region. The statistical errors in counting the electron tracks amount to ~10%, when the subtraction of the background is taken into account.

The over-all uncertainty in the half-life given by the present measurement is thus believed to be ~15%.

It probably should be added that a very small fraction of the electron tracks starting from the neutron beam seemed to have a kind of cluster at the point of origin (Fig. 15). Although this could possibly be attributed to the decay proton, it must be emphasized that the measurements described above give no evidence for this.

VII. CONCLUSIONS

The cloud chamber method of measuring the half-life of the free neutron proved successful. The main sources of error seem to be in the flux measurement and in the poor statistics. If the statistical error can be reduced to ~3% and the error in the flux measurement to something like 3–4%, an over-all error of about 6–7% should result.

Although the present determination has not improved on the best previous measurement of the half-life of the neutron, it has about equal accuracy and confirms it by an independent method.

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Nuclear Structure Effects in Internal Conversion Coefficients by Configuration Mixing*

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The nuclear matrix elements which are needed to determine the internal conversion coefficients when a finite nucleus is employed are derived for nuclei for which the shell model wave functions are a good zero-order approximation for low-energy processes. Using configuration mixing, general expressions are derived for these matrix elements. It is shown that the nuclear structure alteration can be ten to twenty percent or more for l-forbidden transitions. Numerical results are given for the $M1$ and $E2$ 279-kev transitions in Tl^{203}.

I. INTRODUCTION

THE internal conversion coefficients convey important information about the atomic nucleus. The point nucleus calculations of the internal conversion coefficients[1] have been extremely valuable in determining the angular momentum and parity of nuclei. However, for large values of the atomic number, Z, corrections must be made for the extended nucleus. Sliv *et al.*[2] have calculated the alteration in the coefficients when the electron wave functions calculated for an extended, rather than a point nucleus, are used, i.e., the static effect. More recently, the conversion coefficients with the static effect included also have been calculated by Rose.[3] But without rather unphysical assumptions about the nuclear currents, the calculation of the internal conversion coefficients for a finite nuclear size requires the knowledge of the nuclear wave functions in order to calculate certain nuclear matrix elements (or the demonstration that they are unimportant).[4]

This dependence of the conversion coefficients on the nuclear wave functions makes the use of the experimental results less straightforward for determining nuclear properties. However, it has the advantage that an accurate measurement of the coefficients can give additional information about the details of nuclear structure. In particular, conversion coefficients can now help to provide information about the accuracy of nuclear models.

In the original work of Church and Weneser and in the more recent work of Green and Rose[5] the internal conversion coefficients are given by the power series expressions which separate the alterations which proceed

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[1] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. **76**, 1883 (1949); and unpublished tables by M. E. Rose *et al.*

[2] L. A. Sliv and I. M. Band, Leningrad Physico-Technical Institute Reports 1956 and 1958 [translation: Reports 57ICCK1 and 58ICCL1, issued by Physics Department, University of Illinois, Urbana, Illinois (unpublished)].

[3] M. E. Rose, *Internal Conversion Coefficients* (North-Holland Publishing Company, Amsterdam, 1958).

[4] E. L. Church and J. Weneser, Phys. Rev. **104**, 1382 (1956).

[5] T. A. Green and M. E. Rose, Oak Ridge National Laboratory Report ORNL-2395 (unpublished); Phys. Rev. **110**, 105 (1958).

from the use of electron wave functions associated with an extended nucleus from those depending directly on the nuclear wave functions. To the extent that the electron wave functions do not depend on the precise shape of the nuclear charge density, this work makes possible the calculation of the nuclear structure dependence without laborious recalculation of electron wave functions.

It is the aim of the present work to consider nuclei for which the shell model supplies a good zero-order approximation for nuclear wave functions, at least in so far as they are needed to determine certain nuclear matrix elements. Since the largest effects are to be expected for transitions in which selection rules diminish the gamma-transition probability, it is important to derive the matrix elements for nuclear wave functions perturbed by configuration mixing.

In Sec. II, general expressions are given which enable one to readily calculate the matrix elements for any order of electric or magnetic 2^L-pole transitions for any nucleus which is suitably described by a configuration-mixed wave function. The method used for deriving the configuration-mixed wave function follows closely that of Arima, Horie, and Sano.[6]

The application of these results to $M1$ and $E2$ transitions is done in Sec. III. Numerical results are obtained for the 279-kev transition in Tl^{203}. When the alteration of the conversion coefficients is large, the results depend to some extent upon the radial nuclear wave functions employed. Therefore, a study is made, using harmonic oscillator wave functions, which should indicate the dependence on the radial nuclear wave functions.

II. NUCLEAR MATRIX ELEMENTS WITH CONFIGURATION MIXING

The expressions of Green and Rose for the internal conversion coefficients are based on the usual assumptions that the electron charge and current are given by the Dirac operators, and obey a conservation of charge equation. The many-electron aspects are approximated by a screening model. A phenomenological nuclear charge and current distribution is employed, with the assumption that these are also related by a charge conservation equation, and that M dependence of the matrix element of the nuclear current is given by an angular momentum conserving Clebsch-Gordan coefficient. Both of these latter two assumptions are valid for the nuclear charge density and current used in the present work.

For the present work, it is convenient to express the conversion coefficients as the ratio of the internal conversion coefficients with nuclear structure included to those obtained when the currents are limited to the nuclear surface as assumed by Sliv. From Eqs. (47), (61), and (62) of reference 5, the ratio of the internal conversion coefficients to those of Sliv are given by

$$\frac{\beta_L}{\beta_L{}^0}=\frac{|1-\rho_{-L}e^{i\varphi_{-L}}\Sigma_{-L}|^2+y_L}{|1-\rho_{-L}e^{i\varphi_{-L}}\Sigma_{-L}{}^0|^2+y_L}, \tag{1}$$

and

$$\frac{\alpha_L}{\alpha_L{}^0}=\frac{|1-\omega_{-L-1}e^{i\tau_{-L-1}}\Sigma_{-L-1}|^2+|U_L-\omega_L e^{i\tau_L}\Sigma_L|^2}{|1-\omega_{-L-1}e^{i\tau_{-L-1}}\Sigma_{-L-1}{}^0|^2+|U_L-\omega_L e^{i\tau_L}\Sigma_L{}^0|^2}, \tag{2}$$

where $\beta_L(\alpha_L)$ is the magnetic (electric) 2^L-pole conversion coefficient as derived by Green and Rose, while $\beta_L{}^0(\alpha_L{}^0)$ is the magnetic (electric) 2^L-pole coefficient which results from the surface current model. The other quantities in Eqs. (1) and (2) are

$$\begin{aligned}
\Sigma_{-L}&=\sum_{n=0}^{\infty}\frac{a_n(-L)}{a_0(-L)}R(3+2n;1),\\
\Sigma_{-L}{}^0&=\sum_{n=0}^{\infty}\frac{a_n(-L)}{a_0(-L)},\\
\Sigma_{-L-1}&=\sum_{n=0}^{\infty}\{\bar{b}_n(-L-1)S(3+2n;1)\\
&\qquad+\bar{c}_n(-L-1)T(3+2n;1)\},\\
\Sigma_L&=\sum_{n=0}^{\infty}\{\bar{b}_n(L)S(3+2n;1)+\bar{c}_n(L)T(3+2n;1)\},\\
\Sigma_L{}^0&=\sum_{n=0}^{\infty}\left\{\bar{b}_n(L)+\left(\frac{L}{L+1}\right)^{\frac{1}{2}}\bar{c}_n(L)\right\};
\end{aligned} \tag{3}$$

