U}(1,23)+\mathcal{U}(1,23)P_{13}$ the mixed-exchange term. For the states considered in this paper, detailed calculations (see Appendix) show that the pure-exchange term, when summed over all triples of nucleons, is exactly equivalent to the nonexchange term, so that (9) may be written

$$\chi_{12}\mathcal{U}(1,23)\chi_{13}=[g^2+(1-g)^2]\mathcal{U}(1,23)+g(1-g)[P_{12}\mathcal{U}(1,23)+\mathcal{U}(1,23)P_{13}]. \quad (10)$$

Since (10) is unchanged by the substitution $(1-g)$ for g, we see that the splittings are symmetric about the Serber value, $g=\frac{1}{2}$. For all the states we shall consider, the mixed-exchange term produces doublet splittings having the opposite sign to those produced by the nonexchange term, so that for g values between 0 and 1 cancellation occurs. For He^5, the mixed-exchange term, again when summed over all triples of nucleons, can be shown to be equal to $-8/5$ times the nonexchange term, independent of τ/α. Thus the minimum splitting for He^5 occurs at the Serber value, $g=\frac{1}{2}$. A similar situation occurs with the two-body vector force (see Sec. II).

Figure 5 serves mainly, when used with Fig. 1, to fix the scale of the Li^7 P-doublet splitting. Figure 6 gives the ratio of the P-doublet splitting in N^{15} to that in Li^7 for the three-body vector force of Eq. (3). It is seen that the predicted ratio is relatively insensitive to both the range and the shape of the force. It is also fairly insensitive to the exchange parameter, g, provided one stays away from the neighborhood of the Serber mixture, $g=\frac{1}{2}$. For $g=0$ or 1, for both Yukawa- and Gaussian-shape potentials, the predicted ratio varies from $\sim$12 at $\tau/\alpha=0$ to 17 at $\tau/\alpha=\infty$, compared to the experimental value of 13.3. Correction for the smaller value of α^{-1} for Li^7 as compared to that for N^{15}, would reduce the predicted value of the doublet splitting ratio by 15% for large τ/α values, and by a smaller amount for small τ/α values. Somewhat larger values of the

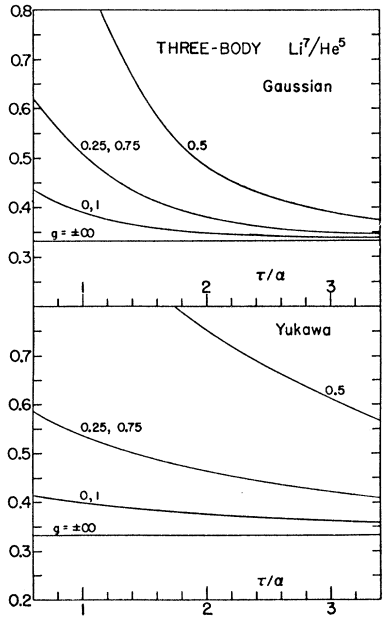

FIG. 5. Ratio of the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in Li^7 to the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in He^5 as a function of τ/α and the exchange parameter g, for three-body vector forces of Gaussian and Yukawa shape. The curves are used with Fig. 1 to fix the scale of the Li^7 P-doublet splitting.

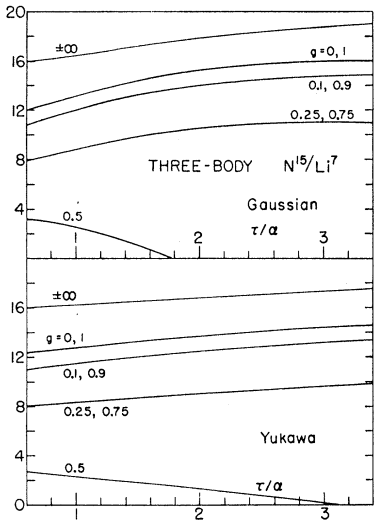

FIG. 6. Ratio of the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in N^{15} to the $^2P_{\frac{3}{2}}$-$^2P_{\frac{1}{2}}$ splitting in Li^7 as a function of τ/α and g, for three-body vector forces of Gaussian and Yukawa shape. The experimental value of the ratio is 13.3.

