PHYSICAL REVIEW VOLUME 114, NUMBER 2 APRIL 15, 1959

Kinetic Equation for a Plasma with Unsteady Correlations*

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(Received July 9, 1958; revised manuscript received December 11, 1958)

As a generalization of the Boltzmann equation, the kinetic equation for a plasma is derived in the form of a generalized Fokker-Planck equation, by considering unsteady correlations, including non-Markovian and nonlinear behavior. Both the binary and ternary correlations are used for many kinds of particles with different temperatures. The coefficients of the kinetic equation depend on the law of interaction for a pair of particles and are influenced by relaxation. The effective potential of friction consists of two parts: the static part corresponds to the Debye potential and is isotropic, the dynamical part is axially symmetrical about the direction of motion, and causes a dynamical friction. The results show that the friction is proportional to velocity for slow particles, and inversely proportional to the square of velocity for fast particles. This tendency of the fast particles to overcome repulsion is a property connected with the "run-away" of electrons. A criterion for maximum friction is derived. The triplet interaction, which mainly affects the shielding phenomena, assures the convergence of the coefficients in case of distant interaction. Since the length scales of interaction are well determined in this way, the kinetic equation can be expected to be valid over a longer range than does the Boltzmann equation. The large scale agrees with the Debye radius, when the shielding term is linearized, as should be expected. When time relaxation is left aside and linearization is made, the kinetic equation degenerates to the classical Fokker-Planck equation with convergent coefficients.

I. INTRODUCTION

THE object of the present paper is to develop a method leading to a kinetic equation for a system of charged particles, interacting according to the Coulomb law. As a result, friction and diffusion of particles enter into the equation in explicit form. The essential features and also the principal difficulties of the problem are: (a) long-range phenomena, (b) nonlinear behavior, and (c) non-Markovian behavior. The complicated dynamical process is necessarily of a stochastic nature. The Boltzmann equation is one method of representing the latter, but it is not generally adequate when long range forces are involved. Moreover, the nonlinear behavior is expressed in a complicated integral form, so that linearization is necessary in applications.

The Fokker-Planck equation evades this mathematical difficulty by incorporating the essential nonlinear behavior simply into the coefficients of the equation. Such an equation is suitable for application to long-range forces. The problem of treating particles undergoing numerous weak deflections was originally encountered in Brownian motion of large molecules which are thermally agitated by the smaller field molecules. Originally the Fokker-Planck equation was derived for this purpose, and was formulated for a distribution function at a given instant of time.[1,2] Later it was extended by Kolmogoroff,[3] Tchen,[4] and Chandrasekhar[5] for a transition function at two instants of time. However, the process was Markovian which implies dependence of the future on the present, but not on the past. Further, since their approach was phenomenological, the coefficients could not be determined explicitly.

In order to determine the coefficients, some dynamical behavior must be considered. Early successful attempts concerning the dynamics of a plasma introduced the collision concept by means of the Boltzmann equation.[6,7]

The use of the integrated Liouville equation by Gasiorowicz, Neuman, and Riddell[8] opened up a broad basic method for treating the dynamical behavior of a plasma. A generalization including nonuniform distributions of test and field particles can be effected, and it forms the essential scheme of the problem of generalizing the Boltzmann equation into a form suitable for application to a plasma. Recently the method of Bogoliubov[9] has been frequently recommended for this purpose. Following Bogoliubov's method, Tolmachev[10] introduced a chain of linked distributions at different instants, and derived the corresponding Fokker-Planck equation. The results were divergent at extreme distances of interaction. Temko[11] made use of a ternary correlation, thus introducing sufficient nonlinearity to insure convergence at large distances. However, two different expressions for ternary correlations were necessary, which were inconsistent from the point of view of symmetry: thus the results imply an unjustifiable asymmetry in the polarization effect. At close interaction, the latter method still involved divergence, which must be eliminated by a somewhat arbitrary cutoff.

* Supported partially by the Office of Scientific Research.

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As mentioned earlier, the classical Fokker-Planck equation is based on the Markovian process. For a plasma this condition is usually not met, so that the interaction will depend not only on the distribution functions, but also on the evolution of correlations in time and space. In the present paper a method for deriving a kinetic equation is developed, which is more general than the Boltzmann equation. Under certain circumstances, the resulting kinetic equation can be written in the form of a generalized Fokker-Planck equation, involving time derivatives of higher orders in the distribution functions. These derivatives account for the non-Markovian behavior. The coefficients are found in explicit form. The convergence at large distances follows automatically, as a result of the internal interaction mechanism itself.

In Sec. II the Liouville equation is integrated. The binary and ternary correlations are discussed in Sec. III. In Sec. IV the equation of correlation is integrated, which leads to the appearance of the time effect, important in describing the non-Markovian behavior. To solve the equation thus obtained, the Fourier transformation is applied, and various transport functions are investigated (Sec. V). After some simplifications, the kinetic equation and the generalized Fokker-Planck equation are obtained (Sec. VI), in a form which permits the calculation of the dynamical friction, the friction potential, and the diffusion in explicit form (Sec. VII–X).

The development shows that the nonlinear behavior occurs both in the distribution function and in the shielding phenomena, respectively, as a consequence of binary and ternary interactions.

II. DERIVATION OF B-B-G-K-Y EQUATION

The B-B-G-K-Y equation is the integrated form of the Liouville equation. Although it has been derived by various authors (Bogoliubov, Born, Green, Kirkwood, and Yvon), it is rederived in the present section. This will give at the same time an opportunity of discussing the mechanism giving rise to the nonlinear behavior.

Let $D(t,\mathbf{x}_1,\cdots,\mathbf{x}_N)$ be the distribution function for the dynamical states of the whole system of N particles, where t is the time, and $\mathbf{x}_i=(\mathbf{q}_i,\mathbf{p}_i)$ are the coordinates in phase space (position and momentum) of the particle i. D is obtained by taking the usual statistical average over the initial states of the system.[10] The fundamental equation which determines the behavior of the dynamical system in phase space may be written as the Liouville equation

$$\partial D/\partial t=[H;D], \tag{1}$$

where $[\cdots]$ are the Poisson brackets:

$$[H;D]=\sum_{1\leq i\leq N}\left\{\frac{\partial H}{\partial \mathbf{q}_i}\cdot\frac{\partial D}{\partial \mathbf{p}_i}-\frac{\partial H}{\partial \mathbf{p}_i}\cdot\frac{\partial D}{\partial \mathbf{q}_i}\right\}.$$

Here and in the following, $\partial/\partial\mathbf{q}$, $\partial/\partial\mathbf{p}$, and $d\mathbf{p}$ represent $\text{grad}_\mathbf{q}$, $\text{grad}_\mathbf{p}$, and volume element in momentum space, according to the notation often adopted in the literature.[12] In the following the indicial notation will also be used, when it appears to be more convenient. H is the Hamiltonian representing the sum of the energies of the individual particles and of the potentials of pair interaction:

$$\phi_{ij}=\phi(|\mathbf{q}_i-\mathbf{q}_j|).$$

Thus

$$H=\sum_{1\leq i\leq N}\left\{\frac{p_i^2}{2m_i}+U_i\right\}+\sum_{1\leq i<j\leq N}\phi_{ij},$$

where $U_i=U(\mathbf{q}_i)$ is the external potential arising from an external force or wall effect, and m_i is the mass of particle i.

With these definitions, Eq. (1) can be written as follows:

$$\frac{\partial D}{\partial t}+\sum_{1\leq i\leq N}\frac{\mathbf{p}_i}{m_i}\cdot\frac{\partial D}{\partial \mathbf{q}_i}-\sum_{1\leq i\leq N}\frac{\partial U_i}{\partial \mathbf{q}_i}\cdot\frac{\partial D}{\partial \mathbf{p}_i}=\sum_{1\leq i<j\leq N}\frac{\partial \phi_{ij}}{\partial \mathbf{q}_i}\cdot\frac{\partial D}{\partial \mathbf{p}_i}.$$

In this form the significance of the Liouville equation is to determine the transport of D in time and space, the external force $-\partial U_i/\partial\mathbf{q}_i$ acting on each individual particle "i," while the internal interaction takes place between all possible pairs i, j, with a prescribed potential ϕ_{ij}. Since D is a probability, it is normalized to unity:

$$\int\cdots\int D d\mathbf{x}_1\cdots d\mathbf{x}_N=1.$$

Here and in the following, the integration extends over the whole available domain of the phase space.

Sometimes it is useful to study the motion not of all the N particles contained in the volume V, but of a set of s particles; we therefore introduce the distribution function $F_s(t,\mathbf{x}_1,\cdots,\mathbf{x}_s)$, such that

$$\frac{1}{V^s}F_s(t,\mathbf{x}_1,\cdots,\mathbf{x}_s)d\mathbf{x}_1\cdots d\mathbf{x}_s =d\mathbf{x}_1\cdots d\mathbf{x}_s\int\cdots\int D(t,\mathbf{x}_1,\cdots,\mathbf{x}_N)d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N$$

is the probability that the dynamical state of the group of s particles be located in $d\mathbf{x}_1$, $\cdots$, $d\mathbf{x}_s$, respectively, regardless of the dynamical states of the remainder set of $N-s$ particles $s+1$, $\cdots$, N. The differential equation determining F_s can be obtained by integrating Eq. (1) with respect to $d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N$, and multiplying by V^s.

[12] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1939 and 1952), Chap. I.

Simplifications can be made by the use of the following relations:

$$\int [H(\mathbf{x}_i);D]d\mathbf{x}_i=0, \tag{2a}$$

$$\int\int [\phi_{ij};D]d\mathbf{x}_i d\mathbf{x}_j=0, \tag{2b}$$

where $H(\mathbf{x}_i)$ is the Hamiltonian for the individual particles:

$$H(\mathbf{x}_i)=p_i^2/2m_i+U(\mathbf{q}_i).$$

Writing Eq. (1) in the form

$$\frac{\partial D}{\partial t}=\sum_{1\le i\le N}[H(\mathbf{x}_i);D]+\sum_{1\le i<j\le N}[\phi_{ij};D],$$

and integrating with respect to $d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N$, we obtain

$$\begin{aligned}\frac{1}{V^s}\frac{\partial F_s}{\partial t}&=\sum_{1\le i\le N}\int\cdots\int[H(\mathbf{x}_i);D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{1\le i<j\le N}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&=\sum_{1\le i\le s}\int\cdots\int[H(\mathbf{x}_i);D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{s+1\le i\le N}\int\cdots\int[H(\mathbf{x}_i);D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{1\le i<j\le s}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{s+1\le i<j\le N}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{\substack{1\le i\le s\\ s+1\le j\le N}}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N.\end{aligned} \tag{3}$$

On the right-hand side, the second and fourth terms vanish, because of Eq. (2). The first and third terms can be combined to give

$$\begin{aligned}&\sum_{1\le i\le s}\int\cdots\int[H(\mathbf{x}_i);D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&+\sum_{1\le i<j\le s}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&=\int\cdots\int[\sum_{1\le i\le s}H(\mathbf{x}_i)+\sum_{1\le i<j\le s}\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&=[\sum_{1\le i\le s}H(\mathbf{x}_i)+\sum_{1\le i<j\le s}\phi_{ij};F_s/V^s].\end{aligned}$$

Because of the symmetry of D with respect to $\mathbf{x}_1,\cdots,\mathbf{x}_N$ the last term of Eq. (3) can be written as follows:

$$\begin{aligned}&\sum_{\substack{1\le i\le s\\ s+1\le j\le N}}\int\cdots\int[\phi_{ij};D]d\mathbf{x}_{s+1}\cdots d\mathbf{x}_N\\&=(N-s)\sum_{1\le i\le s}\int\cdots\int[\phi_{i,s+1};D]d\mathbf{x}_{s+1}d\mathbf{x}_{s+2}\cdots d\mathbf{x}_N\\&=\frac{N-s}{V}\int\sum_{1\le i\le s}[\phi_{i,s+1};F_{s+1}/V^s]d\mathbf{x}_{s+1},\end{aligned}$$

and represents the nonlinear behavior. Hence we find

$$\frac{\partial F_s}{\partial t}=[H_s;F_s]+\frac{N-s}{V}\int\sum_{1\le i\le s}[\phi_{i,s+1};F_{s+1}]d\mathbf{x}_{s+1}, \tag{4}$$

with

$$H_s=\sum_{1\le i\le s}H(\mathbf{x}_i)+\sum_{1\le i<j\le s}\phi_{ij}.$$

In the special case of $s=1$, we have

$$\begin{aligned}\frac{\partial F_1}{\partial t}&=\frac{\partial H_1}{\partial \mathbf{q}_1}\cdot\frac{\partial F_1}{\partial \mathbf{p}_1}-\frac{\partial H_1}{\partial \mathbf{p}_1}\cdot\frac{\partial F_1}{\partial \mathbf{q}_1}\\&+\frac{N-1}{V}\int\left\{\frac{\partial\phi_{12}}{\partial\mathbf{q}_1}\cdot\frac{\partial F_2}{\partial\mathbf{p}_1}-\frac{\partial\phi_{12}}{\partial\mathbf{p}_1}\cdot\frac{\partial F_2}{\partial\mathbf{q}_1}\right\}d\mathbf{x}_2,\end{aligned} \tag{5}$$

and for $s=2$,

$$\begin{aligned}\frac{\partial F_2}{\partial t}&=[H(\mathbf{x}_1)+H(\mathbf{x}_2)+\phi_{12};F_2]\\&+\frac{N-2}{V}\int\sum_{1\le i\le 2}[\phi_{i3};F_3]d\mathbf{x}_3.\end{aligned} \tag{6}$$

In the following we shall need Eqs. (5) and (6) only. Thus we shall not consider equations for correlations of higher order.

