TABLE I. Imaginary part of the central optical-model potential, V_{CI}, in Mev ($\lambda=0.857$).

E_{lab} (Mev)	50	100	140	200	260
V_{CI}, with exclusion principle	64[a]	73	77	80	84
V_{CI}, without exclusion principle	90	103	106	107	108

[a] The value of λ_s has been extrapolated.

The integration (3) has been made numerically and the results, expressed in terms of the mean free path $\bar{\lambda}_s$, are shown in Fig. 2. For comparison the mean free paths obtained (a) from an isotropic angular distribution (with the same k dependence), and (b) from complete neglect of the exclusion principle, are also plotted.

Finally, Table I shows the imaginary part of the central optical-model potential, according to formula (1), at several different energies, together with the corresponding values for the case of no Pauli principle effect.

III. CONCLUSION

The large value of the nucleon-antinucleon cross section has long been known to imply a very short mean free path for antinucleons in nuclear matter. The exclusion-principle effect considered here increases somewhat the mean free path, but not enough to change the conclusion that nearly all antinucleon interactions occur on the nuclear surface.

In the same energy range the nucleon effective mean free path is larger than 5×10^{-13} cm, showing a striking difference between the nucleon-nucleus and antinucleon-nucleus interaction.

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Spectral Representations in Perturbation Theory. II. Two-Particle Scattering*

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In this paper a particular term in the perturbation expansion for the two-particle scattering amplitude is examined. We consider the real plane defined by the square of the total four-momentum and the square of the momentum transfer, and show that the scattering amplitude is an analytic function of both variables in a certain connected region in this plane. The precise boundary of the region is found. The purpose of this work is to find some conditions that integral representations of the scattering amplitude must satisfy, with the hope that such examples may aid the study of such integral representations in general.

We also apply our general result to some particular cases of physical interest.

IN a recent note[1] we discussed some properties of the vertex operator in perturbation theory, corresponding to the three-vertex (triangular) Feynman diagram. For simplicity, we restricted our discussion to the case of six (possibly different) scalar fields, and studied the matrix element as a function of the three independent kinematic invariants of the problem, which in our case were chosen to be the squares of the three incident four-momenta. We then studied the properties of the matrix element as a function of one of these invariants, keeping the other two fixed at values corresponding to physical particles satisfying a number of "stability conditions." The function so defined was shown to be analytic in a cut plane, and our objective was to determine the exact position of the first branch point on the real axis.

In this paper we investigate in a similar fashion the matrix element corresponding to the four-vertex square diagram of Fig. 1, which describes a contribution in perturbation theory to the so-called "four-point function." We again restrict ourselves to the case of a number of interacting scalar fields; since the spins of the particles in no way affect the analytic properties of the scattering amplitude, spin may, for the purpose of this study, be ignored.

In this problem we may distinguish six independent kinematic invariants: the squares of the four incident four-momenta, the square of the total incident momentum, and the square of the momentum transfer, for example.

We keep the first four variables at fixed values corresponding to physical particles, and subject to a number of "stability conditions." Our limited objective is to show that the matrix element is an analytic function of the two remaining variables in a certain connected region in the plane where both are real, and to determine the precise boundary of this region. The

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[1] Karplus, Sommerfield, and Wichmann, Phys. Rev. 111, 1187 (1958). This will be referred to as Paper I. References to the literature may be found there.

thresholds in the spectral representation of the scattering amplitude may be inferred from this information. Our hope is that a few examples will aid in the study of general spectral representations, especially when several stable particles occur in the theory.

Our discussion is applied to some "realistic" cases of physical interest.[2]

Consider the diagram in Fig. 1, where the lines represent scalar particles. For the sake of symmetry we take the four-momenta p_{12}, p_{23}, p_{34}, and p_{41} of the four particles to be "ingoing," so that the law of momentum conservation takes the form

$$p_{12}+p_{23}+p_{34}+p_{41}=0. \tag{1}$$

If a constant factor is ignored, the Feynman amplitude for this diagram is given by the integral (real momenta)

$$F=\int_0^1 d\alpha_1\int_0^1 d\alpha_2\int_0^1 d\alpha_3\int_0^1 d\alpha_4 \times\frac{\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{D_1{}^2}, \tag{2}$$

$$D_1=\sum_{i=1}^4 \alpha_i m_i{}^2-\sum_{i<j=1}^4 \alpha_i\alpha_j p_{ij}{}^2, \tag{3}$$

where we have used the abbreviations

$$p_{13}=p_{12}+p_{23}=-(p_{34}+p_{41}),$$
$$p_{24}=p_{23}+p_{34}=-(p_{41}+p_{12}),$$
$$p_{ij}=p_{ji}.$$