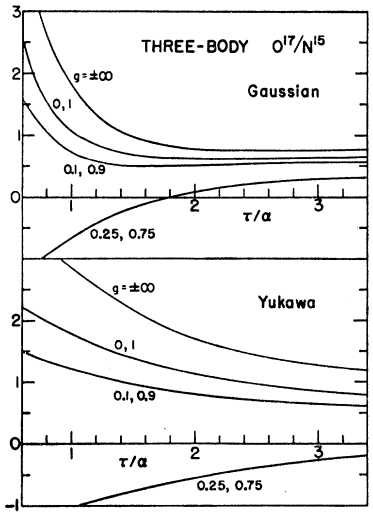

FIG. 7. Ratio of the $^2D_{5/2}$-$^2D_{\frac{3}{2}}$ splitting in O^{17} to the $^2P_{\frac{1}{2}}$-$^2P_{\frac{3}{2}}$ splitting in N^{15} as a function of τ/α and g for three-body vector forces of Gaussian and Yukawa shape. The experimental value of the ratio is 0.80.

ratio can be obtained by going to negative or large positive values of g.

The O^{17}/N^{15} doublet splitting ratio, given in Fig. 7, is somewhat more sensitive to the choice of the parameters τ/α and g, but again agreement with the experimental value of 0.8–1.0 can be obtained for both the Gaussian and Yukawa shapes provided g does not lie in the region $0.1<g<0.9$. Indeed for g near 0.5, the O^{17} D-doublet splitting has the wrong sign. One sees from Figs. 6 and 7 that for both the Yukawa and Gaussian shapes, there exist many g and τ combinations that will simultaneously fit the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios. Examples of three-body vector potentials that give good fits for these ratios are, for $g=0$ or 1, Gaussian: $V_0=-21$ Mev, $\tau^{-1}=1.95$ fermis; Yukawa: $V_0=-163$ Mev, $\tau^{-1}=1.17$ fermis. Here V_0 has been chosen to fit the N^{15} P doublet splitting.

V. GAMMEL-THALER TENSOR POTENTIAL

We have seen in the preceding section that a phenomenological three-body vector force can easily be chosen to fit the Li^7, N^{15}, and O^{17} doublet splittings simultaneously. Since, as discussed in Sec. III and reference 14, such a force can be expected to be a reflection of certain higher order effects of the tensor force, it is interesting to see whether the three-body vector force predicted from the tensor force in this manner is actually of the proper type to fit the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios.

It is clear that, if the theory of Sec. III and reference 14 is roughly correct, the tensor force of conventional meson theory[35] would yield a three-body vector force of incorrect exchange character. Meson theory yields a tensor force with the asymptotic exchange character, $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$, which corresponds to $g=\frac{2}{3}$. From Figs. 6 and 7, we see that such a choice of g gives results completely incompatible with experiment. It has the further disadvantage that the magnitude of the doublet splittings is reduced to a very small value—the He^5 splitting, for example, being reduced to $\frac{1}{5}$ the value obtained for $g=0$ or 1.

Similar results occur with the Gammel-Thaler[6] phenomenological potential, which contains a tensor force having an exchange character close to the Serber mixture. The theory of Sec. III cannot be applied directly to the Gammel-Thaler tensor potential since the odd and even tensor forces have different ranges. However a slight extension of the theory suffices. Let t_e be the even-state two-body tensor potential and t_o the odd-state tensor potential. Then the over-all tensor potential may be written

$$t=\tfrac{1}{2}[(t_e+t_o)+(t_e-t_o)P],$$

where P is the Majorana space-exchange operator. The three-body vector force resulting from such a two-body tensor force can then be written as

$$\begin{aligned}\mathcal{U}=&\tfrac{1}{2}[\mathcal{U}_{ee}(1,23)+\tfrac{1}{2}\{P_{12}\mathcal{U}_{ee}(1,23)+\mathcal{U}_{ee}(1,23)P_{13}\}]\\&+\tfrac{1}{2}[\mathcal{U}_{oo}(1,23)-\tfrac{1}{2}\{P_{12}\mathcal{U}_{oo}(1,23)+\mathcal{U}_{oo}(1,23)P_{13}\}]\\&+\tfrac{1}{4}[P_{12}\{\mathcal{U}_{eo}-\mathcal{U}_{oe}\}-\{\mathcal{U}_{eo}-\mathcal{U}_{oe}\}P_{13}],\end{aligned}\quad(11)$$

