Elementary Particles in a Finite World Geometry

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To account for the elementary particles a new physical geometry may be needed. A previous suggestion that this geometry may be finite is followed up by determining the representations of the orthogonal (Lorentz-like) group. Because of the existence of a new type of orthogonality-preserving transformation some of the representations are multiple-valued. A change of value is identified with a gauge transformation and electric charge is recognized as a certain number determining the many-valuedness of the representation. This charge number reverses sign under space inversion. The charged pions and sigma particles are correlated with some of the new representations.

1. INTRODUCTION

THE most exciting implication of recent discoveries and not so recent difficulties in physics is that the Euclidean concept of physical geometry breaks down for the subatomic world. Obviously this breakdown cannot be directly tested for there are no measuring rods small enough. This lack indeed leaves it even possible that no metric exists in the small. To account for the elementary particles and their properties a new world geometry should be attempted, one which is like ordinary geometry in the large but in the subatomic is non-Euclidean, non-Riemannian or even nonmetric.

For such a geometry the quantum numbers Q (electric charge), I (isotopic spin), and U (d'Espagnat-Prentki number), or Y (the hyper-charge) characterizing the elementary particles should, like spin, arise naturally as characteristic of representations of geometric transformations and should be eigenvalues of operators for these transformations. This is just to say that the geometry should embrace isotopic spin space.

The Euclidean requirement in the large is a difficulty, but merely to get a different geometry is simple. Instead of taking the coordinates of world geometry from the real field (which leads to ordinary Euclidean geometry) we take them from some other ring.[1] To satisfy the difficult requirement that the geometry be Euclidean in the large, a large enough part of the world ring must be like part of the real number system. (For the particular world ring used in this paper, how to do this is known.)

It is necessary to start out with a simplified world ring for the practical reason that otherwise it will be hard to see the physical meaning of the mathematical expressions. We shall at first even oversimplify the problem so that not all our demands will be met, and we naturally begin with restrictions which reduce the amount of unfamiliarity as much as is consistent with some novelty. We shall try a world ring which is a commutative field, and in order to be sure of eliminating divergency difficulties we shall furthermore assume that it is of finite order. (Hence there will be no infinite sums and no infinities can arise.) So long as the order of the world ring is sufficiently large (and it must be enormous to provide for the number of points in the world), there can be no objection in principle to its being finite. Experiment has never forced the conclusion that the number of points is infinite, only that it is very large and atomic physics has always contained the element of discreteness which is one aspect of finiteness. It might seem a strange assumption that there should be some particular number singled out as special in the physical world and yet we already know of such, namely the fine structure constant. In fact, the existence of the fine structure constant is itself the best argument for choosing the world ring to be finite.

All finite fields are known.[2–4] The order of any finite field is a power of a prime, that is, a number of the form p^n where p is a prime and n is an integer. For each such number there is a field and only one field of that order. Such a number field is also known as a Galois field and is denoted by $\mathrm{GF}(p^n)$. If the geometrical coordinates of each point are taken from such a field, that is, if

$$x_\mu \varepsilon \, \mathrm{GF}(p^n),$$

the resulting geometry is a finite geometry[5] with a finite number of points and lines. There is a fundamental length but not a smallest length for the latter concept loses its meaning. It should be clearly understood that this is not a cubic lattice theory.

We shall here consider only the case[6–8] $n=1$ though it will later be seen that this geometry is too simple. We shall find the representations of the orthogonal group associated with the Minkowski metric form and among these will be a new type, multivalued. With the assumption that the various particle field functions transform according to these various representations, it will be possible to recognize the charge number Q as a new

[1] R. H. Bruck, Am. Math. Monthly **62**, Part II, 2 (1955).

[2] L. E. Dickson, *Linear Groups* (B. G. Teubner, Leipzig, 1901).

[3] B. L. van der Waerden, *Modern Algebra* (Frederick Ungar Publishing Company, New York, 1943).

[4] A. A. Albert, *Fundamental Concepts of Higher Algebra* (University of Chicago Press, Chicago, Illinois, 1956).