For convenience we introduce the variables y_{kl} by

$$p_{kl}{}^2=m_k{}^2+m_l{}^2-2m_km_ly_{kl}. \tag{4}$$

With these variables we have

$$D_1=\sum_{k=1}^4 \alpha_k{}^2m_k{}^2+2\sum_{k=1}^4\sum_{l=k+1}^4 \alpha_k\alpha_l y_{kl}m_km_l. \tag{5}$$

We subject the four variables associated with the single-particle invariants to the stability conditions

$$y_{12}>-1,\quad y_{23}>-1,\quad y_{34}>-1,\quad y_{41}>-1, \tag{6}$$

which state that at each vertex the external mass is less than the sum of the two masses to which it is coupled. One might impose further conditions of this nature by restricting these variables to values less than 1; this would correspond to triangular conditions, according to which any mass is less than the sum of any two other masses to which it is *directly* coupled. We will, however, use only conditions (6), which we assume to hold throughout the remaining part of this paper.

[2] R. Oehme and J. G. Taylor [Phys. Rev. **113**, 371 (1959)] have treated a variety of general scattering processes without resort to perturbation theory.

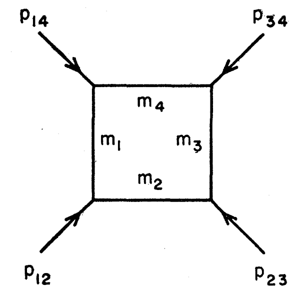

FIG. 1. Feynman diagram for general scattering process in fourth order.

Our objective now is to study the integral in Eq. (2) as a function of the remaining two variables y_{13} and y_{24}; to determine the connected region in the corresponding real plane in which the amplitude is an analytic function of both variables; and to describe the boundary of this region, say,

$$y_{13}>\bar{y}_{13}(y_{24}). \tag{7}$$

Since inspection of Eq. (2) shows that the function is analytic when any one variable is complex while the others are real, it follows from Eq. (7) that the singularities of F in the complex y_{13} plane are then confined to part of the real y_{13} axis,

$$y_{13}\leqslant\bar{y}_{13}.$$

A spectral representation for F that displays this information is

$$F(y_{13},y_{24})=\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\bar{y}_{13}(y_{24})}\frac{\mathrm{Im}F(t+i\epsilon,\,y_{24})}{t-y_{13}}dt. \tag{8}$$

To continue the analysis, the following fractional linear transformation is convenient:

$$\alpha_k=(x_k/m_k)/\sum_{l=1}^4 (x_l/m_l),\qquad \sum_{k=1}^4 x_k=1.$$

It may be solved for the x_k:

$$x_k=\alpha_km_k/\sum_{l=1}^4 \alpha_lm_l.$$

The region of integration remains unchanged under this transformation, and in place of Eq. (2) we obtain

$$F=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\int_0^1 dx_4 \times\frac{\delta(1-x_1-x_2-x_3-x_4)}{m_1m_2m_3m_4D^2}, \tag{9}$$

$$D=\sum_{k=1}^4 x_k{}^2+2\sum_{k=1}^4\sum_{l=k+1}^4 x_kx_ly_{kl}.$$

In Appendix A, the analogous transformations are applied to the triangular vertex diagram considered in Paper I, and the results are rewritten in the new notation. This forms a useful introduction to the study of the four-point function, since the case of the vertex function is much simpler.

Some properties of F may be inferred directly when D is rewritten:

$$D=(x_1+x_3-x_2-x_4)^2+2x_1x_3(y_{13}-1)+2x_2x_4(y_{24}-1)\\+2[x_1x_2(1+y_{12})+x_2x_3(1+y_{23})+x_3x_4(1+y_{34})\\+x_1x_4(1+y_{14})]. \quad (10)$$

The stability conditions (6) then imply that, in the region of the x integration, the denominator D cannot vanish if

$$y_{13}>1 \quad\text{and}\quad y_{24}>1. \quad (11a)$$

On the other hand, if

$$y_{13}<-1 \quad\text{and/or}\quad y_{24}<-1, \quad (11b)$$

the denominator D vanishes at some point in the region of integration.

Thus $F(y_{13},y_{24})$ is an analytic function of both variables in the region defined by the inequalities (11a). This plane region is included in a larger connected plane region R in which F is an analytic function of both variables. We shall find the boundary of this region R and show that R consists of all points (y_{13},y_{24}) such that in the range of the x integration the expression D does not vanish.