where $\mathcal{U}_{ee}$ is the vector force formed from t_e alone in both even and odd states, i.e., in Eq. (3), $f(r)$ is replaced by $f_e(r)$ and V_0 by V_0^e. According to Eq. (4), $V_0^e\propto(T_0^e)^2$, where T_0^e is the strength of t_e. Similarly, $\mathcal{U}_{oo}$ is the vector force formed in the same manner from t_o alone, with strength $V_0^o\propto(T_0^o)^2$, while $\mathcal{U}_{eo}$ is the vector force formed by taking $f(r_{12})=f_e(r_{12})$, $f(r_{13})=f_o(r_{12})$ in Eq. (3), with strength $\propto T_0^eT_0^o$. The first line of Eq. (11) will be recognized as just the three-body vector force resulting from placing $t_o=0$, i.e., it is the vector force arising from a Serber tensor force. The second line of (11) is the vector force arising from the "anti-Serber" force, placing $t_e=0$, while the third line of (11) represents an interference term. Due to the special symmetry properties of the interference term, it can be shown to vanish for all the states we are concerned with. (See the Appendix for the matrix elements of this term.)

The Gammel-Thaler tensor potential is of Yukawa shape, with parameters[36]

$$\begin{aligned}&T_0^e=-159.40\text{ Mev}, &&\tau_e=1.0494\times10^{13}\text{ cm}^{-1},\\&T_0^o=22.0\text{ Mev}, &&\tau_o=0.80\times10^{13}\text{ cm}^{-1}.\end{aligned}\quad(12)$$

The Gammel-Thaler potential also includes a repulsive core of radius $r_0=0.4\times10^{-13}$ cm, but we shall omit this cutoff since the vector splitting comes mainly from the tail of the potential (see Fig. 1). Since V_0^e is some 50× greater than V_0^o, we might expect the first line of (11) to dominate completely over the second line of (11). However, this is mitigated by two effects: Firstly, the

[35] L. Hulthén and M. Sugawara, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. 39.

[36] We use the parameters of the Gammel-Thaler potential given by Brueckner, Gammel, and Weitzner (reference 16).

larger range of t_e has a large effect due to the strong dependence of the vector splitting on τ/α (Fig. 1), and secondly, the nonexchange and mixed-exchange terms tend to cancel in the first line of (11), while they reinforce in the second line, the "anti-Serber" term, of (11). These two effects, however, are not quite sufficient to compensate for the large value of $V_0{}^e$ compared to $V_0{}^o$, and the results for the doublet splitting ratios due to the Gammel-Thaler tensor parameters of (12) are $N^{15}/Li^7=4.8$ and $O^{17}/N^{15}=-2.7$, i.e., the O^{17} doublet splitting has the wrong sign. These results are similar to what one would get using a tensor force of the form (2) with $g=\frac{2}{3}$, the same as the meson-theoretical tensor potential, too close to the Serber mixture to give acceptable results. For the same reason, the predicted Li^7 P-doublet splitting is only about $\frac{1}{10}$ the experimental value.

Of course the derivation of the three-body vector force from the nucleon-nucleon tensor force that we have used is extremely crude, and a more accurate theory might give an effective value of g further removed from the Serber value, thus giving better predicted values for the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios, and larger doublet splittings. In this connection it is interesting to note that doubling of the strength of $T_0{}^o$, or a decrease in τ_o from 0.80×10^{13} cm^{-1} to 0.62×10^{13} cm^{-1} in the Gammel-Thaler tensor potential, would give results in approximate agreement with experiment.

VI. CONCLUDING REMARKS

We have seen that while a one- or two-body vector force cannot explain the N^{15}/Li^7 and O^{17}/N^{15} doublet splitting ratios, a phenomenological three-body vector force does so in a manner remarkably independent of the shape, range, or exchange character chosen for the three-body vector force, provided one stays away from the Serber exchange mixture. Our attempt, however, to derive the three-body vector force so indicated from the nucleon-nucleon tensor parameters, which was the original motivation for introducing a three-body vector force, was not successful.