with

$$\begin{aligned}
R(a;b)&=\int dx\,\Phi_L{}^L(x)\left(\frac{x}{R}\right)^a\Big/\int dx\,\Phi_L{}^L(x)\left(\frac{x}{R}\right)^b,\\
S(a;b)&=\int dx\,\Phi_{L-1}{}^L(x)\left(\frac{x}{R}\right)^a\Big/\\
&\qquad\int dx\,\Phi_{L-1}{}^L(x)\left(\frac{x}{R}\right)^b,\\
T(a;b)&=\int dx\,\Phi_{L+1}{}^L(x)\left(\frac{x}{R}\right)^a\Big/\\
&\qquad\int dx\,\Phi_{L-1}{}^L(x)\left(\frac{x}{R}\right)^b,
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\Phi_\lambda{}^L(x)C(J_iLJ_f;M_i-MM_f)\\
&\qquad=(-1)^{1+L-\lambda+M}x^2\int d\Omega\,\mathbf{J}_N\cdot\mathbf{T}_{L\lambda}{}^{M*}(\Omega),
\end{aligned} \tag{5}$$

with the reduced tensor $\mathbf{T}_{JL}{}^M$ defined[7] in terms of the

[6] A. Arima and H. Horie, Progr. Theoret. Phys. (Kyoto) **12**, 623 (1954); H. Horie and A. Arima, Phys. Rev. **99**, 778 (1955); Arima, Horie, and Sano, Progr. Theoret. Phys. (Kyoto) **17**, 567 (1957).

[7] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957), p. 106.

spherical vectors ξ_m as

$$\mathbf{T}_{JL}{}^{M}=\sum_m C(J1L;M-mm)Y_L{}^{M-m}\xi_m. \quad (6)$$

In Eqs. (1), (2), (3), and (4) the quantities ρ_{-L}, φ_{-L}, γ_{-L}, τ_L, τ_{-L-1}, a_n, $\bar{b}_n$, $\bar{c}_n$, and U_L are independent of the nuclear wave function, and are computed in reference 5 for electron wave functions calculated for a uniform nuclear charge distribution. In Eq. (5), $\mathbf{J}_N$ is the matrix element of the nuclear current between the initial state with angular momentum J_i and the final nuclear state with angular momentum J_f; the M_i and M_f are the respective z-components of angular momentum for the initial and final nuclear state.

The magnitude of the nuclear structure effect depends on the size of the quantities R in the magnetic transitions or S and T in the electric transitions. Since the coefficients ρ_{-L} are small, in fact less than 0.01 for Z less than 100 and energy less than 2.5 Mev, the ratios, R, must be considerably greater than unity if the results for the magnetic case are to be observable. An analogous situation prevails for electric transitions.

As pointed out by Church and Weneser, the most obvious situation in which a large value for R might be expected is one in which the γ-ray matrix element, which is proportional to $\int\Phi_L{}^{L}(x)xdx$, is reduced. In regions where the shell model gives a good first order approximation to the wave function for low-energy processes, the region to which the present work is confined, such a situation is to be found. In certain nuclei, transitions are observed experimentally which should not occur because of an orbital angular momentum selection rule, if the shell model wave functions are the correct ones for the states involved.[8] One explanation for this is that the shell model wave functions are altered by configuration mixing, so that there is a finite transition probability, but one which is considerably smaller than an ordinary single-particle transition probability. Although the transition probability depends critically upon the wave functions, Arima *et al.*[6] have been quite successful in predicting γ-ray lifetimes for many l-forbidden $M1$ transitions. Since the matrix elements needed for the calculation of the internal conversion coefficients are physically very similar, these wave functions should be quite good for use in the present work. Moreover, such wave functions have also been successfully used to predict magnetic and quadrupole moments, which also involve similar matrix elements.

Odd-even nuclei are considered, in which the shell model description of the transition is that the odd proton or neutron orbit is altered by a unit change in occupation number. There are two possibilities. The first of these, called "like-core" in reference 6, are transitions in which one particle in the initial odd orbit undergoes a transition to the orbit which is odd in the final state, thereby decreasing the number of particles in the initial odd orbit by one. The second type of transitions, the "unlike-core" transitions, are those in which a particle in an orbit which is even in the initial configuration undergoes a transition to the orbit which is odd in the initial configuration, thereby increasing the number of particles in the initial odd configuration by one.

Calling j the angular momentum of the initial state and j' that of the final state, with m and m' their respective z-components, l_1 the orbital angular momentum of one of the incomplete spin-orbit doublets in the core, $j_1=l_1+\frac{1}{2}$ and $j_2=l_1-\frac{1}{2}$ the angular momentum of particles in the lower and higher energy levels, respectively, of the doublet, the wave functions can be expressed as

$$\Psi_i(jm)=\Psi_0(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j^{p+1}(j)j'^{q}(0);jm)+\sum_J\beta_J\Psi_J([j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](J)j^{p}(0)j'^{q+1}(j');jm)$$
$$+\text{similar term for each incomplete doublet in the core,} \quad (7a)$$

$$\Psi_f(j'm')=\Psi_0'(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j^{p}(0)j'^{q+1}(j');j'm')$$
$$+\sum_J\beta_J'\Psi_J'([j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](J)j^{p+1}(J)j'^{q}(0);j'm')+\cdots, \quad (7b)$$

for like-core transitions, and similar wave functions (with $p+1$ and $q+1$ replaced by $p-1$ and $q-1$) for unlike-core transitions. In Eqs. (7a) and (7b), n_1 and n_2 are the (even) occupation numbers of the j_1 and j_2 orbits, respectively; the β_J and β_J' are the mixture coefficients in the initial and final states, obtained as in reference 6 by perturbation theory:

$$\beta_J=-\frac{1}{\Delta E}(\Psi_0(jm),\sum_{ik}v_{ik}\Psi_J(jm)),$$
$$\beta_J'=-\frac{1}{\Delta E}(\Psi_0'(j'm'),\sum_{ik}v_{ik}\Psi_J'(j'm')). \quad (8)$$

These are evaluated for like-core and unlike-core transitions needed in this present work in the Appendix.

The usual single-particle nuclear current operator, with a convection and spin part, is used. This neglects the finite proton size and, more importantly, the two- and more-particle terms introduced by nuclear correlations. The success of Arima *et al.* in calculating lifetimes of excited states, magnetic moments, and quadrupole moments of nuclei is an indication that these corrections to the nuclear current operator might not be important in low-energy processes. However, assuming a specific form for exchange currents, the corrections could be readily calculated by the methods described below. This is not done in the present work.

Six types of matrix elements must be evaluated in order to find the nuclear structure effects in the con-

[8] E.g., see L. Spruch and A. Rotenberg, Phys. Rev. **103**, 365 (1955), for references.

version coefficients:

$$I_0{}^C=\int\int d\Omega\int dV_n\,\Psi_0'^*\mathbf{J}^C\Psi_0\cdot\mathbf{T}_{L\lambda}{}^{*M},$$

$$I_L{}^C=\int\int d\Omega\sum_J\int dV_n\Psi_0'^*\mathbf{J}^C\Psi_J\cdot\mathbf{T}_{L\lambda}{}^{*M}\beta_J,$$

$$I_L{}'^C=\int\int d\Omega\sum_J\int dV_n\Psi_J'^*\mathbf{J}^C\Psi_0\cdot\mathbf{T}_{L\lambda}{}^{*M}\beta_J',$$

$$I_0{}^S=\int\int d\Omega\int dV_n\Psi_0'^*\mathbf{J}^S\Psi_0\cdot\mathbf{T}_{L\lambda}{}^{*M},\tag{9}$$

$$I_L{}^S=\int\int d\Omega\sum_J\int dV_n\Psi_0'^*\mathbf{J}^S\Psi_J\cdot\mathbf{T}_{L\lambda}{}^{*M}\beta_J,$$

$$I_L{}'^S=\int\int d\Omega\sum_J\int dV_n\Psi_J'^*\mathbf{J}^S\Psi_0\cdot\mathbf{T}_{L\lambda}{}^{*M}\beta_J'.$$

Some of these matrix elements differ slightly for unlike-core transitions, compared to like-core transitions. In this section the general form for these matrix elements is derived for any electric or magnetic multipole, with either type of transition.