[5] D. J. Struik, *Analytic and Projective Geometry* (Addison-Wesley Press, Cambridge, Massachusetts, 1953).

[6] G. Jarnefelt, Veröffentlichungen des Finnischen Geodätischen Institutes, No. 36 (1949).

[7] G. Jarnefelt and P. Kustaanheimo, Skandinaviske Matematikerkongress i Trondheim **11**, 166 (1949).

[8] G. Jarnefelt, Ann. Acad. Sci. Fennicae Ser. A.I., No. 96 (1951).

quantum number characteristic of the representation, though not as an operator for geometric transformations. It will then appear that the correct conjugation corresponding to space inversion is not just the parity operator but includes also reversal of charge. It is "combined inversion."

It is our plan to discuss U in another paper and to derive there the Gell-Mann–Nishijima scheme for charged particles.

2. FINITE WORLD GEOMETRY AND THE FINITE WORLD FIELD

A number of years ago it was already conjectured by Jarnefelt[6–8] that physical geometry might be finite with the coordinates elements of a finite field of prime order p:

$$x_\mu \varepsilon\ \mathrm{GF}(p). \tag{2.1}$$

This finite field consists simply of the integers taken *modulo* the prime p. If p is large enough, the points of the geometry are so numerous and so "close together" as to be experimentally indistinguishable from a continuum. This in itself, however, is not enough to guarantee an approximation to Euclidean geometry, for the elements of a finite field are not ordered. They cannot be classified into positive and negative, nor can they be compared in magnitude. In the corresponding geometry the points on a line do not lie in order, and line segments cannot be said to be longer than or shorter than other line segments. Finite geometry does not simply become Euclidean to a better and better approximation as p is increased.

The way around this obstacle was seen by Kustaanheimo.[9] It is not necessary to require that the whole finite geometry be an approximation to Euclidean geometry. All that is needed is for a large enough part of finite world geometry to have the Euclidean properties. In terms of the coordinate field (the number field from which the coordinates are taken), this means that a large enough subset of the elements should be ordered. This may be accomplished by first of all remarking that half the nonzero elements are squares (quadratic residues) and half are not squares (nonresidues), and under multiplication the property of being a square or a not-square is analogous to the property of being positive or negative.[10,11] However the analogy breaks down for addition since it is not necessarily true that the sum of two squares is a square. Nevertheless, if p can be chosen so that the first N integers are squares (*mod* p), N being enormously large, and if then the term "positive" is applied to squares and the term "negative" to not-squares, then the usual arithmetical rules of signs will hold over a very large subset of the elements. If furthermore p can be such that -1 is "negative," then these elements can be "ordered." An element is "greater than" another if their difference is "positive." This is not a true ordering for the relation is not transitive throughout the whole field, but it is transitive within the large subset and this ordering might be called a "local ordering."

In this way the first $N(N\sim q_k\sim \ln p)$ numbers become "ordered" if the prime is chosen to be of the form

$$p=8x\prod_{i=1}^{k} q_i-1, \tag{2.2}$$

where x is an odd integer and $\prod q_i$ is the product of the first k odd primes. Dirichlet's theorem[12] guarantees the existence of a prime of this form. The argument, making use of quadratic reciprocity[13] is that this makes -1 "negative" and makes 2 and the first k odd primes "positive."[14]

With the coordinate field so chosen, the geometry would appear to be ordinary Euclidean geometry up to very large and down to very small distances. Jarnefelt[15] has estimated roughly that for ordinary geometry to hold from 10^{-13} cm to 2×10^9 light years p is of an order of magnitude given by

$$p\sim 10^{10^{81}}. \tag{2.3}$$

Once the limits of "ordinary" dimensions are exceeded, however, the geometry becomes quite different because the coordinate field is not completely ordered.

Exemplifying the novelties which arise in such a theory are the solutions ($p+1$ of them[16]) which exist for the equation

$$\alpha^2+\beta^2=-1. \tag{2.4}$$

For these solutions, at least one of α, β is outside the range of "ordinary" arithmetical behavior.