In view of the success of a phenomenological three-body vector force in fitting the N^{15}/Li^7 and O^{17}/N^{15} ratios, it would be desirable to extend the calculations to other nuclei. This is particularly simple in the case of Li^6. Here we can compare the 2.33-Mev splitting of the $3+$, $T=0$ and $2+$, $T=0$ states, which presumably are members of the ^3D multiplet, with, say, the P-doublet splitting in Li^7. For a one-body vector force of fixed strength, the predicted ratio would be 3, in the limit of LS coupling,[2,37,38] compared to the experimental value of 4.9. A complication here is that the tensor force would also split the members of the ^3D multiplet. However, according to the analysis of Pinkston and Brennan,[38] the effective tensor force in Li^6 must be much weaker than the nucleon-nucleon tensor force, so weak as not to affect the level structure of Li^6 appreciably.[39] We shall therefore omit the tensor force completely. It is perhaps significant that the same theory that predicts the three-body vector force, also predicts a weakened tensor force.[13,14]

The three-body vector force matrix elements for the ^3D states of Li^6 have been given by Lyons[13] (see also the Appendix). We find that, for $g=0$ or 1, the calculated value of the Li^6 splitting ratio is 2.9, not significantly different from the value predicted by a one-body force. A nonexchange two-body vector force gives the value 2.7. That all three types of vector force make approximately the same prediction is not surprising, since for such light nuclei as Li^6 and Li^7 most of the vector interaction is between the p particles and the closed s-shell, and this can be considered as due to an effective one-body vector force that should have essentially equal strengths in Li^6 and Li^7. The calculated ratio for the three-body vector force assumed that the single-particle radius parameter, α, is the same for both Li^6 and Li^7. Correction for the 3% difference shown by the electron-scattering data[22] would reduce the predicted value to 2.4, while use of the Coulomb energy data[23] would reduce the predicted value of the Li^6/Li^7 ratio to ~1.5. The large experimental ^3D splitting in Li^6 is thus hard to understand on any theory of the vector force. From the point of view of the three-body vector force theory, it might be due to $\bar{E}$ [Eq. (4)] being considerably smaller for Li^6 than for Li^7 (see Sec. III). The only calculations of $\bar{E}$ are those reported in reference 12. These do indicate such an effect, but only $\frac{1}{4}$–$\frac{1}{2}$ the size needed to explain the entire discrepancy.

An interesting result of the Li^6 calculations is that the off-diagonal matrix element of the three-body vector force between the $^3S_1$ and $^1P_1$ states is very much fmaller when compared to the diagonal matrix elements for the ^3D states than is the case for a one-body vector sorce. This would provide a natural explanation of the small quadrupole moment of Li^6 without requiring cancellation by a tensor force, as is needed for a one-body vector force.[38]

If the three-body vector force is the main vector force in nuclei, the question of how much two-body vector force is also present in the effective nuclear potential cannot, unfortunately, be answered by the calculations of the present paper. The N^{15}/Li^7 doublet splitting ratio is sensitive to the possible dependence of V_0 on A (see Sec. III), and the O^{17}/N^{15} ratio is influenced by the uncertainty in the magnitude of the Ehrmann-Thomas shift. If the strength of the three-body vector force, V_0, is the same for Li^7 and N^{15}, then the results of Sec. IV

[37] G. E. Tauber and T. Y. Wu, Phys. Rev. **93**, 295 (1954).
[38] W. T. Pinkston and J. G. Brennan, Phys. Rev. **109**, 499 (1958).
[39] In any event, an "attractive" effective tensor potential would increase the discrepancy between theory and experiment since it tends to depress the $^3D_2$ state below the $^3D_3$ state [T. Regge, Nuovo cimento **11**, 285 (1954); J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953)]. A "repulsive" effective tensor force would, of course, reduce the discrepancy.

would permit a two-body vector force strong enough to cause, by itself, a splitting of the ^2P states of Li^7 of ~ 200 kev, with, however, a normal ordering of the levels, i.e., the $\frac{1}{2}$ state *below* the $\frac{3}{2}$ state. If, however, V_0 increased, say, by 50% on going from Li^7 to N^{15}, then the results would imply a two-body vector force of about the same strength as for the case of constant V_0, but now of opposite sign. Evidence on the trend of V_0 with A could be obtained by extending the calculations to the f-shell (Ca^{41}), though the labor would be formidable.