By use of the fractional parentage coefficients,[9] the matrix element for $I_0{}^C$ can be simplified. From Eq. (9), for like-core transitions,

$$I_0{}^C=\int\int d\Omega\,\mathbf{T}_{L\lambda}{}^{*M}\cdot(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j^p(0)j'^{q+1}(j');\,j'm'|J^C|\,j_1{}^{n_1}(0)j_2{}^{n_2}(0)j^{p+1}(j)j'^q(0);\,jm)\tag{10}$$

becomes, using RIII (27),

$$=(j'^q(0)j'j'][j'^{q+1}j')(j^{p+1}(j)[[j^p(0)jj)[(p+1)(q+1)]^{\frac12}\int\int d\Omega(j'm'|\,\mathbf{J}^C|\,jm)\cdot\mathbf{T}_{L\lambda}{}^{*M}$$

$$=\left[\frac{(2j+1-p)(2j'+1-q)}{(2j+1)(2j'+1)}\right]^{\frac12}\int\int d\Omega(j'm'|\,\mathbf{J}^C|\,jm)\cdot\mathbf{T}_{L\lambda}{}^{*M},$$

where the phases of the fractional parentage coefficients have been disregarded, since they cancel out of any final result. Using Eq. (A5) from the Appendix for the single-particle matrix element, the final result for like-core transitions is

$$I_0{}^C=\left[\frac{(2j+1-p)}{(2j+1)}\frac{(2j'+1-q)}{(2j'+1)}\right]^{\frac12}\frac{\hbar e}{2mci}(-1)^{M+\frac12-\lambda+l-j'}C(jLj';m,-M)\left[\frac{2\lambda+1}{4\pi}\right]^{\frac12}[(2L+1)(2j+1)]^{\frac12}W(jlj'l';\tfrac12L)$$

$$\times\Big\{R_{j'}\tau^p\Big[-(l+1)^{\frac12}W(l'l-1L1;\lambda1)(l'\|\mathcal{C}^{(\lambda)}\|l+1)\left(\frac{dR_j}{dx}-l\frac{R_j}{x}\right)+l^{\frac12}W(l'l-1L1;\lambda l)(l'\|\mathcal{C}^{(\lambda)}\|l-1)$$

$$\times\left(\frac{dR_j}{dx}+\frac{l+1}{x}R_j\right)\Big]-(-1)^{\lambda-L}\Big[-(l'+1)^{\frac12}W(ll'+1L1;\lambda l')(l'+1\|\mathcal{C}^{(\lambda)}\|l)\left(\frac{dR_{j'}{}^*}{dx}-l'\frac{R_{j'}}{x}\right)$$

$$+(l')^{\frac12}W(ll'-1L1;\lambda l')(l'-1\|\mathcal{C}^{(\lambda)}\|l)\left(\frac{dR_{j'}{}^*}{dx}+\frac{l'+1}{x}R_{j'}{}^*\right)\Big]\tau^pR_j(x)\Big\}.\tag{11}$$

The unlike-core transition matrix element corresponding to Eq. (11) can, in a similar manner, be written

$$I_0{}^C=(pq)^{\frac12}\int\int d\Omega\,\mathbf{T}_{L\lambda}{}^{*M}(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j'^{q-1}(j')[j^{p-1}(j)j]\times(0);\,j'm'|\,\mathbf{J}^C|\,j_1{}^{n_1}(0)j_2{}^{n_2}(0)[j'^{q-1}(j')j'](0)j^{p-1}(j);\,jm),\tag{12}$$

which can be related to Eq. (11) by

$$I_0{}^C(\text{unlike-core})=\frac{(pq)^{\frac12}}{[(2j+1-p)(2j+1-q)]^{\frac12}}I_0{}^C(\text{like-core}).\tag{13}$$

Using similar methods, the integral $I_L{}^C$,

$$I_L{}^C=\sum_J\beta_J\int\int d\Omega\,\mathbf{T}_{L\lambda}{}^{*M}\cdot(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j'^{q+1}(j')j^p(0);\,j'm'|\,\mathbf{J}^C|\times[j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](J)j'^{q+1}(j')j^p(0);\,jm),\tag{14}$$

[9] G. Racah, Phys. Rev. **62**, 438 (1942) (referred to as RII), and Phys. Rev. **63**, 367 (1943) (referred to as RIII).

is expressed as a single-particle matrix element by use of the fractional parentage coefficients, as in the section above, and by Clebsch-Gordan coefficients. With the use of Eq. (A4),

$$I_L{}^C=(-1)^{M+1-\lambda}\frac{\hbar e}{2mci}C(jLj';m,-M)\beta_L[n_1(2j_2+1-n_2)]^{\frac{1}{2}}\left[\frac{2j+1}{2j'+1}\right]^{\frac{1}{2}}\left[\frac{2\lambda+1}{4\pi}\right]^{\frac{1}{2}}W(j_2l_1j_1l_1;\tfrac{1}{2}L)$$
$$\times\Big\{R_{j1}\tau^p\Big[-(l_1+1)^{\frac{1}{2}}W(l_1l_1+1L1;\lambda l_1)(l_1\|\mathcal{C}^{(\lambda)}\|l_1+1)\left(\frac{dR_{j2}}{dx}-l_1\frac{R_{j2}}{x}\right)+l_1{}^{\frac{1}{2}}W(l_1l_1-1L1;\lambda l_1)(l_1\|\mathcal{C}^{(\lambda)}\|l_1-1)$$
$$\times\left(\frac{dR_{j2}}{dx}+(l_1+1)\frac{R_{j2}}{x}\right)\Big]-(-1)^{\lambda-L}\Big[-(l_1+1)^{\frac{1}{2}}W(l_1l_1+1L1;\lambda l_1)(l_1+1\|\mathcal{C}^{(\lambda)}\|l_1)\left(\frac{dR_{j1}}{dx}-l_1\frac{R_{j1}}{x}\right)$$
$$+l_1{}^{\frac{1}{2}}W(l_1l_1-1L1;\lambda l_1)(l_1-1\|\mathcal{C}^{(\lambda)}\|l_1)\left(\frac{dR_{j1}}{dx}+(l_1+1)\frac{R_{j1}}{x}\right)\Big]\tau^pR_{j2}\Big\}. \quad (15)$$

Following the same reasoning as in the section above, it is easy to see that, except for the alteration in the mixture coefficients,

$$I_L{}^C(\text{unlike-core})=I_L{}^C(\text{like-core}). \quad (16)$$

In a similar manner, one can show that

$$I_L{}'^C=-\left(\frac{\beta_L{}'}{\beta_L}\right)\left[\frac{2j'+1}{2j+1}\right]^{\frac{1}{2}}I_L{}^C. \quad (17)$$

It is important to recognize that the final results for $I_L{}^C$ and $I_L{}'^C$ include only the value of $J=L$; i.e., the only admixed configuration which contributes for any spin-orbit doublet in the core is the one in which the total angular momentum of the two odd orbits of the doublet equals the order of the multipole. This holds for both electric and magnetic multipole transitions. As a consequence, very few mixture coefficients are required to obtain the nuclear structure alteration. This not only reduces the labor involved in carrying out the calculations, but makes the final results for the alteration of the conversion coefficients more accurate and far less arbitrary than they would be if more admixtures were involved. As will be shown in the next section, this often means that only one mixture is important in giving the nuclear structure effect.