III. SYMMETRICAL RELATION FOR TERNARY CORRELATIONS

The integrated Liouville equations (5) and (6) can be extended to describe a mixture of particles of different kinds totalling N particles. The different kinds are distinguished by subscripts a, b, c, etc. We shall thus change the notations for the distribution function and the correlation functions, in such a way that the distribution function of a single particle belonging to a group will be denoted by F_a, F_b, etc., with a single subscript. This is to replace the symbol F_1 used in Eqs. (5) and (6). The binary correlation function for a pair of particles belonging to a and b will be denoted by F_{ab}, and the ternary correlation function for three particles by F_{abc}, in the place of F_2 and F_3 used in Eqs. (5) and

(6). Thus we have

$$\frac{\partial F_a}{\partial t}=\frac{\partial H_a}{\partial \mathbf{q}_a}\cdot\frac{\partial F_a}{\partial \mathbf{p}_a}-\frac{\partial H_a}{\partial \mathbf{p}_a}\cdot\frac{\partial F_a}{\partial \mathbf{q}_a}+\sum_b \frac{N_b}{V}\int\int\frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_a}d\mathbf{p}_b d\mathbf{q}_b, \quad (7)$$

$$H_a=\frac{p_a^2}{2m_a}+U(\mathbf{q}_a),$$

$$\frac{\partial F_{ab}}{\partial t}=\left(\frac{\partial H_a}{\partial \mathbf{q}_a}+\frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\right)\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_a}+\left(\frac{\partial H_b}{\partial \mathbf{q}_b}+\frac{\partial \phi_{ab}}{\partial \mathbf{q}_b}\right)\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_b}-\frac{\partial H_a}{\partial \mathbf{p}_a}\cdot\frac{\partial F_{ab}}{\partial \mathbf{q}_a}-\frac{\partial H_b}{\partial \mathbf{p}_b}\cdot\frac{\partial F_{ab}}{\partial \mathbf{q}_b}+\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \times\left\{\frac{\partial \phi_{ac}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{abc}}{\partial \mathbf{p}_a}+\frac{\partial \phi_{bc}}{\partial \mathbf{q}_b}\cdot\frac{\partial F_{abc}}{\partial \mathbf{p}_b}\right\}, \quad (8)$$

where N_a is the number of particles of kind a with mass m_a and charge e_a, etc. The summation with respect to b in Eq. (7) covers all kinds of particles, including $b=a$. The same is true with the summation in Eq. (8). Thus ϕ_{aa} is the interaction potential of a pair of particles of the same kind a.

If the volume of the vessel is large, and the region considered is far from the walls, the distribution functions can be taken as spatially homogeneous; i.e., $F_a(t,\mathbf{p}_a)$, $F_b(t,\mathbf{p}_b)$, and $F_c(t,\mathbf{p}_c)$ do not depend on the coordinates $\mathbf{q}_a$, $\mathbf{q}_b$, and $\mathbf{q}_c$. Only the differences of the coordinates enter into the correlation functions $F_{ab}(t,\mathbf{p}_a,\mathbf{p}_b,\mathbf{q}_a-\mathbf{q}_b)$, etc. Under these circumstances the system of Eqs. (7) and (8) becomes simply

$$\frac{\partial F_a}{\partial t}=\sum_b \frac{N_b}{V}\int\int d\mathbf{q}_b d\mathbf{p}_b \frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_a} \quad (9)$$

$$\frac{\partial F_{ab}}{\partial t}+\frac{\partial H_a}{\partial \mathbf{p}_a}\cdot\frac{\partial F_{ab}}{\partial \mathbf{q}_a}+\frac{\partial H_b}{\partial \mathbf{p}_b}\cdot\frac{\partial F_{ab}}{\partial \mathbf{q}_b}=\frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_a}+\frac{\partial \phi_{ab}}{\partial \mathbf{q}_b}\cdot\frac{\partial F_{ab}}{\partial \mathbf{p}_b}+\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c\left\{\frac{\partial \phi_{ac}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{abc}}{\partial \mathbf{p}_a}+\frac{\partial \phi_{bc}}{\partial \mathbf{q}_b}\cdot\frac{\partial F_{bac}}{\partial \mathbf{p}_b}\right\}. \quad (10)$$

In Eqs. (9) and (10) we have taken $U_a=0$, so that $\partial H_a/\partial \mathbf{q}_a=0$, and similarly $\partial H_b/\partial \mathbf{q}_b=0$.

F_{ab} may depend on $\mathbf{q}_a-\mathbf{q}_b$ in an arbitrary way, according to the relative velocity between the pair of particles.

We can write

$$F_{ab}(t,\mathbf{p}_a,\mathbf{p}_b,\mathbf{q}_a-\mathbf{q}_b)=F_a(t,\mathbf{p}_a)F_b(t,\mathbf{p}_b)+F_{ab}'(t,\mathbf{p}_a,\mathbf{p}_b,\mathbf{q}_a-\mathbf{q}_b). \quad (11)$$

The Maxwellian distributions will be denoted by F_a^0, F_b^0, etc. Equation (11) may be considered as the definition of F_{ab}', which vanishes asymptotically, when the distance between the pair of particles increases indefinitely.

After substitution of Eq. (11), we can rewrite Eqs. (9) and (10) as follows

$$\frac{\partial F_a}{\partial t}=\sum_b \frac{N_b}{V}\int\int d\mathbf{q}_b d\mathbf{p}_b \frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{ab}'}{\partial \mathbf{p}_a}, \quad (12)$$

$$\frac{dF_{ab}'}{dt}=\frac{d}{dt}(F_{ab}-F_aF_b)=\frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial (F_aF_b)}{\partial \mathbf{p}_a}+\frac{\partial \phi_{ba}}{\partial \mathbf{q}_b}\cdot\frac{\partial (F_aF_b)}{\partial \mathbf{p}_b}+\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial \phi_{ac}}{\partial \mathbf{q}_a}\cdot\frac{\partial}{\partial \mathbf{p}_a}(F_{abc}-F_bF_{ac})+\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial \phi_{bc}}{\partial \mathbf{q}_b}\cdot\frac{\partial}{\partial \mathbf{p}_b}(F_{bac}-F_aF_{bc})+\frac{\partial \phi_{ab}}{\partial \mathbf{q}_a}\cdot\frac{\partial F_{ab}'}{\partial \mathbf{p}_a}+\frac{\partial \phi_{ba}}{\partial \mathbf{q}_b}\cdot\frac{\partial F_{ab}'}{\partial \mathbf{p}_b}, \quad (13)$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\frac{\mathbf{p}_a}{m_a}\cdot\frac{\partial}{\partial \mathbf{q}_a}+\frac{\mathbf{p}_b}{m_b}\cdot\frac{\partial}{\partial \mathbf{q}_b}.$$

It is to be noted that the product F_aF_b on the right-hand side of Eq. (11) does not contribute to Eqs. (9) and (12), since

$$\int d\mathbf{q}_b\, \partial\phi_{ab}/\partial\mathbf{q}_a=0.$$

In the integral term of Eq. (13), F_bF_{ac} and F_aF_{bc} arise from $\partial(F_aF_b)/\partial t$ and Eq. (9).

As is usual with nonlinear dynamical systems, the equation for the distribution function cannot be solved independently, a solution requiring a knowledge of the binary correlation function, which in turn is determined by an equation involving ternary correlations, and so forth. Here the system of equations is made determinate by the use of the following symmetrical relation for the ternary correlation function:

$$F_{abc}=F_aF_{bc}+F_bF_{ca}+F_cF_{ab}-2F_aF_bF_c \quad (14)$$

For weak interaction (small ϕ), relation (14) is derived from Eq. (4) by means of an expansion in powers of a small parameter. The derivation of relation (14) and its generalization are given in Appendix A.

With the use of relation (14), the two integral terms in Eq. (13) can be replaced by

$$\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{ac}}{\partial\mathbf{q}_a}\cdot\frac{\partial}{\partial\mathbf{p}_a}(F_aF_{bc}+F_cF_{ab}-2F_aF_bF_c)$$

$$+\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{bc}}{\partial\mathbf{q}_b}\cdot\frac{\partial}{\partial\mathbf{p}_b}(F_bF_{ac}+F_cF_{ba}-2F_bF_aF_c)$$

$$=\frac{\partial F_a}{\partial\mathbf{p}_a}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{ac}}{\partial\mathbf{q}_a}F_{bc}$$

$$+\frac{\partial F_b}{\partial\mathbf{p}_b}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{bc}}{\partial\mathbf{q}_b}F_{ac}$$

$$=\frac{\partial F_a}{\partial\mathbf{p}_a}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{ac}}{\partial\mathbf{q}_a}F_{bc}'$$

$$+\frac{\partial F_b}{\partial\mathbf{p}_b}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{bc}}{\partial\mathbf{q}_b}F_{ac}'.$$

Here we have omitted terms containing F_c, which do not contribute to the integrals, since

$$\int d\mathbf{q}_c\, \partial\phi_{ac}/\partial\mathbf{q}_a=0;\quad \int d\mathbf{q}_c\, \partial\phi_{bc}/\partial\mathbf{q}_b=0.$$

When such a substitution is made, Eqs. (12) and (13) become

$$\frac{\partial F_a}{\partial t}=\sum_b \frac{N_b}{V}\int\int d\mathbf{q}_b d\mathbf{p}_b \frac{\partial\phi_{ab}}{\partial\mathbf{q}_a}\cdot\frac{\partial F_{ab}'}{\partial\mathbf{p}_a}, \tag{15}$$

$$\left\{\frac{\partial}{\partial t}+\frac{\mathbf{p}_a}{m_a}\cdot\frac{\partial}{\partial\mathbf{q}_a}+\frac{\mathbf{p}_b}{m_b}\cdot\frac{\partial}{\partial\mathbf{q}_b}\right\}F_{ab}'=\frac{\partial\phi_{ab}}{\partial\mathbf{q}_a}\cdot\frac{\partial F_a}{\partial\mathbf{p}_a}F_b+\frac{\partial\phi_{ba}}{\partial\mathbf{q}_b}\cdot\frac{\partial F_b}{\partial\mathbf{p}_b}F_a$$

$$+\frac{\partial F_a}{\partial\mathbf{p}_a}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{ac}}{\partial\mathbf{q}_a}F_{bc}'$$

$$+\frac{\partial F_b}{\partial\mathbf{p}_b}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c \frac{\partial\phi_{bc}}{\partial\mathbf{q}_b}F_{ac}'$$

$$+\frac{\partial\phi_{ab}}{\partial\mathbf{q}_a}\cdot\frac{\partial F_{ab}'}{\partial\mathbf{p}_a}+\frac{\partial\phi_{ba}}{\partial\mathbf{q}_b}\cdot\frac{\partial F_{ab}'}{\partial\mathbf{p}_b}. \tag{16}$$

Equation (16) differs from that of Bogoliubov-Temko, where the last two terms are missing. Equations (15) and (16) serve as basis for the dynamical theory of the plasma and the derivation of the kinetic equation. It is to be noted that the right-hand side of Eq. (16) consists of three pairs of terms. The first pair is responsible for the main structure in the form of the Fokker-Planck equation. The second pair (nonlinear terms) controls large distances, and therefore the shielding phenomena. Finally the last pair, governed by the disturbance F_{ab}', refers to small distances.

IV. INTEGRATION OF THE CORRELATION EQUATION

The procedure to be used is to integrate the correlation Eq. (16) for F_{ab}'. Then a simple substitution into Eq. (15) will give the kinetic equation for F_a, in the form of a generalized Fokker-Planck equation.