To prove this assertion let the region of integration be T,

$$T:\ \{x_k\geqq 0,\ x_1+x_2+x_3+x_4=1\}, \quad (12)$$

and let R' be the region in the real (y_{13},y_{24}) plane in which D cannot vanish if x is in T. Then, if the point (y_{13},y_{24}) is in R', so is the point $(y_{13}+\delta_1,\ y_{24}+\delta_2)$ for non-negative δ_1 and δ_2. Furthermore, our previous remarks show that R' is included in the region $y_{13}>-1$, $y_{24}>-1$. Thus, if we let y_{24} be fixed and such that some point of the line $y_{24}=$constant belongs to R', then there is a $\bar{y}_{13}$ such that the point (y_{13},y_{24}) is in R' if and only if $y_{13}>\bar{y}_{13}$. Furthermore, if $y_{13}>\bar{y}_{13}$, the integrand is positive. Therefore the point $(\bar{y}_{13},y_{24})$ is necessarily a singularity of F as a function of y_{13}.[3] On the other hand, F is certainly an analytic function of y_{13} and y_{24} in R'. Hence R and R' are identical.

The problem is thus to find the region R in which the expression D cannot vanish if x is in T. The region T consists of the interior and boundary of an equilateral tetrahedron. The x_k may then be interpreted as barycentric coordinates. We note that at the vertices ($x_k=1$, $k=1$, 2, 3, or 4) we have $D=1$. Our procedure will be as follows: We first find a region R_i such that the expression D is positive on every edge of T if and only if (y_{13},y_{24}) is in R_i. Next, we find a smaller region R_{ii} such that the expression D is positive on every face of T if and only if (y_{13},y_{24}) is in R_{ii}. Finally, we find the region R for which D cannot vanish at all in T. We are thus led to consider three cases.

Case (i).—D can take on nonpositive values on some edge of T. On an edge two x's are equal to zero, e.g., $x_k=x_l=0$. The smallest value that D assumes on such an edge is

$$\min[1;\tfrac{1}{2}(1+y_{kl})]. \quad (13)$$

Because of the stability conditions we need consider only the two edges $x_1=x_3=0$ and $x_2=x_4=0$; the region R_i therefore is

$$R_i:\ \{y_{13}>-1,\ y_{24}>-1\}. \quad (14)$$

Case (ii).—We assume that (y_{13},y_{24}) is in R_i and consider the possibility that D has a nonpositive value on the face of T defined by $x_k=0$. Except for the subscripts, the function D is then the same as D_0, Eq. (A4), which occurs in the vertex problem. The solution from Appendix A may therefore be used. Accordingly, we introduce six angles between 0 and π, θ_{12}, θ_{13}, θ_{14}, θ_{23}, θ_{24}, θ_{34} by the conditions

$$y_{kl}=\cos\theta_{kl}, \quad (15)$$

which are to be used only when they are real, i.e., $|y_{kl}|<1$. Furthermore, let

$$\left.\begin{array}{ll} L_1=-1 & y_{23}+y_{34}\geqslant 0\\ L_1=\cos(\theta_{23}+\theta_{34}) & y_{23}+y_{34}<0\end{array}\right\}, \quad (16)$$

$$\left.\begin{array}{ll} L_2=-1 & y_{14}+y_{34}\geqslant 0\\ L_2=\cos(\theta_{14}+\theta_{34}) & y_{14}+y_{34}<0\end{array}\right\}, \quad (17)$$

$$\left.\begin{array}{ll} L_3=-1 & y_{12}+y_{14}\geqslant 0\\ L_3=\cos(\theta_{12}+\theta_{14}) & y_{12}+y_{14}<0\end{array}\right\}, \quad (18)$$

$$\left.\begin{array}{ll} L_4=-1 & y_{12}+y_{23}\geqslant 0\\ L_4=\cos(\theta_{12}+\theta_{23}) & y_{12}+y_{23}<0\end{array}\right\}, \quad (19)$$

$$\begin{array}{l} L_{24}=\max[L_1,L_3],\\ L_{13}=\max[L_2,L_4].\end{array} \quad (20)$$

We may then conclude that the region R_{ii} is

$$R_{ii}:\ \{y_{13}>L_{13},\ y_{24}>L_{24}\}. \quad (21)$$

The denominator D is positive on all faces of T if and only if (y_{13},y_{24}) is in R_{ii}.