In a similar manner, using the single-particle matrix element from Eq. (A6), one finds that

$$I_0{}^S=\left[\frac{(2j+1-p)(2j'+1-q)}{(2j+1)(2j'+1)}\right]^{\frac{1}{2}}\frac{\hbar e}{2mci}(-1)^{M+\frac{1}{2}+l+l'-i}C(jLj';m,-M)\left[\frac{2\lambda+1}{4\pi}\right]^{\frac{1}{2}}6[(2j+1)(2L+1)]^{\frac{1}{2}}$$
$$\times\Big\{(-1)^\lambda R_{j'}O_\mu\Big[-(l+1)^{\frac{1}{2}}\left(\frac{dR_j}{dx}-l\frac{R_j}{x}\right)(l'\|\mathcal{C}^{(\lambda)}\|l+1)\textstyle\sum_s(-1)^s(2s+1)W(l'l+1L1;\lambda s)W(11sl+1;1l)$$
$$\times W(\tfrac{1}{2}1jl;\tfrac{1}{2}s)W(jsj'l';\tfrac{1}{2}L)+l^{\frac{1}{2}}\left(\frac{dR_j}{dx}+(l+1)\frac{R_j}{x}\right)(l'\|\mathcal{C}^{(\lambda)}\|l-1)\textstyle\sum_s(-1)^s(2s+1)W(l'l-1L1;\lambda s)$$
$$\times W(11sl-1;1l)W(\tfrac{1}{2}1jl;\tfrac{1}{2}s)W(jsj'l';\tfrac{1}{2}L)\Big]-(-1)^{l+l'-L}\Big[-(l'+1)^{\frac{1}{2}}\left(\frac{dR_{j'}}{dx}-\frac{l'R_{j'}}{x}\right)(l'+1\|\mathcal{C}^{(\lambda)}\|l)$$
$$\times\textstyle\sum_s(-1)^s(2s+1)W(ll'+1L1;\lambda s)W(11sl'+1;1l')W(\tfrac{1}{2}1j'l';\tfrac{1}{2}s)W(jlj's;\tfrac{1}{2}L)+l'^{\frac{1}{2}}\left(\frac{dR_{j'}}{dx}+(l'+1)\frac{R_{j'}}{x}\right)$$
$$\times(l'-1\|\mathcal{C}^{(\lambda)}\|l)\textstyle\sum_s(-1)^s(2s+1)W(ll'-1L1;\lambda s)W(11sl'-1;1l')W(\tfrac{1}{2}1j'l';\tfrac{1}{2}s)W(jlj's;\tfrac{1}{2}L)\Big]O_\mu R_j\Big\}, \quad (18)$$

where the sum in each case is over the values of s which are allowed by the triangle relationships satisfied by the W coefficients.[9] O_μ is the magnetic momentum operator, with values of the magnetic moment of the proton or neutron in nuclear magnetons when operating on proton or neutron wave functions, respectively. It is easily seen that

$$I_0{}^S(\text{unlike-core})=\left[\frac{pq}{(2j+1-p)(2j+1-q)}\right]^{\frac{1}{2}}I_0{}^S(\text{like-core}). \quad (19)$$

Using Eq. (6), one finds for like-core transitions

$$I_L{}^S=\beta_L\int\int d\Omega\,\mathbf{T}_{L\lambda}{}^{*M}\cdot(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j'^{q+1}(j')j^p(0);\,j'm'|\,\mathbf{J}^S|[[j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](J)j'^{q+1}(j')j^p(0);\,jm)$$

$$=(-1)^{M+\frac{1}{2}+i_2}\beta_L C(jLj';m,-M)[n_1(2j_2+1-n_2)]^{\frac{1}{2}}\left[\frac{2j+1}{2j'+1}\right]^{\frac{1}{2}}\frac{\hbar e}{2mci}6\left[\frac{2\lambda+1}{4\pi}\right]^{\frac{1}{2}}$$

$$\times\Bigg\{(-1)^{\lambda}R_{j_1}O_\mu\Bigg[\sum_s(-1)^s(2s+1)[-(l+1)^{\frac{1}{2}}]\left(\frac{dR_{j_2}}{dx}-l_1\frac{R_{j_2}}{x}\right)(l_1\|\mathcal{C}^{(\lambda)}\|l_1+1)W(l_1l_1+1L1;\lambda s)$$

$$\times W(11sl_1+1;1l_1)W(\tfrac{1}{2}1j_2l_1;\tfrac{1}{2}s)W(j_2sj_1l_1;\tfrac{1}{2}L)+l_1{}^{\frac{1}{2}}\left(\frac{dR_{j_1}}{dx}+(l_1+1)\frac{R_{j_1}}{x}\right)(l_1\|\mathcal{C}^{(\lambda)}\|l_1-1)\sum_s(-1)^s(2s+1)$$

$$\times W(l_1l_1-1L1;\lambda s)W(11sl_1-1;1l_1)W(\tfrac{1}{2}1j_2l_1;\tfrac{1}{2}s)W(j_2sj_1l_1;\tfrac{1}{2}L)\Bigg]-(-1)^L\Bigg[-(l_1+1)^{\frac{1}{2}}\left(\frac{dR_{j_1}}{dx}-l_1\frac{R_{j_1}}{x}\right)$$

$$\times(l_1+1\|\mathcal{C}^{(\lambda)}\|l_1)\sum_s(-1)^s(2s+1)W(l_1l_1+1L1;\lambda s)W(11sl_1+1;1l_1)W(\tfrac{1}{2}1j_1l_1;\tfrac{1}{2}s)W(j_2l_1j_1s;\tfrac{1}{2}L)$$

$$+l_1{}^{\frac{1}{2}}\left(\frac{dR_{j_1}}{dx}+(l_1+1)\frac{R_{j_1}}{x}\right)(l_1-1\|\mathcal{C}^{(\lambda)}\|l_1)\sum_s(-1)^s(2s+1)$$

$$\times W(l_1l_1-1L1;\lambda s)W(11sl_1-1;1l_1)W(\tfrac{1}{2}1j_1l_1;\tfrac{1}{2}s)W(j_2l_1j_1s;\tfrac{1}{2}L)\Bigg]O_\mu R_{j_2}\Bigg\};\quad(20)$$

and

$$I_L{}^S(\text{unlike-core})=I_L{}^S(\text{like-core}),\quad(21)$$

except for the change in the mixture coefficients, as before. Also, the equation analogous to Eq. (17) holds.

III. NUCLEAR STRUCTURE EFFECTS IN Tl^{203} INTERNAL CONVERSION COEFFICIENTS

Measurements of the internal conversion coefficients of the 279-kev transition in Tl^{203} indicate that there might be a reduction[10–12] in the magnitude of these coefficients beyond that calculated by Sliv with the surface current model. The $M1$ transition is of the l-forbidden type, according to the shell model. The shell model description is a $j=d_{\frac{3}{2}}{}^3(d_{\frac{3}{2}})$ excited state and a $j'=s_{\frac{1}{2}}$ ground state, giving an orbital angular momentum change of two. Since the shell-model wave function should be good as a first order approximation from which a perturbation calculation can be made, the method developed in the previous section should be applicable.