It will be convenient to introduce "complex" numbers

$$z=x+iy, \tag{2.5}$$

where

$$x,\ y \varepsilon\ \mathrm{GF}(p), \tag{2.6}$$

and

$$i^2=-1. \tag{2.7}$$

This set of p^2 "complex" numbers also forms a field, the finite field $\mathrm{GF}(p^2)$.

For a prime of the form

$$p=4n-1, \tag{2.8}$$

as is the case with the prime given by Eq. (2.2), it will

[9] P. Kustaanheimo, Soc. Sci. Fennica, Commentationes Phy.-Math. **15**, 19 (1950).

[10] C. C. MacDuffee, *An Introduction to Abstract Algebra* (John Wiley & Sons, Inc., New York, 1940), Sec. 17, p. 38.

[11] See reference 2, Sec. 61, p. 44.

[12] T. Nagell, *Introduction to Number Theory* (John Wiley & Sons, Inc., New York, 1951), Chap. IV, Sec. 44.

[13] See reference 10, Sec. 19, p. 63.

[14] P. Kustaanheimo, Ann. Acad. Sci. Fennicae Ser. A.I., No. 129 (1952). Kustaanheimo shows that the necessary condition is $p=8x\prod q_i-y^2$, where x and y are integers such that p is prime. However, the essential results of the present stage of the theory are not affected by requiring x to be odd and y^2 to be unity.

[15] See reference 8 and also reference 14.

[16] See reference 2, Sec. 64, p. 46.

readily be seen that

$$i^p = -i. \tag{2.9}$$

Also

$$z^p = x - iy, \tag{2.10}$$

since[17]

$$\left.\begin{matrix} x^p = x \\ y^p = y \end{matrix}\right\} \quad \text{in} \quad \mathrm{GF}(p), \tag{2.11}$$

and all other terms in the expansion of $(x+iy)^p$ contain $p(\equiv 0)$ as a factor.[18] Thus "complex conjugation" may be performed by raising to the power p.

$$z^* \equiv z^p. \tag{2.12}$$

It follows that

$$z^{p+1} = z^* z = x^2 + y^2. \tag{2.13}$$

This number is not necessarily "positive." For numbers α, β which form a solution set of Eq. (2.4), we have

$$\zeta \equiv \alpha + i\beta, \tag{2.14}$$

$$\zeta^* = \zeta^p = \alpha - i\beta, \tag{2.15}$$

$$\zeta^{p+1} = \zeta^* \zeta = \alpha^2 + \beta^2 = -1. \tag{2.16}$$

3. EXTENDED ORTHOGONAL GROUP AND THE SPINOR GROUP

In finite geometry the analog of the Lorentz group is the finite group of linear transformations leaving invariant the quadratic form

$$x_0^2 - x_1^2 - x_2^2 - x_3^2, \tag{3.1}$$

which we shall call the metric form. As in ordinary geometry, these linear transformations are of determinant $+1$ or -1, the former making up a normal subgroup of index two called the proper Lorentz group. This subgroup is of order[19]

$$p^2(p^2+1)(p^2-1). \tag{3.2}$$

Under the transformations of the whole Lorentz group the orthogonality of two vectors is preserved, and in ordinary geometry these are the only linear transformations with this property (apart from the dilatations of space, which we will not consider in this paper). In finite geometry, however, there exist transformations preserving orthogonality but reversing the sign of the metric form. They are outer automorphisms of the Lorentz group.[20] We shall call these the extraordinary orthogonal transformations. They can all be generated from the Lorentz group by a single one of them. An example is

$$\begin{aligned} g:\ & x_0' = -x_3, \quad x_1' = -\alpha x_1 + \beta x_2, \\ & x_3' = +x_0, \quad x_2' = \beta x_1 + \alpha x_2, \end{aligned} \tag{3.3}$$

where

$$\alpha^2 + \beta^2 = -1. \tag{3.4}$$

When these are adjoined to the Lorentz group, the resulting set of operations forms a group which we shall call the extended orthogonal group.