Case (iii).—Let (y_{13},y_{24}) be in R_{ii}. We must now consider the possibility that D can take nonpositive values in the interior of the tetrahedron T, which means that D has a nonpositive minimum there. This occurs if and only if the following conditions are met:

$$\begin{array}{l}\text{(a) } y_{12}<1,\quad y_{23}<1,\quad y_{34}<1,\quad y_{14}<1,\\ \text{(b) } 2\pi<\theta_{12}+\theta_{23}+\theta_{34}+\theta_{41}<2\pi\\ \qquad\qquad +2\min[\theta_{12},\theta_{23},\theta_{34},\theta_{41}];\end{array} \quad (22)$$

[3] It may be remarked that if a function is defined by an integral like (2), and if the variables y_{kl} are not restricted to be real, then the mere fact that for a certain set of y_{kl} the denominator can vanish in the region of integration in no way guarantees that the function has a singularity at this point.

then the region R is‡

$$R:\ \{y_{24}>y_{24}^0,\ y_{13}>L_{13}\} + \{y_{13}>y_{13}^0,\ y_{24}>L_{24}\} + \{\Delta(y_{13},y_{24})>0\}, \quad (23)$$

where

$$\Delta(y_{13},y_{24})=\begin{vmatrix} 1 & y_{12} & y_{13} & y_{14} \\ y_{12} & 1 & y_{23} & y_{24} \\ y_{13} & y_{23} & 1 & y_{34} \\ y_{14} & y_{24} & y_{34} & 1 \end{vmatrix}, \quad (24)$$

and where

$$\Delta(L_{13},y_{24}^0)=0=\Delta(y_{13}^0,L_{24}) \quad (25)$$

define y_{24}^0 and y_{13}^0 uniquely. If the conditions (22) are not satisfied, then

$$R=R_{ii}. \quad (26)$$

The algebraic details are described in Appendix B. The regions R_i, R_{ii}, and R are illustrated in Fig. 2.

The limits used in Eq. (23) may be stated explicitly by solving Eq. (25). They are

$$\begin{aligned} y_{24}^0=S_2&=\frac{\cos\theta_{12}\sin\theta_{34}+\cos\theta_{23}\sin\theta_{14}}{\sin(\theta_{14}+\theta_{34})}, \quad L_{13}=L_2 \\ &=S_4=\frac{\cos\theta_{14}\sin\theta_{23}+\cos\theta_{34}\sin\theta_{12}}{\sin(\theta_{12}+\theta_{23})}, \quad L_{13}=L_4, \end{aligned} \quad (27)$$

and

$$\begin{aligned} y_{13}^0=S_1&=\frac{\cos\theta_{12}\sin\theta_{34}+\cos\theta_{14}\sin\theta_{23}}{\sin(\theta_{23}+\theta_{34})}, \quad L_{24}=L_1 \\ &=S_3=\frac{\cos\theta_{23}\sin\theta_{14}+\cos\theta_{34}\sin\theta_{12}}{\sin(\theta_{14}+\theta_{12})}, \quad L_{24}=L_3. \end{aligned} \quad (28)$$

The function $\bar{y}_{13}$ defined in Eq. (7) may also be directly inferred from Eq. (23) by solving Eq. (24),[4]

$$\begin{aligned} \bar{y}_{13}&=L_{13}, \quad y_{24}>y_{24}^0, \\ &=\frac{1}{1-y_{24}^2}[y_{12}y_{23}+y_{14}y_{34}-y_{24}(y_{12}y_{34}+y_{14}y_{23}) \\ &\qquad -(K_1K_3)^{\frac{1}{2}}], \quad y_{24}^0>y_{24}>L_{24}. \end{aligned} \quad (29)$$

A great simplification results when the scattering process is the elastic scattering of equal particles with equal internal masses. Then we have

$$y_{12}=y_{23}=y_{34}=y_{14}=y. \quad (30)$$

The boundary of the region R for positive y is

$$\bar{y}_{13}=-1 \quad \text{for} \quad y_{24}>-1, \quad y\geqslant 0, \quad (31)$$

‡ *Note added in proof.*—To this description of the region R there must occasionally be added those points $\{y_{13},y_{24}\}$ whose abscissas and ordinates are greater than the abscissa and ordinate of any point satisfying condition (23).

[4] The quantities K_1, K_2, K_3, K_4 are defined by Eq. (B2) and its appropriate permutations.

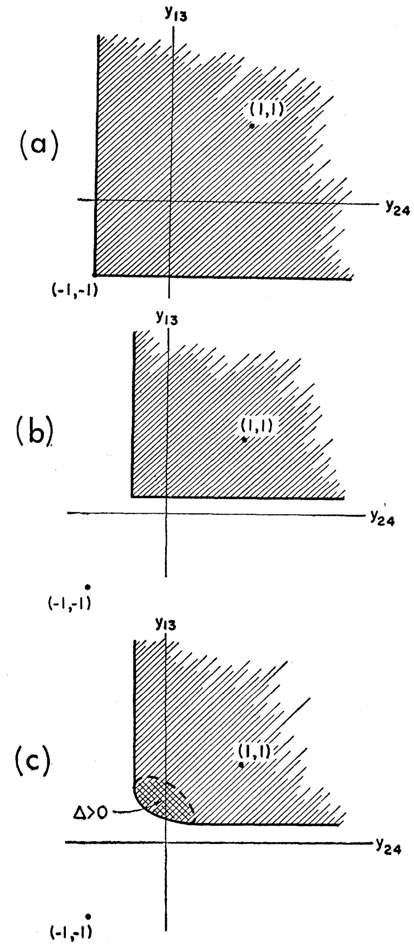

FIG. 2. (a) The region R_i; (b) the region $R_{ii}\neq R_i$; (c) the region $R\neq R_{ii}$.