Part A. $M1$ Transition

The shell-model wave functions for the initial and final states in the 279-kev transition are, respectively,

$$\Psi_i(d_{\frac{3}{2}},m)=\Psi_0(2d_{\frac{3}{2}}{}^3(d_{\frac{3}{2}})3s_{\frac{1}{2}}{}^2(0);\,d_{\frac{3}{2}}m),$$
$$\Psi_f(s_{\frac{1}{2}},m')=\Psi_0'(2d_{\frac{3}{2}}{}^4(0)3s_{\frac{1}{2}};\,s_{\frac{1}{2}}m').\quad(22)$$

The core contains three partially filled spin-orbit doublets. The $1h_{11/2}$ proton orbit is completely filled with twelve particles, while the $1h_{9/2}$ orbit is empty. This is referred to as a "same-type" admixture, since the particle jumping in the core and the excited particle are both protons. The two neutron orbits ("dissimilar-type") contributing are the $1i_{13/2}$ with twelve neutrons, and the $3p_{\frac{1}{2}}$, with four neutrons.

The angular integral of the matrix element,

$$\int\int d\Omega\,\mathbf{J}_N\cdot\mathbf{T}_{1,1}{}^{*M}=\frac{1}{x^2}(-1)^{M+1}\Phi_1{}^1(x)\times C(j1j';m,-M,m'),\quad(23)$$

follows immediately from the equations derived in Sec. II, with $L=\lambda=1$.

Using the results of Sec. II, one finds that

$$\Phi_1{}^1(x)=\frac{\hbar e}{2mci}\frac{1}{\sqrt{\pi}}\Bigg[1.397\left(x^2\frac{d}{dx}(R_{2d}R_{3s})+3x^2\frac{R_{2d}R_{3s}}{x}\right)$$

$$-(1.105\beta_1-0.7812\beta_1')\left(2x^2\frac{dR_{3p}}{dx}-x^2\frac{R_{3p}}{x}\right)R_{3p}$$

$$-(2.252\beta_2-1.592\beta_2')\left(2x^2\frac{dR_{1i}}{dx}-x^2\frac{R_{1i}}{x}\right)R_{1i}$$

$$+(2.335\beta_3-1.651\beta_3')\left(2x^2\frac{dR_{1h}}{dx}\right)R_{1h}$$

$$+(3.262\beta_3-2.307\beta_3')\times\left(2x^2\frac{dR_{1h}}{dx}-x^2\frac{R_{1h}}{x}\right)R_{1h}\Bigg].\quad(24)$$

[10] A. H. Wapstra and G. O. Nijgh, Nuclear Phys. **1**, 245 (1956).
[11] F. K. McGowan and P. H. Stelson, Phys. Rev. **103**, 1133 (1956).
[12] G. O. Nijgh (private communication).

The quantities needed are the ratios

$$R(N,1)=\int\Phi_1{}^1(x)\left(\frac{x}{R}\right)^N dx\Big/\int\Phi_1{}^1(x)\left(\frac{x}{R}\right)dx. \quad (25)$$

The harmonic oscillator functions[13] are used to evaluate these ratios. The radial parameter, ν, is determined by requiring the nuclear wave functions to produce the correct nuclear radius (see Appendix). Using the same radial functions for both orbits of the spin-orbit doublets, the integrals can be easily performed with the use of the relationship:

$$\int r^{N+1}R_1\frac{dR_2}{dr}dr$$

$$=\frac{1}{2N}[l_1(l_1+1)-l_2(l_2+1)-N(N+1)]$$

$$\times\int r^N R_2R_1dr+\frac{M^*}{\hbar^2}\frac{\Delta E}{N}\int r^{N+2}R_2R_1dr, \quad (26)$$

in which $\Delta E=E_1-E_2$ and $M^*=$ the effective mass of a nuclear particle in the shell model.[14]

The six mixture coefficients needed are all derived for the excited doublet coupled to angular momentum unity. The mixture coefficients in the initial state for the two dissimilar type admixtures, in terms of the two-body singlet potential, V_s, and triplet potential, V_t, using Eqs. (B6) and (B8), are

$$\beta_1=\frac{(V_t-V_s)}{\sqrt{3}4\Delta E_{3p}}I(3p3p2d3s),$$
$$\beta_2=\frac{(V_t-V_s)}{\Delta E_{1i}}\frac{3\sqrt{2}}{4(13)^{\frac{1}{2}}}I(1i1i2d3s). \quad (27)$$

From Eqs. (B7) and (B8) in the Appendix, the similar type admixture coefficient is found to be

$$\beta_3=\frac{-V_s}{\Delta E_{1h}}\frac{(15)^{\frac{1}{2}}}{2(11)^{\frac{1}{2}}}I(1h1h2d3s). \quad (28)$$

In Eqs. (27) and (28) the quantity $I(j_1j_2j_3j_4)$ represents the integral $\frac{1}{2}\int R(j_1)R(j_2)R(j_3)R(j_4)r^2dr$. The parameters used in this work are taken the same as in reference 6, i.e., $\Delta E_{1h}=\Delta E_{1i}=2$ Mev, $\Delta E_{3p}=0.5$ Mev, $V_s=-(250\text{ Mev}/A)(\pi^{\frac{1}{3}}/\nu^{\frac{2}{3}})$, $V_t=1.5V_s$. Also, it follows from the Appendix that $\beta'=-\sqrt{2}\beta$. As was the case in reference 6, every β_i is <0.1, consistent with perturbation theory. For the electron matrix elements, the pure shell-model contribution is more than 90% of the total.

[13] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).
[14] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955). M^*/M is altered to account for the finite nucleus in this work.

The ratios are

$$R(3,1)=43.84/(\nu R^2),$$
$$R(5,1)=659.2/(\nu R^2)^2, \quad (29)$$
$$R(7,1)=8840/(\nu R^2)^3.$$

The other quantities needed, as displayed by Eqs. (1) and (3) are a_n/a_0, y_1, ρ_{-1}, and φ_{-1}. Interpolating from the appropriate tables in reference 5, one finds $a_1/a_0=-0.15$, $a_2/a_0=0.02$, $\varphi_{-1}=0$, $y_1=0.0076$, and $\rho_{-1}=0.025$. This gives

$$\Sigma_{-1}'=0.87 \quad (30)$$

as the value of this function for the surface current model of Sliv. Table I gives the values of Σ_{-1} and β/β_S, the ratio of the $M1$ internal conversion coefficient to the value calculated by Sliv, for the most likely values of the nuclear parameter, ν (see Appendix). The second and third electron matrix elements contribute, respectively, 33 to 16% and 8 to 2% of the first one. Therefore these terms are important, but the series seems to be rapidly converging. The number of significant figures does not give the absolute accuracy of the result in Table I, but rather indicates the dependence on the nuclear radial parameter. The experimental results of Nijgh *et al.* give a reduction of 0.644,[12] and those of McGowan and Stelson give 0.65,[11] when compared to the point nucleus calculation. Since Sliv's correction factor is 0.77, this reduction of sixteen percent beyond Sliv's results is of the order of magnitude of the reduction given in Table I.