As in ordinary geometry, there exists a homomorphic group of 2×2 matrices, the spinor group defined as follows.

We first set up the matrix[21,22]

$$\begin{aligned} X &= \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \\ &= x_0 1 + \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3, \end{aligned} \tag{3.5}$$

where σ_1, σ_2, σ_3 are the Pauli matrices. The matrix X has as its determinant the metric form.

$$\|X\| = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{3.6}$$

We now subject the matrix X to the linear transformation

$$X' = a^* X a, \tag{3.7}$$

where a is a 2×2 matrix with coefficients from the "complex" finite field $\mathrm{GF}(p^2)$ and where a^* is the Hermitian conjugate of a. If we set

$$X' = x_0' + \sigma_1 x_1' + \sigma_2 x_2' + \sigma_3 x_3', \tag{3.8}$$

we obtain from (3.7) a linear transformation of the co-ordinates which is real because[23]

$$(a^*)^* = a. \tag{3.9}$$

If it holds true that

$$\|a\|^* \|a\| \equiv \|a\|^{p+1} = 1, \tag{3.10}$$

then the transformation is a Lorentz transformation since

$$\begin{aligned} \|X'\| &= \|a\|^* \|a\| \|X\| \\ &= \|X\| \\ &= x_0^2 - x_1^2 - x_2^2 - x_3^2. \end{aligned} \tag{3.11}$$

Unexpectedly the transformation $-I$, reversing space and time coordinates, is also induced in this way:

$$a(-I) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}, \tag{3.12}$$

where

$$\zeta^* \zeta \equiv \zeta^{p+1} = -1. \tag{3.13}$$

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[17] Fermat's theorem. See reference 10, Sec. 11, p. 24; reference 4, Sec. 13, theorem 20, p. 46; and reference 3, Vol. 1, Sec. 37, particularly p. 118.

[18] See reference 10, Sec. 11, Example 4, p. 26.

[19] See reference 2, Sec. 172, p. 160. When Dickson refers to the orthogonal group he means this subgroup of determinant $+1$.

[20] J. Dieudonné, *On the Automorphisms of the Classical Groups* (Memoirs of the American Mathematical Society, New York, 1951), No. 2, Chap. X and particularly p. 51.

[21] H. Weyl, *Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1950), Chap. III, Sec. 8(c).

[22] H. Boerner, *Darstellungen von Gruppen* (Springer-Verlag, Berlin, 1955), Chap. IX, Sec. 3.

[23] This equation is true for $\mathrm{GF}(p^2)$ but not generally for $\mathrm{GF}(p^n)$ where $n>2$.

Such a situation is not possible with ordinary complex numbers.

The set of matrices a satisfying Eq. (3.10) forms a group homomorphic to the proper Lorentz group and, as shown in Appendix I, this homomorphism is $p+1$ to one. All matrices of the form

$$\omega^\alpha a, \quad \alpha=0,\ 1,\ \cdots p \tag{3.14}$$

where ω is any finite complex number satisfying

$$\omega^{p+1}=1, \tag{3.15}$$

map onto the same Lorentz transformation.

To induce an extraordinary orthogonal transformation, we see from Eq. (3.11) that we must have

$$\|a\|^{p+1}=-1. \tag{3.16}$$

The transformation g [Eq. (3)] arises from

$$a(g)=\begin{pmatrix} 0 & \zeta \\ -1 & 0 \end{pmatrix}. \tag{3.17}$$

It can be seen from Eq. (3.16) that when these are included in the spinor group the homomorphism cannot, as in ordinary geometry, be made two to one by restricting to the unimodular group

$$\|a\|=1. \tag{3.18}$$

As a result we are left with a group of 2×2 matrices satisfying

$$\|a\|^{p+1}=\pm 1, \tag{3.19}$$

and possessing a $p+1$ to one homomorphism onto the proper extended orthogonal group. We shall call this the extended spinor group.