When we consider the right-hand side of Eq. (16), represented by $G(t, \mathbf{q}_a-\mathbf{q}_b, \mathbf{p}_a,\mathbf{p}_b)$, as provisionally given, Eq. (16) can be formally integrated along its characteristics which are given by

$$d\mathbf{q}_a/dt=\mathbf{p}_a/m_a;\quad d\mathbf{q}_b/dt=\mathbf{p}_b/m_b;\quad d\mathbf{p}_a/dt=d\mathbf{p}_b/dt=0.$$

The integral is

$$F_{ab}'(t, \mathbf{q}_a-\mathbf{q}_b, \mathbf{p}_a,\mathbf{p}_b)=\int_0^{t-t_0} d\tau\, G(t-\tau, \mathbf{q}_a-\mathbf{q}_b-\mathbf{g}\tau, \mathbf{p}_a,\mathbf{p}_b)+F_{ab}'[t_0, \mathbf{q}_a-\mathbf{q}_b-\mathbf{g}(t-t_0)],$$

where

$$\mathbf{g}=(\mathbf{p}_a/m_a)-(\mathbf{p}_b/m_b)=\text{relative velocity}.$$

Except for the case of $\mathbf{g}=0$, the initial correlation can be made to vanish at $t_0=-\infty$, since the pair of particles a and b will be independent of each other when their distance is large enough. Thus we have a weakening of correlations toward the infinite past,

$$F_{ab}'[t_0,\mathbf{q}_a-\mathbf{q}_b-\mathbf{g}(t-t_0)]=0 \quad\text{for}\quad t_0=-\infty.$$

Hence we obtain:

$$F_{ab}'(t,\mathbf{q}_a-\mathbf{q}_b,\mathbf{p}_a,\mathbf{p}_b)=\int_0^\infty d\tau\, G(t-\tau,\mathbf{q}_a-\mathbf{q}_b-\mathbf{g}\tau,\mathbf{p}_a,\mathbf{p}_b),$$

or, when G is written out in its full form,

$$F_{ab}'(t,\mathbf{p}_a,\mathbf{p}_b,\mathbf{q}_a-\mathbf{q}_b)=\int_0^\infty d\tau\left\{\frac{\partial\phi_{ab}}{\partial\mathbf{q}_a}\cdot\frac{\partial F_a}{\partial\mathbf{p}_a}F_b+\frac{\partial\phi_{ba}}{\partial\mathbf{q}_b}\cdot\frac{\partial F_b}{\partial\mathbf{p}_b}F_a+\frac{\partial F_a}{\partial\mathbf{p}_a}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c\frac{\partial\phi_{ac}}{\partial\mathbf{q}_a}F_{bc}'+\frac{\partial F_b}{\partial\mathbf{p}_b}\cdot\sum_c \frac{N_c}{V}\int\int d\mathbf{q}_c d\mathbf{p}_c\frac{\partial\phi_{bc}}{\partial\mathbf{q}_b}F_{ac}'+\frac{\partial\phi_{ab}}{\partial\mathbf{q}_a}\frac{\partial F_{ab}'}{\partial\mathbf{p}_a}+\frac{\partial\phi_{ba}}{\partial\mathbf{q}_b}\cdot\frac{\partial F_{ab}'}{\partial\mathbf{p}_b}\right\}_{t-\tau,\mathbf{q}_a-(\mathbf{p}_a/m_a)\tau,\mathbf{q}_b-(\mathbf{p}_b/m_b)\tau} \tag{17}$$

where

$$\{\cdots\}_{t-\tau,\mathbf{q}_a-(\mathbf{p}_a/m_a)\tau,\mathbf{q}_b-(\mathbf{p}_b/m_b)\tau}$$

denotes that the variables t, $\mathbf{q}_a$, $\mathbf{q}_b$ between the brackets are to be replaced by $t-\tau$, $\mathbf{q}_a-(\mathbf{p}_a/m_a)\tau$, and $\mathbf{q}_b-(\mathbf{p}_b/m_b)\tau$, respectively.

One may ask where the irreversibility character of the problem comes in, since the Liouville equation, which has been used as a starting point, is a theorem taken from classical mechanics which therefore represents a reversible relation. The problem of the origin of irreversibility is a general problem in statistical mechanics, and that it has been studied by authors as Kac, Uhlenbeck, etc., to whom one would refer for a deeper analysis. It can be observed, however, that in the present treatment irreversibility is introduced in the integration of the correlation function with respect to the time in Eq. (17). Here it has been supposed that the initial value of F_{ab}' is zero for $t_0=-\infty$, as shown above. Mathematically it would be possible just as well to write the integral in a different form with an integration constant $F_{ab}'(t_0)$ referring to an instant t_0 in the future. However, we do not assume with equal confidence that F_{ab}' goes to zero when t tends to $+\infty$, as our habitual method of reasoning assumes that correlation is brought about by what has happened since some past, and not that correlation is beforehand arranged in such a way that it will automatically disappear in the future.

Such a condition of the weakening of correlation in the infinite past must be distinguished from the condition of the weakening of correlation at large distances. The latter condition is needed in formulating the degeneration of correlations, see Eq. (14) and Appendix A.

V. TRANSPORT FUNCTIONS

In order to analyze the correlation function, we use the Fourier integral representation

$$\phi_{ab}(q)=e_ae_b\int_{-\infty}^{+\infty}d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q})Y(\nu),$$

$$F_{ab}'(\mathbf{q})=e_ae_b\int_{-\infty}^{+\infty}d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q})Z_{ab}(\mathbf{v}),$$

which has the advantage of contracting the various distances of different kind of particles in one single variable $\mathbf{v}$. Then Eqs. (15) and (17) take the following forms

$$\frac{\partial F_a}{\partial t}=-\frac{(2\pi)^3}{V}\sum_b N_be_b^2e_a^2\frac{\partial}{\partial\mathbf{p}_a}\cdot\int d\mathbf{v}\,i\mathbf{v}Y(\nu)\times\int d\mathbf{p}_b\,Z_{ab}(t,\mathbf{p}_a,\mathbf{p}_b,\mathbf{v}),\tag{18a}$$

$$Z_{ab}=Z_{ab}{}^{\mathrm{I}}+Z_{ab}{}^{\mathrm{II}}+Z_{ab}{}^{\mathrm{III}},\tag{18b}$$

with

$$Z_{ab}{}^{\mathrm{I}}=\int_0^\infty d\tau\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)i\nu_hY(\nu)\times\left(\frac{\partial}{\partial p_{ah}}-\frac{\partial}{\partial p_{bh}}\right)\{F_a(t-\tau,\mathbf{p}_a)F_b(t-\tau,\mathbf{p}_b)\},$$

$$Z_{ab}{}^{\mathrm{II}}=\int_0^\infty d\tau\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)i\nu_hY(\nu)\times\frac{(2\pi)^3}{V}\sum_c N_ce_c^2\left\{\frac{\partial F_a}{\partial p_{ah}}\int d\mathbf{p}_c\,Z_{bc}(t-\tau,-\mathbf{v})-\frac{\partial F_b}{\partial p_{bh}}\int d\mathbf{p}_c\,Z_{ac}(t-\tau,\mathbf{v})\right\},\tag{19a}$$

$$Z_{ab}{}^{\mathrm{III}}=e_ae_b\int_0^\infty d\tau\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\int d\mathbf{v}'\,i\nu_h'Y(\nu')\times\left(\frac{\partial}{\partial p_{ah}}-\frac{\partial}{\partial p_{bh}}\right)Z_{ab}(t-\tau,\,\mathbf{p}_a,\mathbf{p}_b,\mathbf{v}-\mathbf{v}').$$

Here the repeated index h denotes a summation. Since Z_{ab} is the Fourier transform of F_{ab}', and $Z_{ab}{}^{\mathrm{II}}$, $Z_{ab}{}^{\mathrm{III}}$ involve Z_{ab}, etc., Eq. (18b) can be considered as an integral indicial equation.

On the right-hand side of Eq. (18b), there are 3 transport terms, each consisting of two parts. The term $Z_{ab}{}^{\mathrm{I}}$ is called convection function in momentum space. Representing a transport of correlation by pair interaction, it causes two fluxes: a flux of diffusion along the momentum of the test particle itself, and a flux producing a friction. These two fluxes represent the main structure of the Fokker-Planck equation. The term $Z_{ab}{}^{\mathrm{II}}$ is a shielding function. It arises from the cross-interaction of a pair of particles with a third one. It represents the nonlinear behavior due to the cooperation of the third ones in overcoming the friction. Finally the term $Z_{ab}{}^{\mathrm{III}}$ is a secondary convection associated with disturbnaces in correlation.

In order to calculate the correlation function and to derive the kinetic equation from Eqs. (18a) and (18b), we shall investigate the three transport functions. In doing so, it is convenient to introduce some new notations, as follows

$$\begin{aligned}
&\mathcal{E}_h(t)=\left(\frac{\partial}{\partial p_{ah}}-\frac{\partial}{\partial p_{bh}}\right)[F_a(t,\mathbf{p}_a)F_b(t,\mathbf{p}_b)],\\
&\epsilon_h(t,F_a,F_b,\mathbf{v})=\int_0^\infty d\tau\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\,\mathcal{E}_h(t-\tau),\\
&C(t,F_a,\mathbf{v})=\nu_h\int d\mathbf{p}_b\,\epsilon_h(t,F_a,F_b,\mathbf{v}),\\
&C^*=\frac{i}{kT_b}F_a{}^0\cong C(F_a{}^0,\mathbf{v}),\ \text{the Maxwellian value,}\\
&H_a{}^b=\int d\mathbf{p}_b\,Z_{ab},\\
&\Lambda_a(t,\mathbf{p}_a,\mathbf{v})=\sum_c N_ce_c^2H_a{}^c(t,\mathbf{p}_a,\mathbf{v}),\\
&Y_0(\nu)=(2\pi^2\nu^2)^{-1};\quad Y_\kappa(\nu)=(2\pi^2)^{-1}(\nu^2+\kappa^2)^{-1},\\
&\kappa^2=\sum_b\frac{4\pi}{VkT_b}N_be_b^2;\quad \gamma=m_b/m_a,\\
&\qquad\gamma^*=(m_b/m_a)(T_a/T_b),\\
&F_a{}^0=(2\pi m_akT_a)^{-\frac{3}{2}}\exp(-p_a{}^2/2m_akT_a),\\
&F_b{}^0=(2\pi m_bkT_b)^{-\frac{3}{2}}\exp(-p_b{}^2/2m_bkT_b),
\end{aligned}\tag{20}$$

where κ is the wave number of Debye shielding, i.e., the product of the reciprocal of the Debye radius with 2π; Y_0 and Y_κ are the Fourier transforms of the Coulomb potential $1/q$ and of the Debye potential $q^{-1}\exp(-\kappa q)$, respectively, while Y denotes the Fourier transform of an arbitrary potential. In $\mathcal{E}_h(t-\tau)$, $t-\tau$ holds for $F_a(t-\tau, \mathbf{p}_a)$ and $F_b(t-\tau, \mathbf{p}_b)$.

We shall calculate the transport functions in two forms: (19a) and

$$\sum_b N_b e_b{}^2 \int Z_{ab}{}^{\mathrm{I}} d\mathbf{p}_b, \text{ etc.} \tag{19b}$$

The latter form is useful in the calculation of $\partial F_a/\partial t$ by means of Eq. (18a).

In principle the procedure is to solve first Z_{cb} from Eq. (18b). Then a substitution of the solution into Eq. (18a) will give the kinetic equation. In view of the special form of the right-hand side of Eq. (18a), it appears simpler to solve not for Z_{ab} but for Λ_a, which is

$$\Lambda_a = \sum_b N_b e_b{}^2 \int Z_{ab} d\mathbf{p}_b.$$

Before going to this end, it is necessary to simplify the transport functions as defined by Eqs. (19a). The details of the transformations will be given in Appendix B. They amount to reduce Z_{bc} of Eqs. (19a) into Z_{ab}, and to localize the time dependence in the singlet distributions, in lieu of Z_{ab}. In the notations of Eqs. (20), the results of transformations of Eqs. (19a) are

$$Z_{ab}{}^{\mathrm{I}} = i\nu_h \epsilon_h Y, \tag{21a}$$

$$Z_{ab}{}^{\mathrm{II}} = \frac{(2\pi)^3}{V} i\nu_h \epsilon_h Y \Lambda_a / F_a, \tag{21b}$$

$$Z_{ab}{}^{\mathrm{III}} = \frac{e_a e_b}{8\pi} \frac{i\nu_h \epsilon_h}{\nu Y} \frac{H_a{}^b}{F_a}, \tag{21c}$$

$$Z_{ab} \equiv Z_{ab}{}^{\mathrm{I}} + Z_{ab}{}^{\mathrm{II}} + Z_{ab}{}^{\mathrm{III}}$$
$$= i\nu_h \epsilon_h Y + \frac{(2\pi)^3}{V} i\nu_h \epsilon_h Y \Lambda_a / F_a + \frac{e_a e_b}{8\pi} \frac{i\nu_h \epsilon_h}{\nu Y} \frac{H_a{}^b}{F_a}. \tag{21d}$$

When Eqs. (21) are integrated with respect to $\mathbf{p}_b$, multiplied by $N_b e_b{}^2$, and summed over b, we can write the transport functions in the forms

$$\sum_b N_b e_b{}^2 \int Z_{ab}{}^{\mathrm{I}} d\mathbf{p}_b = iY \sum_b N_b e_b{}^2 C, \tag{22a}$$

$$\sum_b N_b e_b{}^2 \int Z_{ab}{}^{\mathrm{II}} d\mathbf{p}_b = \frac{(2\pi)^3}{V} iY \sum_b N_b e_b{}^2 (C/F_a) \Lambda_a, \tag{22b}$$

$$\sum_b N_b e_b{}^2 \int Z_{ab}{}^{\mathrm{III}} d\mathbf{p}_b = -(\nu/\nu^*) \Lambda_a, \tag{22c}$$

where ν^* is a wave number of the order of the reciprocal of the Landau parameter e^2/kT, or "molecular diameter," or mean distance of closest approach. More rigorously it depends on singlet distributions. Its origin and the difficulties involved are indicated in Appendix B.