while for negative y it becomes

$$\left.\begin{aligned} \bar{y}_{13}&=2y^2-1 && \text{for} \quad y_{24}>1 \\ &=\frac{4y^2}{1+y_{24}}-1 && \text{for} \quad 1>y_{24}>2y^2-1 \end{aligned}\right\} \quad y<0. \quad (32)$$

We shall examine several scattering processes illustrative of the various cases (Fig. 1). If p_{12} and p_{23} ($-p_{14}$ and $-p_{34}$) are the initial (final) four-momenta, then the total four-momentum squared (W^2) and the square of the four-momentum transfer ($-\Delta^2$) are given by

$$\begin{aligned} W^2&=(p_{12}+p_{23})^2=(p_{14}+p_{34})^2 \\ &>\max[(M_{12}+M_{23})^2,\ (M_{14}+M_{34})^2], \\ \Delta^2&=-(p_{12}+p_{14})^2=-(p_{23}+p_{34})^2 \\ &>\max[-(M_{12}-M_{14})^2,\ -(M_{23}-M_{34})^2]. \end{aligned} \quad (33)$$

The inequalities define the physically accessible regions of W^2 and Δ^2. In terms of y_{13} and y_{24}, we have

$$y_{13}=\frac{m_1^2+m_3^2-W^2}{2m_1m_3}<\min\{y_{12}y_{23}-[(1-y_{12}^2)(1-y_{23}^2)]^{\frac{1}{2}},\quad y_{14}y_{34}-[(1-y_{14}^2)(1-y_{34}^2)]^{\frac{1}{2}}\},\tag{34a}$$

$$y_{24}=\frac{m_2^2+m_4^2+\Delta^2}{2m_2m_4}>\max\{y_{12}y_{14}+[(1-y_{14}^2)(1-y_{12}^2)]^{\frac{1}{2}},\quad y_{23}y_{34}+[(1-y_{23}^2)(1-y_{34}^2)]^{\frac{1}{2}}\}.\tag{34b}$$

Suppose we seek a spectral representation of the scattering amplitude in terms of W^2 for a fixed physical value of Δ^2. Then the inequality (34b) guarantees that the threshold for y_{13} is L_{13}. The "normal" threshold determined by the mass of one real intermediate state is $L_{13}=-1$. Equations (17), (19) show $L_{13}>-1$ only if a sufficient number of y_{12}, y_{23}, y_{34}, y_{14} are sufficiently negative. To see which processes might exhibit "abnormal" thresholds, we list in Table I the values of the y's (and corresponding θ's) for ten possible interactions. We observe that there are only six vertices which have negative y's, and that virtual dissociations of N and π are not among them. Thus we conclude that a spectral representation of F for pion-nucleon scattering as a function of W^2 would have a threshold $\bar{W}^2$ at

$$\bar{W}^2=(M_N+m_\pi)^2,$$

which is considered normal. We next investigate the diagram for pion-deuteron scattering illustrated in Fig. 3(a). Then

$$L_2=L_4=L_{13}=\cos(174°+8°)=-0.999>-1,$$
$$\bar{W}^2=(2M_N)^2-0.002M_N^2,$$

which is just slightly different from the normal threshold. For Σ-nucleon scattering [Fig. 3(b)], we have

$$L_2=L_4=L_{13}=\cos(86°+120°)=-0.899>-1,$$
$$\bar{W}^2=(M_N+M_\Lambda)^2-0.202M_NM_\Lambda.$$

If we do not confine our study to the coupling of particles that have been observed, but also include virtual fields with masses that are limited only by the stability conditions, then the spectral representations have "abnormal" thresholds in many more cases. Some of these have been mentioned in Paper I. Since the general derivations of dispersion relations presuppose a "normal" threshold but do not limit particles in intermediate states to the few kinds that have been observed, these derivations cannot be applied to some cases of interest.

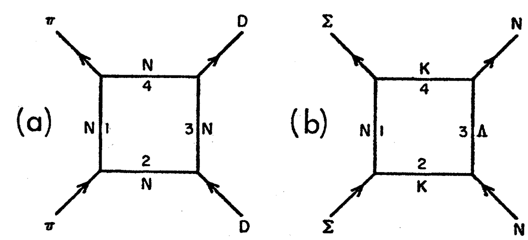

FIG. 3. (a) Feynman diagram for pion-deuteron scattering. (b) Feynman diagram for Σ-nucleon scattering.