4. REPRESENTATIONS OF THE ORTHOGONAL GROUP

Since among the representations should be included the "vector" representation according to which the coordinates transform (the orthogonal group itself), the representations we seek must be matrices having coefficients in GF(p) or GF(p^2),[24] that is, they should be modular representations. We can get the modular representations of the orthogonal group from those of the spinor group because of the homomorphism. The spinor group is a subgroup of the two dimensional general linear homogeneous group GLH$(2,p^2)$ over the complex finite field GF(p^2), and all the irreducible modular representations of this group have been given by Brauer and Nesbitt.[25]

If we denote by ψ_1, ψ_2 the components of the spinor ψ which is transformed by the spinor matrices a and by $a^{(2j)}$, the matrices of dimension $2j+1$ by which the monomials $\psi_1{}^{j+m}\psi_2{}^{j-m}$, $m=0,\ \pm1,\ \cdots \pm j$ transform, then the matrices $a^{(2j)}$ also form a representation for each j, where

$$j=0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ \cdots (p-1)/2. \tag{4.1}$$

($j=0$ signifies the identity representation.)

We denote the complex conjugate of the matrix a by the symbol $\bar{a}$. As shown by Brauer and Nesbitt, the irreducible representations of GLH$(2,p^2)$ are given by

$$\|a\|^n a^{(2j)}\times \bar{a}^{(2k)}, \quad \text{(direct product)} \tag{4.2}$$

where

$$n=0,\ 1,\ \cdots p^2-2.$$

For our subgroup,

$$\|a\|^{p+1}=\pm 1, \tag{4.3}$$

the range of values of n becomes

$$n=0,\ 1,\ \cdots 2p+1, \tag{4.4}$$

or alternatively

$$n=0,\ \pm1,\ \pm2,\ \cdots \pm p,\ p+1, \tag{4.5}$$

the other values merely repeating these representations.

From the homomorphism it is seen that these are also representations of the proper extended orthogonal group. They are not, however, all the representations. Because the homomorphism is not one to one, we may take n to be half-integral and still remain within the range of multivaluedness of the representations.

$$n=0,\ \pm\tfrac{1}{2},\ \pm1,\ \cdots \pm(p+\tfrac{1}{2}),\ p+1. \tag{4.6}$$

We should note by the way that for the extraordinary orthogonal transformation g,

$$\|a\|=\zeta, \tag{4.7}$$

where

$$\zeta^{p+1}=-1, \tag{4.8}$$

and ζ is not a square in GF(p^2). To take the square root, an extension to GF(p^4) is necessary. If we define

$$\zeta^{\frac{1}{2}*}\equiv \zeta^{\frac{1}{2}p}, \tag{4.9}$$

then it is not true that

$$(\zeta^{\frac{1}{2}*})^*=\zeta^{\frac{1}{2}}, \tag{4.10}$$

but rather

$$(\zeta^{\frac{1}{2}*})^*=-\zeta^{\frac{1}{2}}. \tag{4.11}$$

For a Lorentz transformation for which

$$\|a\|^{p+1}=1, \tag{4.12}$$

the factor $\|a\|^n$ merely produces a change to another value of the same representation. It gives no contribution to the spin. By itself it is a one-dimensional spin-zero representation. In analogy with the case of ordinary geometry, we say that the representation (4.2) corresponds to a spin $j+k$.

To represent the improper transformations a doubling of the components is required. Space inversion is induced by the nonlinear transformation

$$X'=\epsilon\bar{X}\epsilon^{-1}, \tag{4.13}$$

[24] Or even in some further extension of the number field if necessary.