The sum of Eqs. (22a), (22b), and (22c) yields

$$\Lambda_a = iY \sum_b N_b e_b{}^2 C$$
$$+ (2\pi)^3 V^{-1} iY \sum_b N_b e_b (C/F_a) \Lambda_a - (\nu/\nu^*) \Lambda_a. \tag{22d}$$

On solving for Λ_a, we obtain

$$\Lambda_a = \sum_b N_b e_b{}^2 H_a{}^b = iY \Delta_a{}^{-1} \sum_b N_b e_b{}^2 C, \tag{23a}$$

where

$$\Delta_a = \Delta(F_a)$$
$$= 1 - (2\pi)^3 V^{-1} iY \sum_b N_b e_b{}^2 C/F_a + (\nu/\nu^*). \tag{23b}$$

Δ_a is the shielding function. When the Coulomb potential is considered,

$$Y = Y_0 = (2\pi^2 \nu^2)^{-1},$$

and when the singlet distribution is linearized to be Maxwellian $F_a = F_a{}^0$, Eq. (23b) reduces to

$$\Delta_0 = \Delta(F_a{}^0) = 1 + \frac{\kappa^2}{\nu^2} + \frac{\nu}{\nu^*}. \tag{23c}$$

It is to be noted that, after such a linearization, the shielding function (23c) determines the long-distance cutoff at the Debye length and the short-distance cutoff at the Landau parameter $1/\nu^*$.

VI. KINETIC EQUATION AND GENERALIZED FOKKER-PLANCK EQUATION

The knowledge of Λ_a found in Eq. (23a) enables us to calculate F_a from Eq. (18a), which can be rewritten as follows:

$$\frac{\partial F_a}{\partial t} = -\frac{(2\pi)^3}{V} e_a{}^2 \frac{\partial}{\partial \mathbf{p}_a} \left[\int d\boldsymbol{\nu}\, i\boldsymbol{\nu} Y(\nu) \Lambda_a(t, p_a, \boldsymbol{\nu}) \right]. \tag{24}$$

It is to be remarked that, according to the integral on the right-hand side of Eq. (24), only odd values of $\Lambda_a(\boldsymbol{\nu})$ may contribute to F_a, since $Y(\nu)$ is necessarily even. Thus in the following the odd terms will be carefully distinguished from the even ones. Such a screening against the even part in Λ_a has the effect of selecting the real values of the collision integral, so that any periodicity must be ruled out in the distribution function.

After substitution for Λ_a, we obtain from Eq. (24)

$$\frac{\partial F_a}{\partial t}=\frac{(2\pi)^3}{V}e_a^2\int d\mathbf{v}\,\nu_k Y^2\frac{\partial}{\partial p_{ak}}\sum_b N_b e_b^2\Delta_a^{-1}C$$
$$=-\frac{\partial}{\partial p_{ak}}\Big[\frac{(2\pi)^3}{V}e_a^2\int d\mathbf{v}\,\nu_k\nu_h Y^2\sum_b N_b e_b^2\Delta_a^{-1}\times\int_0^\infty d\tau\int d\mathbf{p}_b\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\Big(\frac{\partial F_b}{\partial p_{bh}}F_a\Big)_{t-\tau}\Big]$$
$$+\frac{\partial}{\partial p_{ak}}\Big[\frac{(2\pi)^3}{V}e_a^2\int d\mathbf{v}\,\nu_k\nu_h Y^2\sum_b N_b e_b^2\Delta_a^{-1}\times\int_0^\infty d\tau\int d\mathbf{p}_b\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\Big(F_b\frac{\partial F_a}{\partial p_{ah}}\Big)_{t-\tau}\Big].\quad(25)$$

Since the right-hand side of Eq. (25) consists of two terms in the first and the second derivatives of F_a with respect to $\mathbf{p}_a$, Eq. (25) takes the shape of the Fokker-Planck equation. The coefficients before the two derivatives determine the friction and the diffusion. These coefficients involve time integrations of the past history of the distribution functions, with the "memory" kernel depending on relative velocities and distances between particles.

If the distribution $F_a(t-\tau)$ varies slowly with time, a series development is permissible which reduces Eq. (25) to the following differential equation:

$$\frac{\partial F_a}{\partial t}=-\frac{\partial}{\partial p_{ak}}\Big\{\sum_{r=0}^{\infty}\frac{\partial^r}{\partial t^r}[A_k^{(r)}(\Delta_a,F_b)F_a]\Big\}+\frac{\partial}{\partial p_{ak}}\Big\{\sum_{r=0}^{\infty}\frac{\partial^r}{\partial t^r}\Big[B_{kh}^{(r)}\frac{\partial F_a}{\partial p_{ah}}\Big]\Big\}.\quad(26)$$

The values of the coefficients are:

$$A_k^{(r)}(\Delta_a,F_b)=\frac{(2\pi)^3}{V}e_a^2\int d\mathbf{v}\,\nu_k\nu_h Y^2(\nu)\times\sum_b N_b e_b^2\beta_h^{(r)}(F_b)/\Delta_a,$$
$$B_{kh}^{(r)}(\Delta_a,F_b)=\frac{(2\pi)^3}{V}e_a^2\int d\mathbf{v}\,\nu_k\nu_h Y^2(\nu)\times\sum_b N_b e_b^2\beta'^{(r)}(F_b)/\Delta_a,\quad(27a)$$

where

$$\beta_h^{(r)}(F_b)=\int_0^\infty d\tau\int d\mathbf{p}_b\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\frac{(-\tau)^r}{r!}\frac{\partial F_b}{\partial p_{bh}}=\gamma^{-1}\frac{\partial\beta'^{(r)}}{\partial p_{ah}},\quad(27b)$$

$$\beta'^{(r)}(F_b)=\int_0^\infty d\tau\int d\mathbf{p}_b\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\frac{(-\tau)^r}{r!}F_b.$$

The terms contributed by $r\neq 0$ represent the non-Markovian behavior.

Since the integrated Liouville equations (7) and (8), from which the kinetic equation is derived in the forms (25) or (26), are nonlinear, the latter is generally expected to be nonlinear too. This is seen from the dependence of Δ_a on F_a, and from the dependence of the coefficients A, B on F_a, through the summation over all particles including the kind a. However, no significant difference can be expected, when the nonequilibrium distribution in Δ_a is replaced by an equilibrium distribution, since the long range interaction is represented by the summation term in Eq. (25), and each term of the sum is a small deviation from the Maxwellian distribution only.

Moreover, in the kinetic Eq. (26), the particle a may be considered as a test particle, the motion of which is nonstationary. It is embedded in a cloud of other particles b (field particles), which interact with each other and with a. For the same reason, we shall assume that the field particles are in equilibrium with a distribution $F_b=F_b^0$, and shall investigate the motion of the test particle belonging to the plasma. As a consequence, Eqs. (25) and (26) become linearized.

Furthermore, if we confine ourselves to the terms with $r=0$, by neglecting the non-Markovian behavior, Eq. (26) degenerates into the following classical Fokker-Planck equation:

$$\frac{\partial F_a}{\partial t}=-\frac{\partial}{\partial p_{ak}}\{A_k^0(\Delta_0,F_b^0)F_a\}+\frac{\partial}{\partial p_{ak}}\Big\{B_{kh}^0(\Delta_0,F_b^0)\frac{\partial F_a}{\partial p_{ah}}\Big\}.\quad(28)$$

Equations (25), (26), and (28) are different forms of the fundamental kinetic equation for a plasma. Equation (25) is the integral form of the kinetic equation, while in Eq. (26) the integral form is replaced by a series expansion. The nonlinear behavior of Eqs. (25) and (26) is included in the denominator Δ_a and in the coefficients. The non-Markovian behavior results from the relation of the distribution function F_a to the correlation F_aF_b, taken at an earlier time $t-\tau$. Such a memory should be distributed according to the spectrum of the potential function by means of the factor $\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)$. Finally Eq. (28) is the degenerate form of the kinetic equation in the Fokker-Planck type, obtained by keeping only terms with $r=0$ in the series expansion. The coefficients in the Fokker-Planck equation (28) are found to depend on the law of interaction; they will be calculated in Sec. VII.

Equation (28) can also be written in the following form:

$$\frac{\partial F_a}{\partial t}=-\frac{\partial}{\partial p_{ak}}(A_k{}^*F_a)+\frac{\partial^2}{\partial p_{ak}\partial p_{ah}}(B_{kh}{}^*F_a)\quad(29a)$$

by introducing

$$A_k^* = A_k^0 + \partial B_{kh}^0/\partial p_{ah}, \quad B_{kh}^* = B_{kh}^0. \tag{29b}$$

In the following, A_k^* and A_k^0 will be called, respectively, *dynamical friction* and *friction by polarization; B_{kh}^** or B_{kh}^0 will be called *diffusion*. By taking the moment of Eq. (29a), the dynamical friction A_k^* is obtained.

VII. COEFFICIENTS OF THE FOKKER-PLANCK EQUATION

In order to study the friction and the diffusion in a plasma, we shall investigate the coefficients A_k^0 and B_{kh}^0 in the Fokker-Planck equation (28) by means of the Maxwellian approximation. They can be calculated by the use of formulas (27a), rewritten as follows

$$A_k^0 = \frac{(2\pi)^3}{V} e_a^2 \int d\mathbf{v}\, \nu_k \nu_h Y^2(\nu) \sum_b N_b e_b^2 \beta_h^0(F_b^0)/\Delta_0,$$
$$B_{kh}^0 = \frac{(2\pi)^3}{V} e_a^2 \int d\mathbf{v}\, \nu_k \nu_h Y^2(\nu) \sum_b N_b e_b^2 \beta'^0(F_b^0)/\Delta_0, \tag{30}$$

where

$$\beta_h^0(F_b^0) = \int_0^\infty d\tau \int d\mathbf{p}_b \exp(-i\mathbf{v}\cdot\mathbf{g}\tau) \frac{\partial F_b^0}{\partial p_{bh}} = \gamma^{-1} \frac{\partial \beta'^0}{\partial p_{ah}},$$
$$\beta'^0(F_b^0) = \int_0^\infty d\tau \int d\mathbf{p}_b \exp(-i\mathbf{v}\cdot\mathbf{g}\tau) F_b^0. \tag{31a}$$

With the substitution of F_b^0, Eqs. (31a) become

$$\beta_h^0(F_b^0) = -\frac{i\nu_h}{m_b} \int_0^\infty d\tau\, \tau \exp\left[-\frac{\nu^2 k T_b}{2m_b}\tau^2 - i\nu_k \frac{p_{ak}}{m_a}\tau\right],$$
$$\beta'^0(F_b^0) = \int_0^\infty d\tau \exp\left[-\frac{\nu^2 k T_b}{2m_b}\tau^2 - i\nu_k \frac{p_{ak}}{m_a}\tau\right]. \tag{31b}$$

After some transformations of Eqs. (31b) and substitution into Eqs. (30), the latter are reduced to

$$A_k^0 = -\frac{2e_a^2}{\pi V} (kT_a)^{-\frac{3}{2}} m_a^{-1} p_{ah} \times \int d\mathbf{v} \frac{\nu_k \nu_h}{\nu^5 \Delta_0} \sum_b N_b e_b^2 m_b^{\frac{1}{2}} (T_a/T_b)^{\frac{3}{2}} U_1, \tag{32}$$

$$B_{kh}^0 = \frac{2e_a^2}{\pi V} (kT_a)^{-\frac{1}{2}} \int d\mathbf{v} \frac{\nu_k \nu_h}{\nu^5 \Delta_0} \sum_b N_b e_b^2 m_b^{\frac{1}{2}} (T_a/T_b)^{\frac{1}{2}} U_1,$$

where

$$U_1 = (\pi/2)^{\frac{1}{2}} \exp(-u^2/2),$$
$$u = \sqrt{2}\zeta \cos\theta,$$
$$\zeta = (m_b T_a / m_a T_b)^{\frac{1}{2}} \zeta_a, \quad \zeta_a = p_a (2m_a k T_a)^{-\frac{1}{2}},$$
$$\cos\theta = p_{ah} \nu_h / p_a \nu. \tag{33}$$

θ is the angle between $\mathbf{p}_a$ and $\mathbf{v}$. The transformation of Eqs. (31b) and the intermediate calculations leading to Eqs. (32) are referred to Appendix C.

Some relations connecting the two coefficients can be derived as follows. First, we can write the relation

$$A_k^0 = -2\zeta_a (p_{ah}/p_a) B_{kh}^0, \tag{34a}$$

which is derived immediately from Eqs. (32) for the case $T_a = T_b$. Another relation is obtained, if Eqs. (30) are written in the form

$$A_k^0 = \frac{(2\pi)^3}{V} e_a^2 \sum_b N_b e_b^2 \gamma^{-1} \frac{\partial T_{kh}}{\partial p_{ah}},$$
$$B_{kh}^0 = \frac{(2\pi)^3}{V} e_a^2 \sum_b N_b e_b^2 T_{kh},$$

where

$$T_{kh}(\gamma) = \int d\mathbf{v}\, \nu_k \nu_h Y^2 \beta'^0 \Delta_0^{-1}.$$

If T_{kh} can be represented by a power function

$$T_{kh} = \gamma^{*r} \mu_{kh}^{(r)},$$

where $\mu_{kh}^{(r)}$ does not depend on the field particles b, we obtain

$$A_k^0 = \frac{\partial B_{kh}^0}{\partial p_{ah}} \frac{K^2(r-1)}{K'^2(r)}. \tag{34b}$$

This equation is valid for $T_a \neq T_b$ or $T_a = T_b$. The notations K and K' are defined by Eqs. (36b).