TABLE I. Values of y and corresponding θ for ten possible interactions. The table is read with reference to a particular trilinear coupling, for example $\Sigma\,\Lambda\,\pi$. The parameter y and angle θ for the process $\Sigma\rightarrow\Lambda+\pi$ which are

$$\cos\theta=y=(M_\Lambda^2+m_\pi^2-M_\Sigma^2)/2M_\Lambda m_\pi,$$

are found in the line opposite Σ in the box containing Σ, Λ, π together. There is no need to distinguish particles from antiparticles.

	y	θ		y	θ
D	−0.995	174°	K	0.14	82°
N	0.999	3°	K	0.14	82°
N	0.999	3°	π	0.96	16°
N	0.075	86°	Λ	−0.13	97°
N	0.075	86°	N	0.55	57°
π	0.989	8°	K	0.90	26°
Σ	−0.50	120°	Σ	−0.31	108°
Λ	0.58	54°	N	0.66	49°
π	0.994	6°	K	0.92	23°
Σ	0.058	86.5°	Ξ	−0.24	104°
Σ	0.058	86.5°	Λ	0.57	55°
π	0.993	7°	K	0.93	21°
Ξ	0.052	87°	Ξ	−0.07	94°
Ξ	0.052	87°	Σ	0.44	64°
π	0.995	6°	K	0.93	22°

APPENDIX A

Consider the expression (4) in Paper I for the vertex operator. The substitutions

$$\begin{aligned}-q^2=M_1^2&=m_b^2+m_c^2-2m_bm_cy_{23},\\ M_2^2&=m_a^2+m_c^2-2m_am_cy_{13},\\ M_3^2&=m_a^2+m_b^2-2m_am_by_{12},\end{aligned}\tag{A1}$$

$$\begin{aligned}x_1&=m_a\alpha/(m_a\alpha+m_b\beta+m_c\gamma),\\ x_2&=m_b\beta/(m_a\alpha+m_b\beta+m_c\gamma),\\ x_3&=m_c\gamma/(m_a\alpha+m_b\beta+m_c\gamma),\end{aligned}\tag{A2}$$

enable us to rewrite the integral as

$$F=\int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\times\frac{\delta(1-x_1-x_2-x_3)}{(x_1m_bm_c+x_2m_am_c+x_3m_am_b)D_0},\tag{A3}$$

$$D_0=x_1^2+x_2^2+x_3^2+2(x_1x_2y_{12}+x_1x_3y_{13}+x_2x_3y_{23}).\tag{A4}$$

The problem is then to investigate under what conditions the denominator D_0 can vanish in the range of integration.

We compare this problem with the problem in Case (ii) in the text of this paper to determine the conditions under which the expression D can take nonpositive values on the particular face $x_4=0$, and note that the two problems are identical. We are thus to consider D_0 when x_1, x_2, x_3 satisfy the conditions $x_1\geqslant 0$, $x_2\geqslant 0$, $x_3\geqslant 0$, and $x_1+x_2+x_3=1$. We may interpret this region, F_4, as the face of an equilateral triangle described by the barycentric coordinates x_1, x_2, and x_3. As in Case (i) in the text, we impose the conditions on the y_{kl} which state that D_0 cannot vanish on any edge of this triangle, i.e.,

$$y_{12}>-1,\quad y_{13}>-1,\quad y_{23}>-1. \tag{A5}$$

In the case of the vertex operator, two of these conditions can always be interpreted as stability conditions that the physical particles have to satisfy.

Since D_0 is positive on the boundary of the triangle F_4, D_0 can vanish in the interior only if it has a nonpositive minimum there. This minimum may be determined by differentiation, and one must then find the condition that this minimum in fact lies inside F_4. We may distinguish two cases:

$$y_{12}+y_{13}\geqslant 0, \tag{A6a}$$

$$y_{12}+y_{13}<0 \quad \text{(in this latter case } y_{12}<1,\ y_{13}<1\text{)}. \tag{A6b}$$

To study the first case, we may assume $y_{12}\leqslant y_{13}$. Let

$$\lambda=\min[1,y_{12}]. \tag{A7}$$

Then we may write

$$D_0\equiv(x_2-x_3+\lambda x_1)^2+(1-\lambda^2)x_1^2+2[x_2x_3(1+y_{23}) +x_1x_2(y_{12}-\lambda)+x_1x_3(y_{13}+\lambda)]. \tag{A8}$$

Each term in this expression is non-negative, and hence in this case D_0 cannot vanish if $y_{23}>-1$.