The remarkable agreement with experiment obtained by Talmi and others,[40] on the assumption that shell-model nucleon-nucleon forces are solely two-body in nature, is not necessarily in conflict with the possible presence of three-body vector forces. As discussed in reference 14, for the case of a few particles outside closed shells, integration over the closed shells will yield effective one- and two-body vector forces that will dominate over the remaining three-body interactions among the outer nucleons. Thus for three-body forces one can expect a conspicuous breakdown of the Talmi procedure only in the case of a large number of particles outside a small closed core, such as occurs in the p shell. Here indeed the Talmi analysis does lead to disagreement with experiment, but what part of the disagreement is due to three-body forces and what part is due to a departure from jj coupling, another assumption of the Talmi analysis, is an open question.

APPENDIX

We wish to evaluate the sum (6) for He^5 ($m=4$) and O^{17} ($m=16$), and the expression (8) for N^{15} ($m=4$, $n=12$). We label the single-particle states in the following order: first in increasing order of l, each set of given l in decreasing order of m_l, each set of given l and m_l in decreasing order of m_s ($+\frac{1}{2}$, then $-\frac{1}{2}$), and finally each set of given l, m_l, and m_s in decreasing order of m_τ ($+\frac{1}{2}$ for neutron, $-\frac{1}{2}$ for proton). This follows Shortley's procedure.[32] Each term $[ijk]$ of (6) and (8) is a sum of 36 matrix elements but takes on a simple form after the spin and isotopic spin integrations are performed. The spin and isotopic spin integrations can be performed for a general term of the form $[a_l b_m c_n]$ where a, b, or c stand for the l, m_l values of the single-particle state, and the subscripts give the m_s, m_τ values of the state. For the subscripts we use the numbers 1–4, in the order given above, e.g., the subscript 2 means a proton with spin up. Since the operators are independent of isotopic spin, the matrix elements are invariant to the simultaneous subscript substitution, $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. The results of performing the spin and isotopic spin integrations for the four types of three-body vector operators of the text are given in Table I. The operators are $\mathcal{U}_n$, the non-exchange operator, $\mathcal{U}_{ex}$, the pure-exchange operator, $\mathcal{U}_m$, the mixed-exchange operator, all defined in Eq. (9), and $\mathcal{U}_i$, the interference operator defined in Eq. (11) [the last line of Eq. (11), without the factor $\frac{1}{4}$]. In Table I the notation $abc|def$ represents the spatial matrix element $\langle a_1 b_2 c_3|\mathcal{O}|d_1 e_2 f_3\rangle$, where for the operators $\mathcal{U}_n$, $\mathcal{U}_{ex}$, and $\mathcal{U}_m$, $\mathcal{O}$ is the operator

$$4iV_0 f(r_{12}) f(r_{13}) (\mathbf{r}_{12}\cdot\mathbf{r}_{13}) (\mathbf{r}_{12}\times\mathbf{r}_{13})_z / r_{12}^2 r_{13}^2,$$

and the subscripts 1, 2, 3 on the state labels a, b, c are particle labels. For the operator $\mathcal{U}_i$, $f(r_{12})f(r_{13})$ is to be replaced by $f_e(r_{12})f_o(r_{13})$ (see discussion in Sec. V). The labels a, b, etc. now refer to the spatial parts of the single-particle wave functions, and are assumed normalized to unity. The matrix elements $abc|def$ have, for the case of $\mathcal{U}_n$, $\mathcal{U}_{ex}$, $\mathcal{U}_m$, the symmetry properties,

$$abc|def = -def|abc = dfe|acb.$$

The sums in Eqs. (6) and (8) can now be readily performed. As mentioned in the text, for the states we are concerned with, the operator $\mathcal{U}_{ex}$ gives results identical with $\mathcal{U}_n$, and $\mathcal{U}_i$ vanishes. We obtain