When the masses and temperatures for the two kinds of particles are identical, Eq. (34b) reduces to the well-known relation

$$A_k^0 = \partial B_{kh}^0 / \partial p_{ah},$$

and hence Eq. (29b) becomes

$$A_k^* = 2A_k^0. \tag{34c}$$

VIII. FRICTION BY POLARIZATION AND DYNAMICAL FRICTION

Since the two frictions are related by simple expressions (29b) and (34c), we shall calculate only the friction by polarization A_k^0 from the first of Eqs. (32), which can be rewritten as follows, after substitution for U_1 from notation (33):

$$A_k^0 = -\frac{2e_a^2}{\pi V m_a} (kT_a)^{-\frac{3}{2}} \sum_b N_b e_b^2 (m_b)^{\frac{1}{2}} (T_a/T_b)^{\frac{3}{2}} I_k,$$

where

$$I_k = (\pi/2)^{\frac{1}{2}} \int d\mathbf{v} \frac{\nu_k \nu_h p_{ah}}{\nu^5 \Delta_0} \exp(-\zeta^2 \cos^2\theta).$$

We introduce spherical coordinates with $\mathbf{p}_a$ along the polar axis, and write

$$\nu_k = \nu(\alpha_{1k} \cos\theta + \alpha_{2k} \sin\theta \cos\phi + \alpha_{3k} \sin\theta \sin\phi),$$

where α_{1k}, α_{2k}, α_{3k} are the direction cosines of ν_k. Carrying out the integration with regard to ϕ, one obtains

$$I_k=(2\pi^3)^{\frac{1}{2}}p_{ak}L\int_{-1}^{1}dz\, z^2\exp(-\zeta^2z^2).$$

Here

$$z=\cos\theta,$$

and

$$L=\int_0^\infty d\nu\frac{\nu}{\nu^2+\kappa^2+\nu^3/\nu^*} \tag{35a}$$

$$=\ln(1+\nu^{*2}/\kappa^2)^{\frac{1}{2}}\cong\ln(\nu^*/\kappa).$$

The term ν^3/ν^* plays the role of a cutoff at the upper limit ν^*. Carrying out the integration with regard to z, we obtain

$$I_k=-2^{-\frac{1}{2}}\pi^2p_{ak}L\zeta^{-1}I_0, \tag{35b}$$

where

$$I_0=\frac{d}{d\zeta}(\zeta^{-1}\,\mathrm{erf}\zeta).$$

Hence

$$A_k^0=\tfrac{1}{2}e_a^2(p_{ak}/p_a)L\sum_b K_b^2(0)I_0, \tag{36a}$$

where ζ is defined by Eqs. (33), and

$$K_b^2(r)=\frac{4\pi}{VkT_b}N_be_b^2\left(\frac{m_b}{m_a}\frac{T_a}{T_b}\right)^r,$$

$$K_b'^2(r)=\frac{4\pi}{VkT_a}N_be_b^2\left(\frac{m_b}{m_a}\frac{T_a}{T_b}\right)^r,$$

$$K^2(r)=\sum_b K_b^2(r),\quad K'(r)=\sum_b K_b'^2(r),$$

$$\kappa^2=K^2(0),\qquad \kappa'^2=K'^2(0). \tag{36b}$$

The wave number κ of Debye shielding was defined in Eqs. (20) too.

The ratio p_{ak}/p_a indicates that the friction is opposed to the direction of the motion. The following asymptotic formulas can be used

$$I_0=\frac{d}{d\zeta}(\zeta^{-1}\,\mathrm{erf}\zeta)$$

$$=-(4/3\sqrt{\pi})\zeta(1-\tfrac{3}{5}\zeta^2),\ \text{for small}\ \zeta;$$

$$=-\zeta^{-2}[1-(2/\sqrt{\pi})\zeta\exp(-\zeta^2)],\ \text{for large}\ \zeta.$$

Thus Eq. (36a) reduces to

$$A_k^0=-\frac{2}{3\pi^{\frac{1}{2}}}e_a^2\frac{p_{ak}}{p_a}LK^2(\tfrac{1}{2})\zeta_a\left\{1-\frac{3}{5}\frac{K^2(\frac{3}{2})}{K^2(\frac{1}{2})}\zeta_a^2\right\}$$

$$\text{for small}\ \zeta_a, \tag{37a}$$

and

$$A_k^0=-\tfrac{1}{2}e_a^2\frac{p_{ak}}{p_a}L\zeta_a^{-2}$$

$$\times\left\{K^2(-1)-\frac{2}{\sqrt{\pi}}\zeta_a\sum_b K_b^2(-\tfrac{1}{2})\exp(-\gamma^*\zeta_a^2)\right\}$$

$$\text{for large}\ \zeta_a. \tag{37b}$$

When the mass of electron is assumed negligible as compared to the mass of ion, Eq. (36a) degenerates into the following formula of friction for ion a.

$$A_k^0=\tfrac{1}{2}e_a^2(p_{ak}/p_a)L\frac{4\pi N_ae_a^2}{VkT_a}\frac{d}{d\zeta_a}(\zeta_a^{-1}\,\mathrm{erf}\zeta_a)$$

$$=-(p_{ak}/p_a)L\frac{2\pi N_ae_a^4}{VkT_a}\times\begin{cases}(4/3\sqrt{\pi})\zeta_a, & \text{for small}\ \zeta_a\\ \zeta_a^{-2}, & \text{for large}\ \zeta_a.\end{cases} \tag{37c}$$

Here ζ_a is the dimensionless velocity defined by Eqs. (33). The friction for slow particles increases linearly with increasing velocity, as in Brownian motion, while the friction of fast particles decreases inversely as the square of the velocity. Thus the fast particles are not so much hindered by the plasma cloud as are the slow ones, a property compatible with the conditions of fusion.

The maximum value of expression (37c) is determined by the condition

$$\frac{d^2}{d\zeta_a^2}(\zeta_a^{-1}\,\mathrm{erf}\zeta_a)=0.$$

This gives

$$\zeta_a=0.97.$$

A criterion for maximum friction is thus obtained: The friction is largest, when the kinetic energy and the thermal energy are about equal.

IX. FRICTIONAL POTENTIAL

On the basis of the frictional force A_k^0, we introduce a friction potential ψ, which can be represented by the Fourier expansion:

$$\psi=\int d\mathbf{v}\,\eta(\mathbf{v})\exp(i\mathbf{v}\cdot\mathbf{q}_a).$$

We have

$$A_k^0=-(\partial\psi/\partial q_{ak})_{q_a=0}$$

$$=-\int d\mathbf{v}\,\eta(\mathbf{v})i\nu_k.$$

Now according to Eqs. (C4) of Appendix C,

$$A_k^0=-(2\pi e_a)^2\int d\mathbf{v}\,\nu_kY^2\Delta_0^{-1}$$

$$\times\{\tfrac{1}{2}i\kappa^2+D_a[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})]\}$$

where both odd and even terms are retained in the integrand, it follows that

$$\eta(\mathbf{v})=-i(2\pi e_a)^2 Y^2\Delta_0^{-1}\{\tfrac{1}{2}i\kappa^2+D_a[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})]\}$$

$$\psi=-i(2\pi e_a)^2\int d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q}_a)Y^2\Delta_0^{-1}\times\{\tfrac{1}{2}i\kappa^2+D_a[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})]\}.$$

The notation D and P have been defined in Eqs. (C1). The friction potential can be written as a sum:

$$\psi=\psi_0+\psi_1,$$

where

$$\psi_0=e_a^2\int d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q}_a)2\pi^2\kappa^2Y^2/\Delta_0 \tag{38a}$$

is independent of velocity and is the equilibrium potential, while

$$\psi_1=-2ie_a^2\int d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q}_a)2\pi^2Y^2\Delta_0^{-1}D_a\times[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})] \tag{38b}$$

depends on the velocity and is thus a nonequilibrium contribution to the potential.

A. Equilibrium Potential

By taking the Coulomb force in Eq. (38a), we find the following equilibrium potential:

$$\psi_0=\frac{e_a^2}{2\pi^2}\int d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q}_a)\frac{\kappa^2}{\nu^2(\kappa^2+\nu^2+\nu^3/\nu^*)}=\frac{e_a^2}{q_a}(1-e^{-\kappa q_a})\quad\text{for}\quad \nu^*\to\infty. \tag{39a}$$

It is to be noted that the shielding effect of the plasma cloud is indicated by the exponential function. The equilibrium potential is isotropic, as expected.

B. Nonequilibrium Potential Field about a Moving Particle

In A we have found that the equilibrium potential is isotropic about the test particle a. If the particle a moves with a velocity p_{ak}/m_a in the k direction, it is expected that an excess of particles of charge equal to e_a will accumulate ahead of the moving particle a and, conversely, that an excess of particles of charge opposite to e_a will accumulate behind it. Under such circumstances, polarization occurs, and the potential field becomes asymmetrical. In the present section we shall investigate the asymmetrical distribution of the nonequilibrium potential field about a moving particle of low energy.

If the Coulomb expression for Y is used and the values for D_a, P_1, and P_2 are substituted from Eqs. (C1), Eq. (38b) reduces to

$$\psi_1=-\tfrac{1}{2}ie_a^2\pi^{-\frac{3}{2}}\zeta_aK^2(\tfrac{1}{2})J, \tag{39b}$$

where

$$J=\int d\mathbf{v}\exp(i\mathbf{v}\cdot\mathbf{q}_a)\nu^{-4}\Delta_0^{-1}p_{ak}\nu_k/p_a\nu. \tag{40}$$

After some simplifying transformations which we refer to Appendix D, we reduce Eq. (40) to

$$J=i\pi\frac{\mathbf{p}_a\cdot\mathbf{q}_a}{p_aq_a}(2\kappa^2q_a)^{-1},$$

which, substituted into Eq. (39b), yields

$$\psi_1=\frac{\pi^{-\frac{1}{2}}}{4}\frac{e_a^2}{q_a}\frac{K^2(\frac{1}{2})}{\kappa^2}\frac{\mathbf{p}_a\cdot\mathbf{q}_a}{p_aq_a}\zeta_a. \tag{41}$$

Note that the usual Coulomb factor e_a^2/q_a occurs in both potentials (39a) and (41). However, the nonequilibrium potential has a directional dependence, which results in an ellipsoidal anisotropy. The shielding effect of the plasma cloud depends on the mass ratio in the nonequilibrium potential, in contradistinction to the case of equilibrium potential.

X. DIFFUSION

The coefficient of diffusion in momentum space is given by the second of Eqs. (32), which after slight transformation may be rewritten in the following form:

$$B_{kh}^0=\frac{e_a^2}{2^{\frac{3}{2}}\pi^2}p_a\zeta_a^{-1}\sum_b K_b'^2(\tfrac{1}{2})I_{kh}, \tag{42}$$

where

$$I_{kh}=\int d\mathbf{v}\frac{\nu_k\nu_h}{\nu^5\Delta_0}U_1=(\pi/2)^{\frac{1}{2}}\int d\mathbf{v}\frac{\nu_k\nu_h}{\nu^5\Delta_0}\exp(-\zeta^2\cos^2\theta), \tag{43a}$$

where $K_b'^2$ is the shielding wave number defined by Eqs. (36b).

After integration with the use of polar coordinates, Eq. (43a) is transformed into:

$$I_{kh}=\frac{\pi^2}{\sqrt{2}}L\left[-(3\alpha_{1k}\alpha_{1h}-\delta_{kh})\frac{1}{2\zeta}\frac{d}{d\zeta}(\zeta^{-1}\operatorname{erf}\zeta)+(\delta_{kh}-\alpha_{1k}\alpha_{1h})\zeta^{-1}\operatorname{erf}\zeta\right], \tag{43b}$$

where $\boldsymbol{\alpha}_1$ is a unit vector along the direction of $\mathbf{p}_a$.