[25] R. Brauer and C. Nesbitt, Ann. Math. **42**, 556 (1941), Sec. 30.

where $\bar{X}$ means the complex conjugate of X and

$$\epsilon=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.14}$$

In the four-component form, the representations are

$$\begin{pmatrix} a & 0 \\ 0 & \epsilon\bar{a}\epsilon^{-1} \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & \|a\|^* a^{*-1} \end{pmatrix}, \tag{4.15}$$

and space inversion is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{4.16}$$

In this 4-component theory, however, covariants transforming like scalar, pseudoscalar, or tensor do not exist in general, for their existence depends on the condition[21]

$$\|a\|=1, \tag{4.17}$$

which cannot be applied here. Because of this, the Brauer-Weyl[26] method of deriving the four-component spinor representation cannot be used in this theory, nor can the full array of covariant interactions arise as in the usual theory. In particular, only V, A arise.

5. GAUGE TRANSFORMATIONS AND THE CHARGE NUMBER

The multivaluedness of the representation consists in the fact that the $p+1$ spinor matrices related by

$$a \rightarrow \omega^\alpha a, \quad \alpha=0, 1, \cdots p \tag{5.1}$$

where

$$\omega^*\omega=\omega^{p+1}=1 \tag{5.2}$$

all represent the same element of the orthogonal group.

This means that $\omega^\alpha 1$ is just as much a representation of the identity I of the orthogonal group as 1 itself is. Hence the transformation on the spinor,

$$\psi \rightarrow \omega^\alpha \psi, \tag{5.3}$$

is to be considered as having the same physical effect as multiplying by the identity. Since, from Eq. (5.2), ω is like a phase factor, what we have here corresponds to a gauge transformation and gauge invariance.

Under a gauge transformation (5.1), the general representation [Eq. (4.2)] undergoes the transformation

$$\|a\|^n a^{(2j)} \times \bar{a}^{(2k)} \rightarrow \omega^{Q\alpha} \|a\|^n a^{(2j)} \times \bar{a}^{(2k)}, \tag{5.4}$$

where

$$Q=2(n+j-k). \tag{5.5}$$

From the form of this transformation we recognize that the representation label Q is to be identified with charge number. It is an integer (positive, negative, or zero) in agreement with the experimental fact that all charges are multiples of a basic unit.

The usual unimodular spinor representation corresponds to the representation

$$\|a\|^{-\frac{1}{2}} a,$$

and from the present point of view is a neutral representation artificially charged.

The 4-component representations of Eq. (4.15) transform under a gauge transformation not simply by multiplication by a phase factor but by a phase factor for the first two components and by its inverse for the last two components, like a phase factor containing a γ_5 operator.

The two singly charged representations of "simplest" form are

$$a, \quad n=0, \quad j=\tfrac{1}{2}, \quad k=0, \tag{5.6}$$

and

$$\|a\|^{\frac{1}{2}}, \quad n=\tfrac{1}{2}, \quad j=0, \quad k=0, \tag{5.7}$$

the former being of spin $\frac{1}{2}$ and the latter of spin zero. It seems natural to identify this latter with the π^+ meson and because of the similarity of the two in isotopic spin space we identify the former with the Σ^+ particle.

6. COMBINED INVERSION

We can see from Eqs. (4.15) and (4.16) that under space inversion the representation is changed into one similar to its complex conjugate. Under a guage transformation

$$a \rightarrow \omega^{Q\alpha} a, \tag{6.1}$$

the complex conjugate representations transform by

$$\bar{a} \rightarrow \omega^{-Q\alpha} \bar{a}, \tag{6.2}$$

since

$$\omega^*=\omega^{-1} \tag{6.3}$$

from Eq. (5.2), which is characteristic of a particle of the opposite sign of the charge.

In this theory the mirror image of a charged particle has the opposite charge. The space inversion operator is the "combined inversion" of Landau. We will designate it by CP. We will identify two conjugate representations related by CP as particle and antiparticle.[27–30] Hence the representations according to which the $\pi^-(\equiv\bar{\pi}^+)$ and the $\bar{\Sigma}^+$ particles transform are

$$\|a\|^{*\frac{1}{2}} \equiv \|a\|^{\frac{1}{2}p}, \tag{6.4}$$

and

$$\epsilon\bar{a}\epsilon^{-1} \equiv \|a\|^* a^{*-1}, \tag{6.5}$$

respectively.