Two checks on the accuracy of this calculation are the magnetic moment of the ground state and the lifetime of the transition. The magnetic moment as calculated in reference 6 was within about ten percent of the experimental value. A more sensitive test is the lifetime. An equation for the reduced matrix element which is equivalent to that given by Arima *et al.*, but more convenient in using the results given above, is

$$\mathfrak{M}=2\left(\frac{2j+1}{3}\right)^{\frac{1}{2}}\sum_i\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)}\frac{l_1}{2l_1+1}\right]^{\frac{1}{2}}$$
$$\times(g_s-g_l)_i\beta_i,$$

where the sum is over all the partly filled doublets and $(g_s-g_l)=4.585$ or -3.826 for admixed protons or neutrons, respectively. For the 279-kev level in Tl^{203} the experimental value for the reduced matrix element is given by $\mathfrak{M}_{\text{exp}}{}^2=4.4\times10^{-11}/\tau$, where τ is the experimental mean life.

The results of this paper give $\mathfrak{M}=0.52$, in agreement with the results of Arima *et al.* The experimental values vary, and are made even more uncertain by the uncertainty in the $E2$ to $M1$ ratio, but the magnitude of $\mathfrak{M}_{\text{exp}}$ is two to three times smaller than the calculated value, a result discussed in reference 6, where the calculation for this lifetime gave one of the poorest results of the cases treated. This probably reflects the

choice of the wave functions used, and could indicate that the conversion coefficient is reduced more than the amount indicated in Table I. However, a recent calculation[15] predicts a twenty percent admixture of collective excitations to the wave functions of Tl^{203}. This would not alter the magnetic moment very much, but could alter both the gamma lifetime and the probability for electron emission. For this reason it is not certain how the reduction given in Table I would be affected, although the magnitude of the $E2$ lifetime, which is about four times the single-particle value, is correctly given only when the collective contributions are included.

Part B. $E2$ Transition

The wave functions needed for the calculation of the $E2$ matrix elements are identical to those in part A, except that the excited spin-orbit doublets in the admixed configurations in both the initial and final states are coupled to angular momentum two (2), instead of one (1). To evaluate the possible alteration in the $E2$ internal conversion coefficient, one needs the two angular integrals

$$\int d\Omega\, \mathbf{J}_N\cdot\mathbf{T}_{2,1}{}^{*M}=\frac{1}{x^2}(-1)^M C(j2j';m,-M,m')\Phi_1{}^2(x),$$
$$\int d\Omega\, \mathbf{J}_N\cdot\mathbf{T}_{2,3}{}^{*M}=\frac{1}{x^2}(-1)^M C(j2j';m,-M,m')\Phi_3{}^2(x). \tag{31}$$

Proceeding in precisely the same way as in Part A of this section, one can show that

$$\begin{aligned}\Phi_1{}^2(x)=&\frac{1}{(4\pi)^{\frac{1}{2}}}\frac{\hbar e}{2mci}\Big\{0.894x^2R_{3s}\frac{dR_{2d}}{dx}-0.894x^2R_{2d}\\
&\times\frac{dR_{3s}}{dx}+2.68xR_{3s}R_{2d}-3.75\\
&\times\Big[x^2R_{3s}\frac{dR_{2d}}{dx}+x^2R_{2d}\frac{dR_{3s}}{dx}+3xR_{3s}R_{2d}\Big]\\
&+\Big(\frac{\beta_3'}{\sqrt{2}}+\beta_3\Big)\\
&\times\Big[10.3x^2\Big(\frac{dR_{1h}}{dx}\Big)R_{1h}+15.4xR_{1h}{}^2\Big]\Big\}\\
&-\Big(\frac{\beta_1'}{\sqrt{2}}+\beta_1\Big)\Big[4.592x^2\Big(\frac{dR_{3p}}{dx}\Big)R_{3p}+6.889xR_{3p}{}^2\Big]\\
&-\Big(\frac{\beta_2'}{\sqrt{2}}+\beta_2\Big)\\
&\times\Big[4.076x^2\Big(\frac{dR_{1i}}{dx}\Big)R_{1i}+6.114xR_{1i}{}^2\Big]\Big\},\end{aligned} \tag{32}$$

[15] L. Silverberg (private communication).

TABLE I. Ratio of $M1$ internal conversion coefficient to that of Sliv for the 279-kev transition in Tl^{203}, for the most likely values of the nuclear parameter.

νR^2	Σ_{-1}	$\beta/\beta s$
7	4.76	0.81
8	4.28	0.83
9	3.89	0.85
10	3.57	0.87
11	3.30	0.88
12	3.07	0.89
13	2.87	0.9
14	2.69	0.91

and

$$\begin{aligned}\Phi_3{}^2(x)=&\frac{1}{(4\pi)^{\frac{1}{2}}}\frac{\hbar e}{2mci}\Big\{-4.155x^2R_{3s}\frac{dR_{2d}}{dx}-1.793x^2R_{2d}\\
&\times\frac{dR_{3s}}{dx}+8.311xR_{2d}R_{3s}\\
&+\Big(\frac{\beta_3'}{\sqrt{2}}+\beta_3\Big)\Big[15.256x^2\frac{dR_{1h}}{dx}R_{1h}+7.760xR_{1h}{}^2\Big]\\
&+\Big(\frac{\beta_1'}{\sqrt{2}}+\beta_1\Big)\Big[-2.165\Big(x^2R_{3p}\frac{dR_{3p}}{dx}-xR_{3p}{}^2\Big)\Big]\\
&+\Big(\frac{\beta_2'}{\sqrt{2}}+\beta_2\Big)\\
&\times\Big[0.887x^2R_{1i}\frac{dR_{1i}}{dx}-0.449xR_{1i}{}^2\Big]\Big\}.\end{aligned} \tag{33}$$

In Eqs. (32) and (33) the β and β' correspond to the same admixtures as in Part A, except the $J=2$, instead of 1. The equations given in the Appendix, with the values of the parameters used in Part A of this section, give for the mixture coefficients

$$\begin{aligned}\Big(\frac{\beta_2'}{\sqrt{2}}+\beta_2\Big)&=0.0154,\\
\Big(\frac{\beta_3'}{\sqrt{2}}+\beta_3\Big)&=-0.06231,\\
\Big(\frac{\beta_1'}{\sqrt{2}}+\beta_1\Big)&=-0.00679.\end{aligned} \tag{34}$$

From Eqs. (32), (33), and (34), one can determine the ratios needed in Eq. (2). Interpolating from reference 5 to find the other quantities needed for Eq. (2), one can find the conversion coefficients. The results are given in Table II. Once more the number of significant figures in the table does not represent the accuracy of the calculation, but rather indicates the dependence on the radial parameter. From this it is seen that for no reasonable value of the nuclear radial parameter does there arise an

TABLE II. Ratio of $E2$ internal conversion coefficient to that of Sliv for the 279-kev transition in Tl^{203} for the most likely values of the nuclear parameter.

νR^2	Σ_{-2}	Σ_2	α_2/α_{2s}
7	0.793	0.310	0.9995
8	0.702	0.277	0.9998
9	0.631	0.251	1.0001
10	0.569	0.227	1.0002
11	0.524	0.211	1.0004
12	0.483	0.195	1.0006
13	0.442	0.178	1.0007
14	0.418	0.170	1.0008

experimentally determinable alteration of the internal conversion coefficient. This is a reasonable result, since the shell model contribution is the important one in this case, and therefore it does not seem likely that the ratios of matrix elements should be much affected by the admixed configurations. This result depends, of course, upon the shell model wave function being a good approximation from which to start perturbation theory. These results are in disagreement with Nijgh *et al.*,[12] who find a reduction of 11% beyond Sliv, but in agreement with the results of McGowan *et al.*,[11] who obtain the 0.65 $M1$ reduction with no $E2$ reduction, which is essentially the result of Sliv.