On substituting Eq. (43b) into Eq. (42), we obtain

$$B_{kh}^0=\tfrac{1}{4}e_a^2Lp_a\zeta_a^{-2}\times\sum_b K_b'^2(0)\left\{-\tfrac{1}{2}(3\alpha_{1k}\alpha_{1h}-\delta_{kh})\frac{d}{d\zeta}(\zeta^{-1}\operatorname{erf}\zeta)+(\delta_{1h}-\alpha_{1k}\alpha_{1h})\operatorname{erf}\zeta\right\}. \tag{44a}$$

The asymptotic values are

$$B_{kh}^0=\frac{1}{3\sqrt{\pi}}e_a^2Lp_a\zeta_a^{-1}K'^2(\tfrac{1}{2})\times\left\{\delta_{kh}-\frac{K'^2(\frac{3}{2})}{K'^2(\frac{1}{2})}(\tfrac{2}{5}\alpha_{1k}\alpha_{1h}+\tfrac{1}{5}\delta_{kh})\zeta_a^2\right\}\quad\text{for small }\zeta_a,\quad(44\text{b})$$

$$B_{kh}^0=\tfrac{1}{4}e_a^2Lp_a\zeta_a^{-2}K'^2(0)\times\left\{(\delta_{kh}-\alpha_{1k}\alpha_{1h})+\frac{1}{2}\frac{K'^2(-1)}{K'^2(0)}(3\alpha_{1k}\alpha_{1h}-\delta_{kh})\zeta_a^{-2}\right\}\quad\text{for large }\zeta_a,\quad(44\text{c})$$

with

$$\alpha_{1k}=p_{ak}/p_a,\quad \alpha_{1h}=p_{ah}/p_a.$$

One may verify that the above values (44b) and (44c) of B_{kh}^0, together with the values of A_k^0 found in Eqs. (37a) and (37b), satisfy the relation (34b). For that purpose, the derivatives of B_{kh}^0 may be calculated from Eqs. (44b) and (44c) as follows:

$$\begin{aligned}\partial B_{kh}^0/\partial p_{ah}&=-(2/3\sqrt{\pi})e_a^2(p_{ak}/p_a)LK'^2(\tfrac{3}{2})\zeta_a\quad\text{for small }\zeta_a,\\ \partial B_{kh}^0/\partial p_{ah}&=-\tfrac{1}{2}e_a^2(p_{ak}/p_a)LK'^2(0)\zeta_a^{-2}\quad\text{for large }\zeta_a.\end{aligned}\quad(45)$$

The following simplifications are used in the derivation of Eqs. (45)

$$\begin{aligned}&\partial(p_a\zeta_a^{-2}\delta_{kh})/\partial p_{ah}=-\zeta_a^{-2}p_{ak}/p_a,\\ &\partial(\alpha_{1k}\alpha_{1h}p_a\zeta_a^{-2})/\partial p_{ah}=\zeta_a^{-2}p_{ak}/p_a,\\ &\partial(\alpha_{1k}\alpha_{1h}\zeta_a^2)/\partial p_{ah}=4\zeta_a^2p_{ak}/p_a^2,\\ &\partial(\delta_{kh}\zeta_a^2)/\partial p_{ah}=2\zeta_a^2p_{ak}/p_a^2.\end{aligned}$$

By comparing Eqs. (37a), (37b), and (45), we obtain the following relation

$$A_k^0=\frac{K^2(r-1)}{K'^2(r)}\frac{\partial B_{kh}^0}{\partial p_{ah}},$$

$$r=\tfrac{3}{2}\text{ for small }\zeta_a$$
$$=0\text{ for large }\zeta_a.$$

This result was predicted in Eq. (34b).

Formulas (44) indicate that the diffusion coefficient of ions is nearly independent of temperature at low temperature and varies as $T_a^{-\frac{1}{2}}$ at high temperature.

XI. DISCUSSIONS

The development given in Sec. II–IV leading to the kinetic equation (25) may be adapted to the case of a more general interaction potential ϕ_{ij} than the Coulomb interaction. To go to such a generalization, the following modifications must be kept in mind:

1. The product $e_ae_bY_0$ of Eq. (25) must be replaced by the Fourier transform of ϕ_{ij} in its general form.
2. The denominator Δ_a must be calculated anew for the general potential of interaction. In practice, when the potential varies more rapidly than the Coulomb law, the convergence of the integral of the kinetic equation (25) is ensured at distant interaction, even without interference by ternary correlation. Thus we may approximate Δ_a by unity.

The Fokker-Planck equation has been derived earlier by Chandrasekhar, Spitzer, Rosenbluth, etc. The kinetic equation (25) in the general Fokker-Planck form developed here has the following advantages: (a) It considers triple correlations. (b) As a consequence, the coefficients of friction and diffusion are convergent without external cutoffs. (c) The non-Markovian behavior is considered.

XII. ACKNOWLEDGMENT

The author takes this opportunity to express his appreciation to Professor J. M. Burgers of University of Maryland for valuable discussions on questions connected with this work and with reference 11.

APPENDIX A. DEGENERATION OF CORRELATIONS

Under the assumption of weak interaction (small ϕ), the correlation of any order can be degenerated into binary correlations and singlet distributions. In order to derive a formula of degeneration of correlations, we start from Eq. (4) which governs the correlation functions, and which is rewritten in a slightly different form as follows:

$$\frac{dF_s}{dt}=\sum_{1\le i<j\le s}\frac{\partial\phi_{ij}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_s}{\partial\mathbf{p}_i}+\int d\mathbf{x}_{s+1}\sum_{1\le i\le s}\frac{N}{V}\frac{\partial\phi_{i,s+1}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_{s+1}}{\partial\mathbf{p}_i},\quad(\text{A}1)$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_{1\le i\le s}\frac{\mathbf{p}_i}{m_i}\cdot\frac{\partial}{\partial\mathbf{q}_i}.$$

We assume that the interaction is small by writing

$$\phi=(1/\lambda)\psi,$$

where λ is a large quantity. If Eq. (A1) were reduced into dimensionless form, the dimensionless expression of $1/\lambda$ would be

$$1/\lambda=(e^2/kT)(N/V)^{\frac{1}{3}}.$$

But in order to avoid the reduction into dimensionless form, we can carry out the expansion in powers of $1/\lambda$ as follows

$$F_s=\sum_{r=0}^{\infty}\frac{1}{\lambda^r}F_s^{(r)}.\quad(\text{A}2)$$

Such an expansion has been used by Bogoliubov[9] and suggested by Burgers. By substituting Eq. (A2) into Eq. (A1), and by reducing into the same order of magnitude, we obtain

$$\frac{dF_s^{(r)}}{dt}=\sum_{1\leq i<j\leq s}\frac{\partial\psi_{ij}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_s^{(r-1)}}{\partial\mathbf{p}_i}+\sum_{1\leq i\leq s}\int dx_{s+1}\frac{N}{V}\frac{\partial\psi_{i,s+1}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_{s+1}^{(r-1)}}{\partial\mathbf{p}_i},\tag{A3}$$

where r denotes the order of approximation.

For the zero-order approximation ($r=0$), Eq. (A3) reduces into the following equation:

$$dF_s^{(0)}/dt=0$$

from which we obtain the solution

$$F_s^{(0)}=\prod_{1\leq i\leq s}F_1(\mathbf{p}_i).\tag{A4}$$

The time dependence is included in $F_1(\mathbf{p}_i)$ after the integration along the characteristics.

Further, for the first order approximation ($r=1$), Eq. (A3) reduces into

$$\frac{dF_s^{(1)}}{dt}=\sum_{1\leq i<j\leq s}\frac{\partial\psi_{ij}}{\partial\mathbf{q}_i}\cdot\prod_{1\leq l\leq s}\frac{\partial F_1(p_l)}{\partial\mathbf{p}_i}$$

$$=\sum_{1\leq i<j\leq s}\frac{\partial\psi_{ij}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_1(\mathbf{p}_i)}{\partial\mathbf{p}_i}F_1(\mathbf{p}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l).\tag{A5}$$

The integral term of Eq. (A3) is not carried over into Eq. (A5), because the integration of $\partial\psi_{i,s+1}/\partial\mathbf{q}_{s+1}$ with respect to $d\mathbf{q}_{s+1}$ is zero in the present case of $r=1$.

The expression under the summation sign of Eq. (A5) is exactly

$$\frac{dF_2^{(1)}(\mathbf{x}_i,\mathbf{x}_j)}{dt}=\frac{\partial\psi_{ij}}{\partial\mathbf{q}_i}\cdot\frac{\partial F_1(\mathbf{p}_i)}{\partial\mathbf{p}_i}F_1(\mathbf{p}_j),\quad(j\neq i)\tag{A6}$$

so that Eq. (A5) can be rewritten as follows

$$\frac{dF_s^{(1)}}{dt}=\sum_{1\leq i<j\leq s}\frac{d}{dt}F_2^{(1)}(\mathbf{x}_i,\mathbf{x}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l),$$

whence, after integration, we have

$$F_s^{(1)}=\sum_{1\leq i<j\leq s}F_2^{(1)}(\mathbf{x}_i,\mathbf{x}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l).\tag{A7}$$

The integration constant, which would contain the parameters $\mathbf{p}_1,\cdots,\mathbf{p}_s$, must vanish, if the following condition of the weakening of correlations at large distances is to be fulfilled:

$$F_2^{(1)}(\mathbf{x}_i,\mathbf{x}_j)=0\quad\text{if}\quad|\mathbf{q}_i-\mathbf{q}_j|=\infty$$

Consider now the expression

$$\sum_{1\leq i<j\leq s}F_2^{(0)}(\mathbf{p}_i,\mathbf{p}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l).\tag{A8}$$

It can be written as

$$\sum_{1\leq i<j\leq s}F_1(\mathbf{p}_i)F_1(\mathbf{p}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l)=\frac{s(s-1)}{2}\prod_{1\leq l\leq s}F_1(\mathbf{p}_l).\tag{A9}$$

Thus we can write

$$F_s^{(0)}\equiv\prod_{1\leq l\leq s}F_1(\mathbf{p}_l)\tag{A10}$$

$$=\sum_{1\leq i<j\leq s}F_2^{(0)}(\mathbf{p}_i,\mathbf{p}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l)-\left[\frac{s(s-1)}{2}-1\right]\prod_{1\leq l\leq s}F_1(\mathbf{p}_l),\tag{A11}$$

by adding (A8) and subtracting (A9) from the right-hand side of (A10). Finally when Eq. (A7) is added term by term to Eq. (A11), after multiplication of the former equation by $1/\lambda$, we obtain

$$F_s=\sum_{1\leq i<j\leq s}F_2(\mathbf{x}_i,\mathbf{x}_j)\prod_{\substack{1\leq l\leq s\\ l\neq i\neq j}}F_1(\mathbf{p}_l)-\left[\frac{s(s-1)}{2}-1\right]\prod_{1\leq l\leq s}F_1(\mathbf{p}_l).\tag{A12}$$

We conclude that the correlation of any order can be degenerated into F_1 and F_2 as a first approximation by formula (A12). Such a degeneration of correlations serve as a basis for solving the hierarchy of equations of correlations in Sec. III. In fact when $s=3$, Eq. (A12) degenerates into Eq. (14) of Sec. III, which is the crucial equation for breaking the chain of correlations in the problem.

APPENDIX B. TRANSFORMATION OF TRANSPORT FUNCTIONS

In the following paragraphs (a), (b), and (c), we are concerned with some details of transformation of the three transport functions, as defined by Eqs. (19a). The results of transformation are given in Eqs. (21).

(a) Convection Function

By means of the notations (20), the convection function may be written as follows:

$$Z_{ab}^{\mathrm{I}}=i\nu_h\epsilon_h Y.\tag{B1}$$

(b) Shielding Function

The shielding function has the role of shielding the interaction potential at large distances by the plasma cloud.

The shielding function has two parts, one of which, $Z_{bc}\partial F_a/\partial \mathbf{p}_a$, may be considered as a compensating term. We shall assume there that the distance effect is much more important than the effect of anisotropy. Therefore, in such a compensating term, the dependence of the correlation on the distribution functions can assume the following structure:

$$Z_{bc}=F_bF_c\chi(\mathbf{v}), \tag{B2}$$

where $\chi(\mathbf{v})$ should not contain any additional distribution or time variation.

In the equilibrium case, Z_{bc} is found to be isotropic, and $\chi(\nu)$ agrees with the Debye distribution

$$\chi(\nu)=Y_\kappa(\nu), \tag{B3}$$

which is spherical and is shielded at the Debye radius. In the nonequilibrium case, Z_{bc} should be anisotropic, with an ellipsoidal distribution. Thus the assumption (B2) amounts to replace the correlation ellipsoid Z_{bc} by an effective correlation sphere, with a radius and a distribution adjusted to the true correlation. It is to be expected that such a simplification holds for distances not too small, i.e., ν not too large. Since the shielding function is effective at large distances only, the simplification made in Eq. (B2) is reasonable.

From Eq. (B2), there results the following relation:

$$Z_{bc}(t,\mathbf{p}_b,\mathbf{p}_c,-\mathbf{v})=(F_b/F_a)Z_{ac}(t,\mathbf{p}_a,\mathbf{p}_c,\mathbf{v}),$$

so that the shielding function defined by Eq. (19a) can be written as follows

$$\begin{aligned}Z_{ab}{}^{\mathrm{II}}(t,F_a,F_b,\mathbf{v})&=\frac{(2\pi)^3}{V}\sum_c N_ce_c{}^2i\nu_hY(\nu)\\&\times\int_0^\infty d\tau\,\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\mathcal{E}_h(t-\tau,F_a,F_b)\\&\times\int dp_c\frac{Z_{ac}(t-\tau,\mathbf{p}_a,\mathbf{p}_c,\mathbf{v})}{F_a(t-\tau,p_a)}\\&=\frac{(2\pi)^3}{V}i\nu_hY\int_0^\infty d\tau\,\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\mathcal{E}_h(t-\tau)\\&\times\frac{\Lambda_a(t-\tau,\mathbf{p}_a,\mathbf{v})}{F_a(t-\tau,\mathbf{p}_a)},\end{aligned} \tag{B4}$$

where Λ_a is defined by Eqs. (20).

Since the two parts of the shielding function have certain properties of nonlinearity and of coupled symmetry that produce the compensating phenomena, it is important in the following calculations either to leave the correlation function in unknown form in each part or to assume a known form such as Eq. (B2). It would not be appropriate to treat them differently, by taking one correlation as unknown and the other known. Such a practice would disturb the coherent balance of the pair and introduce some unwarranted asymmetry. We shall here retain them as unknowns in the differential equation.