$$\begin{aligned}\langle \mathcal{U}_n\rangle(He^5) &= (2^{15/2}/3\pi^{9/2}) V_0 \alpha^{11} \int\int\int \exp[-2\alpha^2(r_1^2+r_2^2+r_3^2)] \\ &\quad \times f(r_{12}) f(r_{13}) [(\mathbf{r}_{12}\cdot\mathbf{r}_{13})(\mathbf{r}_{12}\times\mathbf{r}_{13})/r_{12}^2 r_{13}^2] \\ &\quad \cdot[3(\mathbf{r}_2\times\mathbf{r}_3)+2(\mathbf{r}_1\times\mathbf{r}_2)] dv_1 dv_2 dv_3,\end{aligned}$$

which we abbreviate as

$$\langle \mathcal{U}_n\rangle(He^5) = 3(\mathbf{r}_2\times\mathbf{r}_3)+2(\mathbf{r}_1\times\mathbf{r}_2).$$

Using the same abbreviated notation,

$$\begin{aligned}\langle \mathcal{U}_m\rangle(He^5) &= -(8/5)[3(\mathbf{r}_2\times\mathbf{r}_3)+2(\mathbf{r}_1\times\mathbf{r}_2)], \\ \langle \mathcal{U}_n\rangle(N^{15}) &= -3(\mathbf{r}_2\times\mathbf{r}_3)-2(\mathbf{r}_3\times\mathbf{r}_1) \\ &\quad -8\alpha^2\{(\mathbf{r}_2\times\mathbf{r}_3)[2r_1^2+2(\mathbf{r}_2\cdot\mathbf{r}_3)-(\mathbf{r}_1\cdot\mathbf{r}_2)] \\ &\quad +(\mathbf{r}_3\times\mathbf{r}_1)[(\mathbf{r}_1\cdot\mathbf{r}_2)+(\mathbf{r}_2\cdot\mathbf{r}_3)]\} \\ &\quad -16\alpha^4\{(\mathbf{r}_2\times\mathbf{r}_3)[4r_1^2(\mathbf{r}_2\cdot\mathbf{r}_3)-(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_1\cdot\mathbf{r}_3)] \\ &\quad +2(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_2\cdot\mathbf{r}_3)\}, \\ \langle \mathcal{U}_m\rangle(N^{15}) &= (24/5)(\mathbf{r}_2\times\mathbf{r}_3)+(16/5)(\mathbf{r}_3\times\mathbf{r}_1) \\ &\quad 16\alpha^2\{(\mathbf{r}_2\times\mathbf{r}_3)[r_1^2+(\mathbf{r}_2\cdot\mathbf{r}_3)-2(\mathbf{r}_1\cdot\mathbf{r}_3)] \\ &\quad +2(\mathbf{r}_3\times\mathbf{r}_1)[(\mathbf{r}_1\cdot\mathbf{r}_2)+(\mathbf{r}_2\cdot\mathbf{r}_3)]\} \\ &\quad -64\alpha^4\{(\mathbf{r}_2\times\mathbf{r}_3)[(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_1\cdot\mathbf{r}_3)-r_1^2(\mathbf{r}_2\cdot\mathbf{r}_3)] \\ &\quad +2(\mathbf{r}_1\times\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_2\cdot\mathbf{r}_3)\}, \\ \langle \mathcal{U}_n\rangle(O^{17}) &= (8/5)\alpha^2[2(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_3)+3(\mathbf{r}_2\times\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3)] \\ &\quad +(64/5)\alpha^4\{(\mathbf{r}_2\times\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3) \\ &\quad \times[2(\mathbf{r}_2\cdot\mathbf{r}_3)+2r_1^2-(\mathbf{r}_1\cdot\mathbf{r}_3)] \\ &\quad +(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_3)[(\mathbf{r}_2\cdot\mathbf{r}_3)+(\mathbf{r}_1\cdot\mathbf{r}_2)]\} \\ &\quad +(128/5)\alpha^6\{(\mathbf{r}_2\times\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3) \\ &\quad \times[4r_1^2(\mathbf{r}_2\cdot\mathbf{r}_3)-(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_1\cdot\mathbf{r}_3)] \\ &\quad +2(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_2)\},\end{aligned}$$

[40] S. Goldstein. and I. Talmi, Phys. Rev. **102**, 589 (1956); **105**, 995 (1957); S. P. Pandya, Phys. Rev. **103**, 956 (1956); I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956); R. D. Lawson and J. L. Uretsky, Phys. Rev. **106**, 1369 (1957); I. Talmi, Phys. Rev. **107**, 326, 1601 (1957).