In the second case we write

$$D_0\equiv(x_1+x_2y_{12}+x_3y_{13})^2+[x_2(1-y_{12}^2)^{\frac{1}{2}} -x_3(1-y_{13}^2)^{\frac{1}{2}}]^2+2x_2x_3\{y_{23}-y_{12}y_{13} +[(1-y_{12}^2)(1-y_{13}^2)]^{\frac{1}{2}}\}. \tag{A9}$$

Hence D_0 is positive if [see Eq. (15)]

$$y_{23}>y_{12}y_{13}-[(1-y_{12}^2)(1-y_{13}^2)]^{\frac{1}{2}}=\cos(\theta_{12}+\theta_{13}). \tag{A10}$$

The condition (A6b) takes the form $\theta_{12}+\theta_{13}>\pi$. We also note that if $y_{23}=\cos(\theta_{12}+\theta_{13})$ and

$$\begin{aligned} x_1&=-\frac{\sin(\theta_{12}+\theta_{13})}{\sin\theta_{12}+\sin\theta_{13}-\sin(\theta_{12}+\theta_{13})}>0,\\ x_2&=\frac{\sin\theta_{13}}{\sin\theta_{12}+\sin\theta_{13}-\sin(\theta_{12}+\theta_{13})}>0,\\ x_3&=\frac{\sin\theta_{12}}{\sin\theta_{12}+\sin\theta_{13}-\sin(\theta_{12}+\theta_{13})}>0,\end{aligned} \tag{A11}$$

then $D_0=0$. Thus the necessary and sufficient condition that D_0 be positive on the face F_4 is given by the inequality (A10). Comparing the results just obtained with the results in the appendix of Paper I, we see that they are identical.

A graphical solution to the problem in Paper I may be obtained as follows: In a plane draw three vectors $\mathbf{m}_a$, $\mathbf{m}_b$, $\mathbf{m}_c$ of lengths m_a, m_b, m_c from a common origin 0, such that $|\mathbf{m}_a-\mathbf{m}_b|=M_3$ and $|\mathbf{m}_a-\mathbf{m}_c|=M_2$. Then the threshold μ^2 of the spectral representation of F as a function of $-q^2$ is $|\mathbf{m}_b-\mathbf{m}_c|^2$, provided that the figure can be drawn at all and provided that the origin 0 lies inside the triangle determined by the end points of $\mathbf{m}_a$, $\mathbf{m}_b$, $\mathbf{m}_c$. Otherwise the threshold is $(m_b+m_c)^2$.

APPENDIX B

In this Appendix we consider Case (iii) of the text. We let (y_{13},y_{24}) be in R_{ii} so that D cannot vanish on any of the faces F_1, F_2, F_3, F_4 of the tetrahedron T. We seek a region R_{ii}' such that D vanishes at some point in the interior of T if and only if (y_{13},y_{24}) is in R_{ii}'. We proceed by finding necessary conditions for the existence of a region R_{ii}'.

Since D is positive on the boundaries of T, it can vanish inside T only if it assumes a nonpositive minimum inside T. We shall show that if such is the case D can be negative only in the two regions $x_k>0$ and $x_k<0$ $(k=1, 2, 3, 4)$ in the four-dimensional Euclidean x space. In particular D is non-negative on the four hyperplanes $x_k=0$.

To prove these remarks we observe that since D is a quadratic function of the x_k, it can have only one extreme value on any hyperplane. In particular, if—as our assumptions state—on the hyperplane $x_1+x_2+x_3+x_4=1$, D has a minimum value within T and is positive on the boundaries of T, then it must be positive throughout the rest of this hyperplane. Now suppose that we evaluate D at some point x' such that $x_1'+x_2'+x_3'+x_4'=c$. Then the hyperplane $x_1+x_2+x_3+x_4=c$, passing through x', is parallel to the hyperplane $x_1+x_2+x_3+x_4=1$, on which lies the point x^0 defined by $x_k'=cx_k^0$. Furthermore $D(x')=c^2D(x^0)$, and since $D(x^0)$ can be negative only for $x_k^0>0$, so then can $D(x')$ be negative only in one of the regions $x_k'>0$ or $x_k'<0$, depending on the sign of c.