As is justifiable on the basis of Eq. (B2), we replace

$$\Lambda_a(t-\tau,\mathbf{p}_a,\mathbf{v})/F_a(t-\tau,\mathbf{p}_a)$$

by

$$\Lambda_a(t,\mathbf{p}_a,\mathbf{v})/F_a(t,\mathbf{p}_a)$$

in Eq. (B4) so that the latter reduces to

$$Z_{ab}{}^{\mathrm{II}}(t,F_a,F_b,\mathbf{v})=\frac{(2\pi)^3}{V}i\nu_hY(\nu)\epsilon_h(t,F_a,F_b,\mathbf{v})\frac{\Lambda_a(t,\mathbf{p}_a,\mathbf{v})}{F_a(t,\mathbf{p}_a)}. \tag{B5}$$

Accordingly, Λ_a/F_a gets separated from the time integral.

(c) Secondary Convection

The secondary convection function $Z_{ab}{}^{\mathrm{III}}$, as defined by the third of Eqs. (19a), is originated from the disturbances of the binary correlation function, and therefore controls close encounters. It can be inferred that the effect of close encounters will be much less than that of the "grazing" deflections, if the density of the gas is low. We shall assume that $Z_{ab}{}^{\mathrm{III}}$ is not an important term, and shall estimate it by allowing rough approximations.

If the assumption (B2) is used in the secondary convection function, we can write in the integrand of the third of Eqs. (19a):

$$\left(\frac{\partial}{\partial p_{ah}}-\frac{\partial}{\partial p_{bh}}\right)Z_{ab}=\mathcal{E}_h\int d\mathbf{p}_b\,Z_{ab}/F_a=\mathcal{E}_hH_a{}^b/F_a,$$

so that the secondary convection function becomes

$$\begin{aligned}Z_{ab}{}^{\mathrm{III}}&=e_ae_b\int_0^\infty d\tau\int d\mathbf{v}'\,\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)i\nu_h{}'Y(\nu')\\&\times\left(\frac{\partial}{\partial p_{ah}}-\frac{\partial}{\partial p_{bh}}\right)Z_{ab}(t-\tau,\mathbf{p}_a,\mathbf{p}_b,\mathbf{v}-\mathbf{v}')\\&=e_ae_b\int_0^\infty d\tau\,\exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\mathcal{E}_h(t-\tau)\\&\times\int d\nu'\,i\nu_h{}'Y(\nu')\frac{H_a{}^b(t-\tau,\mathbf{p}_a,\mathbf{v}-\mathbf{v}')}{F_a(t-\tau,\mathbf{p}_a)}.\end{aligned}$$

By replacing

$$H_a{}^b(t-\tau,\mathbf{p}_a,\mathbf{v}-\mathbf{v}')/F_a(t-\tau,\mathbf{p}_a)$$

by

$$H_a{}^b(t,\mathbf{p}_a,\mathbf{v}-\mathbf{v}')/F_a(t-\tau,\mathbf{p}_a)$$

on the basis of Eq. (B2), we can write $Z_{ab}{}^{\mathrm{III}}$ in the following approximate form:

$$Z_{ab}{}^{\rm III}\cong e_a e_b \int_0^\infty d\tau \exp(-i\mathbf{v}\cdot\mathbf{g}\tau)\,\mathcal{E}_h(t-\tau)$$

$$\times\int d\mathbf{v}'\, i\nu_h' Y(\nu') \frac{H_a{}^b(t,\mathbf{p}_a,\mathbf{v}-\mathbf{v}')}{F_a(t,\mathbf{p}_a)}$$

$$= e_a e_b \epsilon_h \int d\mathbf{v}'\, i\nu_h' Y(\nu') \frac{H_a{}^b(t,p_a,\mathbf{v}-\mathbf{v}')}{F_a}. \quad \text{(B6)}$$

A rough estimate of expression (B6) can be made by using the equilibrium approximation (B3) for $H_a{}^b$. To this end, we write expression (B6) as follows:

$$Z_{ab}{}^{\rm III} = e_a e_b \epsilon_h \frac{H_a{}^b(t,\mathbf{p}_a,\mathbf{v})}{F_a Y_\kappa(\nu)} R_h. \quad \text{(B7)}$$

Here

$$R_h = \int d\mathbf{v}'\, i\nu_h' Y(\nu') \frac{H_a{}^{0b}(t,\mathbf{p}_a,\mathbf{v}-\mathbf{v}')}{F_a}$$

$$= \int d\mathbf{v}'\, i\nu_h' Y(\nu') Y_k(|\mathbf{v}-\mathbf{v}'|)$$

$$= (2\pi^2)^{-2} \int d\nu' \frac{i\nu_h'}{\nu'^2[(\mathbf{v}-\mathbf{v}')^2+\kappa^2]}$$

for the case of a Coulomb field. The integration can be performed by taking spherical polar coordinates with $\mathbf{v}$ along the polar axis. Then we can write

$$R_h = (2\pi^2)^{-2} \int_0^{\nu_1\to\infty} d\nu' \int_0^\pi d\theta \int_0^{2\pi} d\varphi$$

$$\times \sin\theta \frac{i\nu'(\alpha_{1h}\cos\theta + \alpha_{2h}\sin\theta\cos\varphi + \alpha_{3h}\sin\theta\sin\varphi)}{\nu'^2 - 2\nu\nu'\cos\theta + \nu^2 + \kappa^2},$$

where α_{1h}, α_{2h}, α_{3h} are the direction cosines of ν_h. After integration with respect to φ and with $z=\cos\theta$, we find

$$R_h = (i\alpha_{1h}/8\pi)R_0,$$

where

$$R_0 = \left(\frac{2}{\pi}\right)^2 \int_{-1}^{-1} dz\, z \int_0^{\nu_1\to\infty} d\nu' \frac{\nu'}{\nu'^2 - 2z\nu\nu' + \nu^2 + \kappa^2}$$

$$= (2/\pi)[-(v^2-1)^{\frac{1}{2}} + v^2 \sin^{-1}(1/v)], \quad v^2 = 1+\kappa^2/\nu^2$$

$$= 0 \quad \text{for} \quad \nu/\kappa = 0,$$

$$= 1 \quad \text{for} \quad \nu/\kappa = \infty.$$

R_0 increases monotonically from 0 to unity as ν/κ increases. As a convenient value for R_0, we may take

$$R_0 \cong (Y_\kappa/Y)^s,$$

where the exponent s is an empirical parameter. Its value lies between 0 and 1. The final result does not depend appreciably on the particular choice of this value, as will be shown later. It follows that

$$R_h = (i\nu_h/8\pi\nu)(Y_\kappa/Y)^s. \quad \text{(B8)}$$

By substituting Eq. (B8) into Eq. (B7), we obtain

$$Z_{ab}{}^{\rm III} \cong \frac{e_a e_b}{8\pi} \frac{i\nu_h \epsilon_h}{\nu Y} \frac{H_a{}^b}{F_a} \left(\frac{Y_\kappa}{Y}\right)^{s-1}, \quad \text{(B9)}$$

and

$$\sum_b N_b e_b{}^2 \int d\mathbf{p}_b\, Z_{ab}{}^{\rm III} = \frac{e_a}{8\pi} \frac{i}{\nu Y} \left(\frac{Y_\kappa}{Y}\right)^{s-1}$$

$$\times \sum_b N_b e_b{}^3 (C/F_a) H_a{}^b. \quad \text{(B10)}$$

The summation term of the right-hand side of Eq. (B10) can be approximated by

$$\sum_b N_b e_b{}^3 C H_a{}^b \cong \sum_b N_b e_b{}^3 C H_a{}^{0b} \left(\frac{\sum_b N_b e_b{}^2 H_a{}^b}{\sum_b N_b e_b{}^2 H_a{}^{0b}}\right),$$

where $H_a{}^{0b}$ is the equilibrium value of $H_a{}^b$, according to Eq. (B3). We have then

$$\sum_b N_b e_b{}^3 C H_a{}^b \cong \left(\frac{\sum_b N_b e_b{}^3 C}{\sum_b N_b e_b{}^2}\right) \sum_b N_b e_b{}^2 H_a{}^b,$$

so that Eq. (B10) simplifies into:

$$\sum_b N_b e_b{}^2 \int d\mathbf{p}_b\, Z_{ab}{}^{\rm III}$$

$$\cong -(\nu/\nu^*)(Y_\kappa/Y)^{s-1} \sum_b N_b e_b{}^2 H_a{}^b, \quad \text{(B11)}$$

where roughly

$$(\nu^*)^{-1} = e_a{}^2/kT_a. \quad \text{(B12)}$$

By comparing Eqs. (B5) and (B9) it is seen that the role of $Z_{ab}{}^{\rm III}$ must lie in the region of large ν, and is supposed to be overtaken by $Z_{ab}{}^{\rm II}$ in the region of small ν. For this reason the asymptotic expressions (B9) and (B11) for large ν can be taken, they are

$$Z_{ab}{}^{\rm III} \cong \frac{e_a e_b}{8\pi} \frac{i\nu_h \epsilon_h}{\nu Y} \frac{H_a{}^b}{F_a}, \quad \text{(B13)}$$

$$\sum_b N_b e_b{}^2 \int d\mathbf{p}_b\, Z_{ab}{}^{\rm III} \cong -(\nu/\nu^*) \sum_b N_b e_b{}^2 H_a{}^b, \quad \text{(B14)}$$

where the exponent s does not play any role, as was mentioned earlier. Equation (B14) suggests a cutoff of the interaction potential at the Landau distance (B12).

It is to be noted that the quantity ν^* defined by Eq. (B12) gives only a very rough estimate of the cutoff of the interaction potential at small distances. A more detailed account of ν^* may be obtained from Eqs. (B10) and (B11) for the case of particles of like charge. Here the repulsion offers an adequate mechanism of cutoff in the interaction potential. However, difficulties may appear when we deal with particles of opposite charge, because the expression (B10) can be positive, negative or zero. For instance, if we have only two kinds of particles, both with unit charge, one kind positive, the other negative, with the same number density for each of them, we find that $\sum N_b e_b^3$ vanishes. When the positive ions have a multiple charge, while the gas as a whole again is neutral, $\sum N_b e_b^3$ will be positive, and thus (B10) will be positive if e_a is positive (ions), and negative if e_a is negative (that is, when the subscript a belongs to the electrons). In the latter case, Δ_0 of Eq. (23c) might become zero for some large value of ν. It is thus necessary to introduce the value of ν^* as an upper limit for ν. Probably the Landau value kT/e^2 as upper limit will serve equally well.

The effect of the cutoff introduced in this way can be interpreted by observing that it amounts to a change in the Fourier transform of the Coulomb potential function between two particles, of such nature that the potential for small distances (of the order e^2/kT) does not go to infinity as $1/r$, but is limited to a finite value. This has the consequence that the features which might be produced by close encounters between particles are not described in the proper way. But close encounters between particles of opposite charge never can be described in a proper way unless attention is given to the possibility of a chemical combination. On the other hand, from the general structure of the equations it can be inferred that the effect of close encounters will be much less than that of the "grazing" deflections when the density of the gas is low. We shall assume therefore that the cutoff does not introduce a serious error.

APPENDIX C. TRANSFORMATIONS OF THE COEFFICIENTS OF THE FOKKER-PLANCK EQUATION

We shall transform the expressions (30) for the coefficients of the Fokker-Planck equation. Since even values of β^0 and β'^0 in ν will contribute, we shall make a careful distinction between the even and the odd parts, and therefore introduce the following notations:

$$\sigma = T_b/T_a, \tag{C1}$$

$$\zeta_a = p_a(2m_akT_a)^{-\frac{1}{2}},$$

$$\zeta = (m_bT_a/m_aT_b)^{\frac{1}{2}}\zeta_a,$$

$\cos\theta = p_{ah}\nu_h/p_a\nu$, θ being the angle between $\mathbf{p}_a$ and $\mathbf{v}$,

$$u = \sqrt{2}\zeta\cos\theta,$$

$U_1 = (\pi/2)^{\frac{1}{2}}\exp(-u^2/2)$, even in ν,

$U_2 = \exp(-u^2/2)\int_0^u du'\exp(u'^2/2)$, odd in ν,

$$P_1(r) = \sum_b N_be_b^2(m_b)^{\frac{1}{2}}\sigma^r U_1,$$

$$P_2(r) = \sum_b N_be_b^2(m_b)^{\frac{1}{2}}\sigma^r U_2,$$

$$D_a = (8\pi^2/m_a)^{\frac{1}{2}}(VkT_a)^{-1}\zeta_a\cos\theta.$$

Then we can rewrite Eqs. (31b) in the form

$$\begin{aligned}\beta_h^0(F_b^0) &= -(\nu_h/\nu^2kT_b)[i+u(U_1-iU_2)],\\ \beta'^0(F_b^0) &= (m_b/\nu^2kT_b)^{\frac{1}{2}}(U_1-iU_2).\end{aligned} \tag{C2}$$

From Eqs. (C2), it then follows that

$$\begin{aligned}\textstyle\sum_b N_be_b^2\beta_h^0 &= -(V/2\pi)(\nu_h/\nu^2)\\ &\quad\times\{\tfrac{1}{2}i\kappa^2+D_a[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})]\},\\ \textstyle\sum_b N_be_b^2\beta'^0 &= (\nu^2kT_a)^{-\frac{1}{2}}[P_1(-\tfrac{1}{2})-iP_2(-\tfrac{1}{2})].\end{aligned} \tag{C3}$$

Hence $C(F_a^0)$ written in the notation (20) can be calculated. The result is

$$C(F_a^0)/F_a^0 = (i/kT_b) - (U_1 - iU_2) \times (m_akT_a)^{-1}(m_b/\nu^2kT_b)^{\frac{1}{2}}\nu_hp_{ah}(1-T_a/T_b).$$

The last term can be neglected, if T_a and T_b are not very different. This simplification results in the approximate value C^* in notation (20).