TABLE I. Matrix elements of the various three-body vector operators after spin and isotopic spin integration. See text for notation.

Term[a]	$\mathcal{V}_n$	$\mathcal{V}_{ex}$	$\mathcal{V}_m$	$\mathcal{V}_i$
$[a_1b_1c_2]$	0	$(1/10)(2bac\|cba-2bac\|cab+2abc\|cab$ $-2abc\|cba+2bca\|abc-bca\|bac-acb\|abc)$	0	0
$[a_1b_1c_3]$	$(2abc\|bca-abc\|acb$ $-bac\|bca)$	$2bac\|acb-bac\|bca-abc\|acb$	$(-4bac\|acb+2bac\|bca$ $+2abc\|acb)$	0
$[a_1b_2c_3]$	$-bac\|bca$	$-[a_1c_1b_2]-abc\|acb$	$-2abc\|bca$	$-2abc\|bca+2bac\|acb$

[a] $[a_1b_1c_1]=0$; $[a_1b_1c_4]=-[a_1b_1c_2]$; $[a_1b_2c_2]=[b_1c_1a_2]$; $[a_1b_2c_4]=[b_1a_2c_3]$; $[a_1b_3c_3]=-[b_1c_1a_3]$; $[a_1b_3c_4]=-[b_1c_2a_3]$; $[a_1b_4c_4]=[b_1c_1a_2]$.

$$\langle\mathcal{V}_m\rangle(O^{17})=(64/5)\alpha^2(\mathbf{r}_1\times\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_3)-(128/5)\alpha^4\times\{(\mathbf{r}_2\times\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3)[r_1^2+(\mathbf{r}_2\cdot\mathbf{r}_3)-2(\mathbf{r}_1\cdot\mathbf{r}_2)]+2(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_3)[(\mathbf{r}_2\cdot\mathbf{r}_3)+(\mathbf{r}_1\cdot\mathbf{r}_2)]\}-(512/5)\alpha^6\{(\mathbf{r}_2\times\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3)\times[r_1^2(\mathbf{r}_2\cdot\mathbf{r}_3)-(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_1\cdot\mathbf{r}_3)]+2(\mathbf{r}_3\times\mathbf{r}_1)(\mathbf{r}_1\cdot\mathbf{r}_3)(\mathbf{r}_2\cdot\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_2)\}.$$

The above matrix elements refer, by our method of state ordering, to the $^2P_{\frac{3}{2}}$ and $^2D_{\frac{5}{2}}$ states of He^5 and N^{15}, and O^{17}, respectively. For the $^2P_{\frac{3}{2}}$ state of Li^7 we can use the method of reference 12, and obtain

$$\langle\mathcal{V}_n\rangle(Li^7)=(\mathbf{r}_2\times\mathbf{r}_3)+(2/3)(\mathbf{r}_3\times\mathbf{r}_1)+(8/45)\alpha^2[(\mathbf{r}_2\times\mathbf{r}_3)r_1^2+2(\mathbf{r}_1\times\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_2)],$$

$$\langle\mathcal{V}_m\rangle(Li^7)=-(8/15)[3(\mathbf{r}_2\times\mathbf{r}_3)+2(\mathbf{r}_3\times\mathbf{r}_1)]+(16/45)\alpha^2[(\mathbf{r}_2\times\mathbf{r}_3)r_1^2+2(\mathbf{r}_1\times\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{r}_2)].$$

The matrix element $\langle\mathcal{V}_n\rangle$ for the $^3D_3$ state of Li^6 has been given essentially by Lyons.[13] The result is

$$\langle\mathcal{V}_n\rangle(Li^6)=(8/15)[11(\mathbf{r}_2\times\mathbf{r}_3)+7(\mathbf{r}_3\times\mathbf{r}_1)+6\alpha^2(\mathbf{r}_2\times\mathbf{r}_3)r_1^2].$$