The quadratic form D may be written in diagonal form in many ways, some of which can be obtained from the following by suitable interchange of indices:

$$D=(x_1+x_2y_{12}+x_3y_{13}+x_4y_{14})^2+(1-y_{12}^2)\left(x_2+x_3\frac{y_{23}-y_{12}y_{13}}{1-y_{12}^2}+x_4\frac{y_{24}-y_{12}y_{14}}{1-y_{12}^2}\right)^2$$
$$+\frac{K_4}{1-y_{12}^2}\left(x_3+x_4\frac{(y_{34}-y_{13}y_{14})(1-y_{12}^2)-(y_{23}-y_{12}y_{13})(y_{24}-y_{12}y_{14})}{K_4}\right)^2+(\Delta/K_4)x_4^2, \quad \text{(B1)}$$

where

$$K_4=1-y_{12}^2-y_{13}^2-y_{23}^2+2y_{12}y_{13}y_{23}. \quad \text{(B2)}$$

In order that D take a nonpositive value, at least one of the coefficients: 1, $1-y_{12}^2$, $K_4/(1-y_{12}^2)$, Δ/K_4, must be nonpositive. But we have just shown that for (y_{13},y_{24}) in R_{ii}, D takes a nonpositive minimum inside T only if D is positive definite for $x_4=0$. This allows us to conclude that necessary conditions for D to vanish inside T are $|y_{12}|<1$, $K_4>0$, $\Delta\leqslant 0$. By considering other ways of writing D, we find that the region R_{ii}' can be no larger than that defined by[4]

$$|y_{kl}|<1, \quad \text{(B3a)}$$

$$K_l>0, \quad \text{(B3b)}$$

$$\Delta\leqslant 0. \quad \text{(B3c)}$$

The inequality (B3a) permits us to introduce the real angles θ_{12}, θ_{23}, θ_{34}, θ_{14} defined in Eq. (15). It then follows from the structure of the K_i that there exists a region in the (y_{13},y_{24}) plane in which condition (B3b) is satisfied if and only if

$$\theta_{12}+\theta_{23}+\theta_{34}+\theta_{14}-2\min[\theta_{12},\theta_{23},\theta_{34},\theta_{14}]<2\pi. \quad \text{(B4)}$$

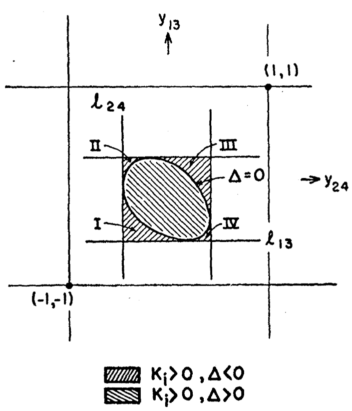

FIG. 4. The regions defined by Eq. (B3).

Because each K_i depends on y_{13} or on y_{24} but not on both, the region defined by (B3b) is a rectangle with edges parallel to the coordinate axes. The curve

$$\Delta=0 \quad \text{(B5)}$$

lies inside this rectangle and is tangent to all four edges. The situation is depicted in Fig. 4.

Now, it follows from the argument after Eq. (12) that the region R_{ii}' must have in common with R_{ii} those parts of its boundary that separate it from regions of smaller y_{13} and (or) y_{24}. For this reason the lines marked l_{13} and l_{24} in Fig. 4 must actually be L_{13} and L_{24}, respectively, if there is to be a region R_{ii}', and only the section marked I in Fig. 4 can be a part of R_{ii}'. The condition that $l_{13}=L_{13}$ and $l_{24}=L_{24}$ is

$$\theta_{12}+\theta_{23}+\theta_{34}+\theta_{41}>2\pi. \quad \text{(B6)}$$

The inequalities (B5) and (B6) are necessary conditions for the existence of a region R_{ii}'. It is simple to show that the conditions are also sufficient by choosing some point in the region I, Fig. 4, and exhibiting a negative D for some values of the x_i in T.

The threshold $\bar{p}_{13}^2$ of a spectral representation of F as a function of p_{13}^2 corresponding to the conditions that allow D to vanish inside T has a graphical construction. Draw, in three dimensions, four vectors $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$, $\mathbf{m}_4$ of lengths m_1, m_2, m_3, m_4 from a common origin 0. Adjust their directions so that

$$|\mathbf{m}_1-\mathbf{m}_2|=M_{12}, \quad |\mathbf{m}_2-\mathbf{m}_3|=M_{23},$$
$$|\mathbf{m}_3-\mathbf{m}_4|=M_{34}, \quad |\mathbf{m}_1-\mathbf{m}_4|=M_{14}, \quad p_{24}^2=|\mathbf{m}_2-\mathbf{m}_4|^2.$$

Then the threshold is $\bar{p}_{13}^2=|\mathbf{m}_1-\mathbf{m}_3|^2$, provided that the figure can be drawn at all and provided that the origin 0 lies inside the tetrahedron determined by the end points of $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$, $\mathbf{m}_4$. If this is not the case, draw figures of the type described at the end of Appendix A with the vectors $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ and $\mathbf{m}_1$, $\mathbf{m}_3$, $\mathbf{m}_4$, and obtain the threshold given there.