7. CONCLUSION

Taking world geometry to be finite, we have found part of the meaning of electric charge. In finite geometry

[26] R. Brauer and H. Weyl, Am. J. Math. **57**, 425 (1935).

[27] L. D. Landau, Nuclear Phys. **3**, 127 (1957).
[28] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957).
[29] A. Salam, Nuovo cimento **5**, 299 (1957).
[30] L. C. Biedenharn and H. E. Rohrschach, Phys. Rev. **107**, 1075 (1957).

some representations of the orthogonal group are many-valued, a change of value being a gauge transformation. The charge number determines the degree of multi-valuedness. Space inversion produces reversal of sign and is therefore "combined inversion."

The hypercharge number U and the Gell-Mann–Nishijima scheme are to be discussed in another paper.

Looking back over the argument, we find that electric charge arises from the existence of linear transformations reversing the sign of the metric form, and these occur because the world coordinate field is not completely ordered. It is strongly suggested that any theory accounting for electric charge must possess this feature.

Looking forward to the future development, we see that there are two main lines of generalization. First of all of course, the charge number is to be not just a representation label but also a coupling constant to the electromagnetic field. These two properties are related through the notion of affine connection.[31] To advance on this line will require a calculus and a generalization of the metric form.

Secondly, to identify Q as a geometric operator will require such an extension of the coordinate algebra as will make isotopic space part of the geometry.

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APPENDIX

In this Appendix we show that the homomorphism relation of the spinor group to the proper Lorentz group is $p+1$ to one.

First of all, the Lorentz transformation induced by

$$X'=a^*Xa \tag{A.1}$$

must be proper, that is, its determinant must be $+1$. This may be shown by the same line of argument as followed by Boerner[22] for the case of ordinary geometry. Secondly, two spinor matrices which differ otherwise than by a multiple ω^n of the identity, where ω is a phase factor, must lead to different Lorentz transformations. For suppose that a_1 and a_2 are two spinor matrices leading to the same Lorentz transformation. Then $a_1a_2^{-1}\equiv a$ must lead to the identity Lorentz transformation,

$$a^*Xa=X. \tag{A.2}$$

Taking $X=1$, we have

$$a^*a=1 \tag{A.3}$$

(i.e., a must be unitary). Hence

$$a^*=a^{-1}, \tag{A.4}$$

and from (A.2)

$$a^{-1}Xa=X, \tag{A.5}$$

so that a must commute with every X and must therefore be a scalar multiple of the identity

$$a=\lambda 1. \tag{A.6}$$

But since a is unitary, we must have

$$\lambda^*\lambda=1, \tag{A.7}$$

that is,

$$\lambda=\omega^n, \quad n=0,\ 1,\ \cdots p \tag{A.8}$$

where

$$\omega^{p+1}=1.$$

Thus

$$a_1=\omega^n a_2. \tag{A.9}$$

There are $p+1$ a's differing from each other by such a phase factor, all of which lead to the same proper Lorentz transformation. If they differ otherwise, they must lead to different Lorentz transformations.

The spinor matrices are of the form

$$a\equiv\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \tag{A.10}$$

with coefficients in $GF(p^2)$, so that each coefficient may take on p^2 different values. They are not independent, however, but are related by the condition

$$\alpha\delta-\beta\gamma=\omega^n, \quad n=0,\ 1,\ \cdots p. \tag{A.11}$$

For $\alpha\neq 0$ there are $p^4(p^2-1)$ such matrices, and for $\alpha=0$ there are $p^2(p^2-1)$, giving a total of

$$p^2(p^2+1)(p^2-1), \tag{A.12}$$

none of them differing (for fixed ω) merely by a phase factor. They must lead to $p^2(p^2+1)(p^2-1)$ different Lorentz transformations. But this is just the order of the proper Lorentz group and hence every proper Lorentz transformation must arise. Since there are $p+1$ spinor matrices which induce each Lorentz transformation, the homomorphism must be $p+1$ to one.

[31] P. G. Bergmann, Phys. Rev. **107**, 624 (1957).