IV. CONCLUSIONS

General formulas are derived for the nuclear matrix elements involved in determining the nuclear structure effect in internal conversion coefficients, using shell model wave functions with configuration interaction. By using the results of Green and Rose, one can apply these matrix elements to find the alteration in the K-shell conversion coefficients for any magnetic or electric 2^L-pole transitions of interest at this time. Moreover, these matrix elements are unaffected by the atomic consideration, and could be used to determine the conversion coefficient alteration in other atomic shells, if the Green-Rose work should be extended.

From a study of the general formulas, and of the special results in Sec. III, it is seen that the internal conversion coefficient is not altered greatly by purely nuclear structure considerations unless the gamma-transition is hindered by a selection rule in this region where the shell model gives a good zero-order approximation for wave functions to determine low-energy matrix elements. The success in predicting magnetic moments, quadrupole moments, and lifetimes indicates that the configuration-mixed wave functions should be quite good in estimating the matrix elements needed in the present work, for the region of nuclei studied here.

The largest contribution from the admixed configurations comes from any similar type spin-orbits which may be included; i.e., for proton transitions, the configuration interaction contribution due to an unfilled spin-orbit proton doublet is more important than the contributions from such unfilled neutron doublets in the core. This means that in some cases the sign of the matrix elements, and to some extent the magnitude of the alteration of the internal conversion coefficient, can be determined by a single admixture.

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APPENDIX A. ANGULAR INTEGRAL WITH THE SINGLE-PARTICLE MATRIX ELEMENT

Let

$$M=M^C+M^S, \tag{A1}$$

where the matrix element is broken into a convection and spin current part,

$$M=\int\int d\Omega(j_f m_f|\mathbf{J}^N|j_i m_i)\cdot\mathbf{T}_{L\lambda}{}^{*M}(\Omega),$$

$$M^C=\int\int d\Omega(j_f m_f|\mathbf{J}^C|j_i m_i)\cdot\mathbf{T}_{L\lambda}{}^{*M}(\Omega), \tag{A2}$$

$$M^S=\int\int d\Omega(j_f m_f|\mathbf{J}^S|j_i m_i)\cdot\mathbf{T}_{L\lambda}{}^{*M}(\Omega),$$

where $\mathbf{J}^C$ and $\mathbf{J}^S$ are the convection and spin parts of the current, respectively. Using the single-particle current operator, the convection part is

$$M_C=\int\int d\Omega\,\mathbf{T}_{L\lambda}{}^{*M}(\Omega)\cdot\psi_f{}^*(x_1\cdots x_A)\frac{e}{2mc}$$
$$\times\{\textstyle\sum_n[\mathbf{p}_n\delta(\mathbf{x}-\mathbf{x}_n)+\delta(\mathbf{x}-\mathbf{x}_n)\mathbf{p}_n]\}$$
$$\times\tau_n\psi_i(x_1\cdots x_A)d^3x_1\cdots d^3x_A, \tag{A3}$$

where τ_n is isotopic spin operator for the nth nucleon. The single-particle wave function is a product of a radial function and a spherical harmonic:

$$\phi_{jm}(x)=R_{j1}(x)Y(\Omega)=R_i\textstyle\sum_\tau$$
$$\times C(1sj;m-\tau,\tau)Y_1{}^{m-\tau}\chi_s{}^\tau. \tag{A4}$$

Using the gradient formula[16] to express the gradient of

[16] See reference 7, p. 124.

the wave function in terms of tensors, and carrying out the angular integral by the use of RII (29) and Eq. (19) of Biedenharn *et al.*,[17] one gets the result

$$M_C=\frac{e\hbar}{2mci}(-1)^{M-2m_i+\frac{3}{2}-\lambda+l_i-j_f}C(j_iLj_f;-M,m_f)\left[\frac{2\lambda+1}{4\pi}\right]^{\frac{1}{2}}[(2L+1)(2j_i+1)]^{\frac{1}{2}}W(j_il_ij_fl_f;\tfrac{1}{2}L)$$

$$\times\Big\{R_{jf}^*(x)\tau_1{}^p\Big[-(l_i+1)^{\frac{1}{2}}W(l_fl_i+1\ L1;\lambda l_i)(l_f\|\mathcal{C}^{(\lambda)}\|l_i+1)\left(\frac{dR_{ji}(x)}{dx}-l_i\frac{R_{ji}(x)}{x}\right)$$

$$+l_i{}^{\frac{1}{2}}W(l_fl_i-1\ L1;\lambda l_i)(l_f\|\mathcal{C}^{(\lambda)}\|l_i-1)\left(\frac{dR_{ji}(x)}{dx}+(l_i+1)\frac{R_{ji}(x)}{x}\right)\Big]$$

$$-(-1)^{-L}\Big[-(l_f+1)^{\frac{1}{2}}W(l_il_f+1\ L1;\lambda l_f)(l_f+1\|\mathcal{C}^{(\lambda)}\|l_i)\left(\frac{dR_{jf}^*(x)}{dx}-l_f\frac{R_{jf}^*(x)}{x}\right)+l_f{}^{\frac{1}{2}}W(l_il_f-1\ L1;\lambda l_f)$$

$$\times(l_f+1\|\mathcal{C}^{(\lambda)}\|l_i)\left(\frac{dR_{jf}^*(x)}{dx}+(l_f+1)\frac{R_{jf}^*(x)}{x}\right)\Big]\tau_1{}^pR_{j1}(x)\Big\}. \quad \text{(A5)}$$

The evaluation of M_S is done in an analogous manner. From Eq. (A2),

$$M_S=\frac{e\hbar}{2mci}\int\int d\Omega[\nabla\phi_i(x)\times\phi_f^*(x)(\mu^p\tau^p+\mu^N\tau^N)\sigma+\nabla\phi_f^*(x)\times\sigma(\mu^p\tau^p+\mu^N\tau^N)\phi_i(x)]\cdot\mathbf{T}_{L\lambda}{}^{*M}, \quad \text{(A6)}$$

where μ^N and μ^p are the magnetic moments in nuclear magnetons of a neutron and proton, respectively. Once more the gradient formulas are used to express $\nabla\phi_f^*$ and $\nabla\phi_i^*$ in terms of the reduced tensors. Using the spherical representation for the σ vectors, one finds by the usual methods that

$$M_S=(-1)^{M+\frac{1}{2}-j_f}\left[\frac{2\lambda+1}{4\pi}\right]^{\frac{1}{2}}\times2\times3[(2j_i+1)(2L+1)]^{\frac{1}{2}}C(j_iLj_f;m_i,-M,m_f)\frac{e\hbar}{2mci}$$

$$\times\Big\{(-1)^\lambda R_{jf}^*O_\mu\Big[\textstyle\sum_s(-1)^s(2s+1)[-(l_i+1)^{\frac{1}{2}}]\left(\frac{dR_{ji}}{dx}-l_i\frac{R_{ji}}{x}\right)(l_f\|\mathcal{C}^{(\lambda)}\|l_i+1)W(l_fl_i+1\ L1;\lambda s)$$

$$\times W(11sl_i+1;1l_i)W(\tfrac{1}{2}1j_il_i;\tfrac{1}{2}s)W(j_isj_f;\tfrac{1}{2}L)+l_i{}^{\frac{1}{2}}\textstyle\sum_s(-1)^s(2s+1)\left(\frac{dR_{ji}}{dx}+(l_i+1)\frac{R_{ji}}{x}\right)(l_f\|\mathcal{C}^{(\lambda)}\|l_i-1)$$