When Eqs. (C3) are substituted into Eqs. (30), the coefficients in the Fokker-Planck equation can be written as follows:

$$\begin{aligned}A_k^0 &= -(2\pi e_a)^2\int d\mathbf{v}\ \nu_kY^2Q_A,\\ B_{kh}^0 &= \frac{(2\pi)^3}{V}e_a^2(kT_a)^{-\frac{1}{2}}\int d\mathbf{v}(\nu_k\nu_h/\nu)Y^2Q_B.\end{aligned} \tag{C4}$$

Here

$$\begin{aligned}Q_A &= \Delta_0^{-1}\{\tfrac{1}{2}i\kappa^2+D_a[P_1(-\tfrac{3}{2})-iP_2(-\tfrac{3}{2})]\},\\ Q_B &= \Delta_0^{-1}[P_1(-\tfrac{1}{2})-iP_2(-\tfrac{1}{2})].\end{aligned} \tag{C5}$$

Separating the even and odd terms in Eqs. (C5), one can reduce Eqs. (C4) to

$$A_k^0 = -(2\pi e_a)^2\int d\mathbf{v}\ \nu_kY^2D_aP_1(-\tfrac{3}{2})\Delta_0^{-1},$$

$$B_{kh}^0 = \frac{(2\pi)^3}{V}e_a^2(kT_a)^{-\frac{1}{2}}\int d\mathbf{v}\ (\nu_k\nu_h/\nu)Y^2P_1(-\tfrac{1}{2})\Delta_0^{-1},$$

or, with the use of notations (C1), we can write

$$A_k^0 = -\frac{2e_a^2}{\pi V}m_a^{-1}(kT_a)^{-\frac{3}{2}}p_{ah}\int d\mathbf{v}\frac{\nu_k\nu_h}{\nu^5\Delta_0} \times\sum_b N_be_b^2m_b^{\frac{1}{2}}(T_a/T_b)^{\frac{3}{2}}U_1,$$

$$B_{kh}^0 = \frac{2e_a^2}{\pi V}(kT_a)^{-\frac{1}{2}}\int d\mathbf{v}\frac{\nu_k\nu_h}{\nu^5\Delta_0}\sum_b N_be_b^2m_b^{\frac{1}{2}}(T_a/T_b)^{\frac{1}{2}}U_1.$$

This result of transformations is used in Eqs. (32).

APPENDIX D. ANISOTROPIC DISTRIBUTION OF THE FRICTION POTENTIAL

The anisotropic part of the friction potential, given by Eq. (39b), contains a geometrical factor J illustrating an ellipsoidal anisotropy. It is defined by Eq. (40), rewritten as follows:

$$J=\int d\mathbf{v}\,\exp(i\mathbf{v}\cdot\mathbf{q}_a)\nu^{-4}\Delta_0^{-1}p_{ak}\nu_k/p_a\nu. \tag{D1}$$

In order to calculate J, we use spherical coordinates with $\mathbf{q}_a$ along the polar axis.

First we have

$$\mathbf{v}\cdot\mathbf{q}_a=\nu q_a\cos\psi.$$

Let p_a, φ', ψ' be the coordinates of $\mathbf{p}_a$. Then

$$\mathbf{p}_a\cdot\mathbf{v}=p_a\nu(\cos\psi\cos\psi'+\sin\psi\sin\varphi\sin\psi'\sin\varphi'+\sin\psi\cos\varphi\sin\psi'\cos\varphi').$$

With these coordinates, Eq. (D1) takes the form

$$J=\int_0^\infty d\nu\int_0^\pi d\psi\int_0^{2\pi}d\varphi\,\nu^{-2}\Delta_0^{-1}\sin\psi e^{i\nu q_a\cos\psi}\times(\cos\psi\cos\psi'+\sin\psi\sin\varphi\sin\psi'\sin\varphi'+\sin\psi\cos\varphi\sin\psi'\cos\varphi'),$$

which, after integration with respect to φ, reduces into

$$J=2\pi\cos\psi'\int_0^\infty\nu^{-2}\Delta_0^{-1}d\nu\int_{-1}^1 dz\,ze^{i\nu q_a z},\quad\text{with}\quad z=\cos\psi.$$

Further integration with respect to z gives

$$J=4\pi i\cos\psi'\int_0^\infty\nu^{-2}\Delta_0^{-1}\left(\frac{\sin\nu q_a}{\nu q_a}-\cos\nu q_a\right)\frac{d\nu}{\nu q_a}, \tag{D2}$$

with

$$\cos\psi'=\mathbf{p}_a\cdot\mathbf{q}_a/p_aq_a.$$

Now denote the integral in the right-hand side of Eq. (D2) by

$$M=\int_0^\infty\nu^{-2}\Delta_0^{-1}\left(\frac{\sin\nu q_a}{\nu^2q_a^2}-\frac{\cos\nu q_a}{\nu q_a}\right)d\nu,$$

and introduce the auxiliary integral

$$S=\int_0^\infty\nu^{-2}\Delta_0^{-1}\sin\nu q_a d\nu\cong\frac{1}{2\kappa}\left\{\int_0^\infty d\nu\frac{e^{-\nu q_a}}{\kappa-\nu}+\int^\infty d\nu\frac{e^{-\nu q_a}}{\kappa+\nu}\right\}.$$

it is easy to verify the following relation:

$$\frac{\partial M}{\partial q_a}=\frac{1}{q_a}(S-2M).$$

As a good approximation we can write

$$M\cong\tfrac{1}{2}S,$$

valid for large q_a. Since the asymptotic value of S is

$$S=(\kappa^2q_a)^{-1},$$

we find

$$M=(2\kappa^2q_a)^{-1}$$

Hence Eq. (D2) becomes

$$J=\pi i\cos\psi'(2\kappa^2q_a)^{-1}.$$

Here the cosine dependence determines the ellipsoidal anisotropy. The result (D3) is used to reduce Eq. (39b) to Eq. (41).

APPENDIX E. GEOMETRICAL TENSOR OF DIFFUSION I_{kh}

The diffusion coefficient B_{kh}^0 is a tensor characterized by the geometrical factor I_{kh}, defined by Eq. (43a).

In order to calculate I_{kh}, we use spherical coordinates, with $\mathbf{p}_a=\boldsymbol{\alpha}_1$ along the polar axis, and $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ as other orthogonal axes. If α_{1k}, α_{2k}, and α_{3k} are projections of $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, and $\boldsymbol{\alpha}_3$ along the directions of p_{ak}, we can write

$$\nu_k=\nu(\alpha_{1k}\cos\theta+\alpha_{2k}\sin\theta\cos\varphi+\alpha_{3k}\sin\theta\sin\varphi).$$

Also

$$\nu_h=\nu(\alpha_{1h}\cos\theta+\alpha_{2h}\sin\theta\cos\varphi+\alpha_{3h}\sin\theta\sin\varphi),$$

and

$$d\mathbf{v}=\nu^2\sin\theta d\theta d\varphi d\nu.$$

Hence

$$I_{kh}=(\pi/2)^{\frac{1}{2}}\int_0^\infty d\nu\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\frac{\nu_k\nu_h}{\nu^3\Delta_0}\exp(-\zeta^2\cos^2\theta).$$

After integration with respect to φ, and putting $\cos\theta=z$, we have

$$I_{kh}=(\pi/2)^{\frac{1}{2}}\int_0^\infty d\nu\int_{-1}^1 dz\,2\pi\nu^{-1}\Delta_0^{-1}\times\{\alpha_{1k}\alpha_{1h}z^2+\tfrac{1}{2}(\alpha_{2k}\alpha_{2h}+\alpha_{3k}\alpha_{3h})(1-z^2)\}\exp(-\zeta^2z^2).$$

Now, since

$$\alpha_{ik}\alpha_{ih}=\delta_{kh},$$

we can write

$$\alpha_{2k}\alpha_{2h}+\alpha_{3k}\alpha_{3h}=\delta_{kh}-\alpha_{1k}\alpha_{1h},$$

so that

$$I_{kh}=(2\pi^3)^{\frac{1}{2}}\int_0^\infty d\nu(\nu\Delta_0)^{-1}\times\int_{-1}^1 dz\exp(-\zeta^2z^2)\{z^2(\tfrac{3}{2}\alpha_{1k}\alpha_{1h}-\tfrac{1}{2}\delta_{kh})+\tfrac{1}{2}(\delta_{kh}-\alpha_{1k}\alpha_{1h})\}.$$

When we replace the integrals in z by the following values:

$$\int_{-1}^1 dz\exp(-\zeta^2z^2)=\sqrt{\pi}\,\mathrm{erf}\zeta/\zeta,$$

and

$$\int_{-1}^1 dz\,z^2\exp(-\zeta^2z^2)=-\frac{\sqrt{\pi}}{2\zeta}\frac{d}{d\zeta}(\mathrm{erf}\zeta/\zeta),$$

and the integral with respect to ν by L, according to

Eq. (35a), we obtain finally

$$I_{kh}=\frac{\pi^2}{\sqrt{2}}L\left\{-(3\alpha_{1k}\alpha_{1h}-\delta_{kh})\frac{1}{2\zeta}\frac{d}{d\zeta}(\text{erf}\zeta/\zeta)+(\delta_{kh}-\alpha_{1k}\alpha_{1h})\,\text{erf}\zeta/\zeta\right\}.$$

This result is used in Eq. (43b). As a partial check of the correctness of this result, we may verify the following relation:

$$I_k=p_{ah}I_{kh},$$

since I_k was calculated in Eq. (35b).

PHYSICAL REVIEW VOLUME 114, NUMBER 2 APRIL 15, 1959

Stark Effect for Cyclotron Resonance in Degenerate Bands

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(Received November 24, 1958)

A calculation of the motion is given for an electron in a simple band subjected to perpendicular magnetic and electric fields. It is shown that the cyclotron resonance frequency is unaffected by the presence of the electric field. For degenerate bands, however, there is a Stark shift of the cyclotron transitions between the low-lying "quantum" states. Calculations using second-order perturbation theory indicate that fractional line shifts of $\Delta\nu/\nu_0 \sim 10\%$ may be obtained under reasonable experimental conditions. This effect may be useful in the study of the valence bands of germanium and silicon.

I. INTRODUCTION

In the presence of a magnetic field $\mathfrak{H}$ the continuum of energy levels for a band coalesces into discrete sub-bands, the so-called Landau levels. An electric field applied perpendicular to $\mathfrak{H}$ perturbs the motion of the carriers in the crystal giving rise to "Stark" energy shifts of these Landau levels. For simple bands, however, an appropriate translation of the coordinate axes can transform the Schrödinger equation to a form free of the electric field. Although this transformation displaces the Landau levels, it will be shown that the selection rules allow transitions only between levels which have undergone equal energy shifts. Consequently, there is no observable effect on the cyclotron resonance lines corresponding to these transitions.

An example where this is not the case is provided by degenerate bands such as the valence bands in germanium and silicon. In the framework of the effective-mass formalism, Luttinger[1] has determined in detail the energy level schemes for these bands in the presence of a magnetic field. Here the situation is described by a system of coupled Schrödinger equations. These calculations predict that the spacing of low-lying energy levels will deviate considerably from the classical cyclotron frequencies. These anomalous "quantum" effects have been observed in germanium by Fletcher, Yager, and Merritt[2] in cyclotron resonance and by Zwerdling, Lax, and Roth[3] in the oscillatory magneto-absorption effect.

If the above system is perturbed by a uniform electric field, the coupled Schrödinger equations no longer admit the simple transformation of the classical case; and, consequently, the low-lying quantum levels may experience a Stark shift. That the energy differences do indeed undergo a shift will be shown in Sec. III using second-order perturbation theory for the approximate "isotropic" case described by Luttinger.[1] In this model the energy surfaces of the valence band are assumed spherical rather than fluted so that the energy levels are independent of the direction of the magnetic field. For a more realistic comparison with experiment the calculations in principle can be extended to an anisotropic case.

Numerical results indicate that the Stark shift may be large enough to aid in the identification and measurement of many of the cyclotron transitions in germanium involving the low-lying anomalous magnetic states.

II. SIMPLE BANDS

Let us consider first the motion of an electron or hole in a simple band subjected to a uniform magnetic field $\mathfrak{H}$ and a uniform electric field $\mathcal{E}$ perpendicular to $\mathfrak{H}$. We shall assume this band to have ellipsoidal energy surfaces similar to those for the conduction bands of silicon and germanium but centered in the Brillouin zone at $\mathbf{k}=0$:

$$\epsilon(\mathbf{k})=\hbar^2\left(\frac{k_x^2+k_y^2}{2m_1}+\frac{k_z^2}{2m_2}\right), \quad (1)$$

where $\mathbf{k}$ is the usual band wave number that varies over the Brillouin zone. In (1) m_1 and m_2 are the effective masses, respectively, along the transverse and principal axes of the energy ellipsoids. It is not difficult to extend the results to bands having displaced minima at $\mathbf{k}=\mathbf{k}_0$.

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