The integration over the coordinates of particle 1 can now be performed by changing to $\mathbf{r}_1$, $\mathbf{r}_{12}$, $\mathbf{r}_{13}$ as the independent variables. We obtain

$$\langle\mathcal{V}_n\rangle(He^5)=(5\times3^{\frac{1}{2}}V_0/4\pi^3)\int\int\exp[-s^2+(\mathbf{s}\cdot\mathbf{t})-t^2]\times f(r_{12})f(r_{13})(\mathbf{s}\cdot\mathbf{t})[(\mathbf{s}\times\mathbf{t})^2/s^2t^2]dv_s dv_t,$$

where $\mathbf{s}=3^{\frac{1}{2}}\mathbf{r}_{12}/\alpha$, $\mathbf{t}=3^{\frac{1}{2}}\mathbf{r}_{13}/\alpha$. The other matrix elements differ from the above by the presence of an additional factor in the integrand. We list below this additional factor:

$\langle\mathcal{V}_m\rangle(He^5)$: $-8/5$,

$\langle\mathcal{V}_n\rangle(N^{15})$: $(1/45)[-118+22s^2-131(\mathbf{s}\cdot\mathbf{t})+8s^4+17s^2t^2-4s^2(\mathbf{s}\cdot\mathbf{t})-29(\mathbf{s}\cdot\mathbf{t})^2]$,

$\langle\mathcal{V}_m\rangle(N^{15})$: $(4/45)[76-34s^2+65(\mathbf{s}\cdot\mathbf{t})+4s^4-5s^2t^2-2s^2(\mathbf{s}\cdot\mathbf{t})-(\mathbf{s}\cdot\mathbf{t})^2]$,

$\langle\mathcal{V}_n\rangle(O^{17})$: $(2/625)[1098-846s^2+621(\mathbf{s}\cdot\mathbf{t})+432s^4-189s^2t^2-252s^2(\mathbf{s}\cdot\mathbf{t})+234(\mathbf{s}\cdot\mathbf{t})^2+202s^6-78s^4t^2+204s^4(\mathbf{s}\cdot\mathbf{t})-399s^2t^2(\mathbf{s}\cdot\mathbf{t})-114s^2(\mathbf{s}\cdot\mathbf{t})^2+205(\mathbf{s}\cdot\mathbf{t})^3]$,

$\langle\mathcal{V}_m\rangle(O^{17})$: $(8/625)[-612+414s^2-459(\mathbf{s}\cdot\mathbf{t})-486s^4+135s^2t^2+90s^2(\mathbf{s}\cdot\mathbf{t})+198(\mathbf{s}\cdot\mathbf{t})^2-178s^6+150s^4t^2-276s^4(\mathbf{s}\cdot\mathbf{t})+327s^2t^2(\mathbf{s}\cdot\mathbf{t})+24s^2(\mathbf{s}\cdot\mathbf{t})^2-55(\mathbf{s}\cdot\mathbf{t})^3]$,

$\langle\mathcal{V}_n\rangle(Li^7)$: $(1/625)[203+8s^2+8(\mathbf{s}\cdot\mathbf{t})]$,

$\langle\mathcal{V}_m\rangle(Li^7)$: $(4/625)[-101+4s^2+4(\mathbf{s}\cdot\mathbf{t})]$,

$\langle\mathcal{V}_n\rangle(Li^6)$: $(4/75)[35+2s^2+2(\mathbf{s}\cdot\mathbf{t})]$.

The 6-dimensional integrals above are readily reduced by going to polar coordinates and integrating over the angles. The resultant 2-dimensional integrals are then most conveniently evaluated in terms of power series. See reference 12 and Lyons[13] for the details.

By the Landé interval rule, the P- and D-doublet splittings are 3 times and $(\frac{5}{2})$ times the matrix elements for the $^2P_{\frac{3}{2}}$ and $^2D_{\frac{5}{2}}$ states, respectively, and the $^3D_3$-$^3D_2$ separation in Li^6 is (7/6) times the matrix element for the $^3D_3$ state.