$$\times W(l_fl_i-1\ L1;\lambda s)W(11sl_i-1;1l_i)W(\tfrac{1}{2}1j_il_i;\tfrac{1}{2}s)W(j_isj_fl_f;\tfrac{1}{2}L)\Big]$$

$$+(-1)^{1-L+l_i+l_f}\Big[-(l_f+1)^{\frac{1}{2}}\left(\frac{dR_{jf}^*}{dx}-l_f\frac{R_{jf}^*}{x}\right)(l_f+1\|\mathcal{C}^{(\lambda)}\|l_i)\textstyle\sum_s(-1)^s(2s+1)W(l_il_f+1\ L1;\lambda s)$$

$$\times W(11sl_f+1;1l_f)W(\tfrac{1}{2}1j_fl_f;\tfrac{1}{2}s)W(j_il_ij_fs;\tfrac{1}{2}L)+l_f{}^{\frac{1}{2}}\left(\frac{dR_{jf}^*}{dx}+(l_f+1)\frac{R_{jf}^*}{x}\right)(l_f-1\|\mathcal{C}^{(\lambda)}\|l_i)$$

$$\times\textstyle\sum_s(-1)^s(2s+1)W(\frac{1}{2}1j_fl_f;\frac{1}{2}s)W(j_il_ij_fs;\frac{1}{2}L)W(l_il_f-1\ L1;\lambda s)W(11sl_f-1;1l_f)\Big]O_\mu R_{ji}\Big\}, \quad \text{(A7)}$$

where $O_\mu=\mu_p\tau^p+\mu_N\tau^N$. Note a factor of $(-1)^{1+l_i+l_f-j_i-j_f}$ is needed if the spin wave functions are uncoupled by $C(\frac{1}{2}lj;m_sm_l)$ [also in (A5)].

APPENDIX B. MIXTURE COEFFICIENTS

From first-order perturbation theory, the mixture coefficients are obtained by Eq. (8), where v_{ik} is the nucleon-nucleon interaction potential. In this work we follow the procedure used in reference 6. The two-body interaction

[17] Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).

potential is of the δ-function type with a singlet and triplet contribution; i.e.,

$$v_{ik}=[V_s(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)/4+V_t(3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_k)/4]\delta(r_i-r_k)\delta\left(\cos\frac{\mathbf{r}_i\cdot\mathbf{r}_k}{r_ir_k}-1\right)\Big/r_ir_k. \tag{B1}$$

The two parameters V_s and V_t are determined from experiment. As is stated in reference 6, the two-body potential can more conveniently be written as a scalar product of two tensors:

$$v_{ik}=\sum_{xkr}(-1)^{x+k-r}V_xv_k(r_i,r_j)(\mathbf{t}_i{}^{xk;r}\mathbf{t}_j{}^{xk;r}), \tag{B2}$$

where $V_0=(V_s+3V_t)/4$ and $V_1=(V_t-V_s)/4$ and

$$t_m{}^{xk;r}=\sum_b C_{b,m-b}{}^{xkr}\sigma_b{}^x\mathcal{C}_{m-b}{}^k, \tag{B3}$$

where $\mathcal{C}_m{}^k=[4\pi/(2k+1)]^{\frac{1}{2}}Y_k{}^m(\Omega)$, and $\sigma_b{}^x$ is unity for $x=0$ and the Pauli spin operator for $x=1$ in the spherical representation. Also, in Eq. (B2),

$$v_k=[(2k+1)/2]\delta(r_i-r_j)/r_ir_j. \tag{B4}$$

Part A. Like-Core Mixtures in the Initial State. Dissimilar Type $M1$

$$-\Delta E\beta=(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j'{}^q(0)j^{p+1}(j);\,jm|\sum v_{ik}|[j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](1)j'{}^{q+1}(j')j^p(0);\,jm). \tag{B5}$$

Using the methods of reference 6 and a theorem due to de-Shalit,[18] one obtains

$$\beta=\frac{3V_1}{8\Delta E}\left[\frac{(n_1)(2j_2+1-n_2)(2j+1-p)(2j'+1-q)(2j_1-1)(2j-1)}{(2j_2+1)(2j+1)(2j'+1)jj_1}\right]^{\frac{1}{2}}I(j_1j_2jj'). \tag{B6}$$

Part B. Like-Core Mixture in the Initial State. Similar Type $M1$

For similar type transition the results differ from Part A in that the Pauli principle must be applied to the matrix elements. The result is

$$\beta=\frac{\sqrt{3}(-V_s)}{16\Delta E}\left[\frac{n_1(2j_2+1-n_2)(2j+1-p)(2j'+1-q)(2j_1-1)(2j-1)}{(2j_2+1)(2j+1)(2j'+1)jj_1}\right]^{\frac{1}{2}}I(j_1j_2jj'). \tag{B7}$$

Part C. Unlike-Core Mixture Coefficients $M1$

Whether the transition is of similar or dissimilar type, or the mixture is in the initial or final state, the theory follows in precisely the same manner. The angular momenta are uncoupled by fractional parentage coefficients and Clebsch-Gordan coefficients. Since all of the matrix elements of the unlike-core type have the relationship to those of the same kind but of the like-core type as Eq. (12) is related to Eq. (10), it turns out that in every case

$$\beta(\text{unlike-core})=\left[\frac{pq}{(2j+1-p)(2j'+1-q)}\right]^{\frac{1}{2}}\beta(\text{like-core}). \tag{B8}$$

Part D. Mixtures in the Final State $M1$

As an example of mixing in the final state, from which one can draw general conclusions, the like-core, similar-type transitions have a matrix element

$$-\Delta E\beta'=(j_1{}^{n_1}(0)j_2{}^{n_2}(0)j'{}^{q+1}(j')j^p(0);\,jm|v_{ik}|[j_1{}^{n_1-1}(j_1)j_2{}^{n_2+1}(j_2)](1)j'{}^q(0)j^{p+1}(j);\,j'm'), \tag{B9}$$

which is to be compared with Eq. (B5). They are identical if the following pairs are interchanged: (p,q), (j,j'), and (l,l'). In doing this, one should keep in mind that there are two possible cases. One may have $j=j'+1$, corresponding to $l=l'+2$; or one may have $j=j'-1$, corresponding to $l=l'-2$. In general, the mixture parameters for the two cases are different, so the coefficient corresponding to $j=j'-1$ must be used (after the three pairs mentioned above are switched) instead of $j=j'+1$, or vice versa. However, in the $M1$ case, the final results are the same in every case. The result is

$$\beta'=-[(2j+1)/(2j'+1)]^{\frac{1}{2}}\beta. \tag{B10}$$

[18] A. de-Shalit, Phys. Rev. **91**, 1479 (1953).

Part E. $E2$ Mixture Coefficients

For each case, the $E2$ coefficient is obtained by techniques similar to those in the corresponding $M1$ case, although the calculations are all somewhat more tedious for the $E2$. The results are given below.

(1) Initial State Admixture, Dissimilar

$j=j'+1$:

$$\beta=\frac{\sqrt{5}}{16}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j-1)}{j_1(j_1-1)(j_1+1)j(j+1)(j-1)}\right]^{\frac{1}{2}}(3V_0+2jj_1V_1), \quad \text{(B11)}$$

$j=j'-1$:

$$\beta=\frac{\sqrt{5}}{16}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+3)}{j_1(j_1-1)(j_1+1)j(j+1)(j+2)}\right]^{\frac{1}{2}}[-3V_0+2(j+1)j_1V_1]. \quad \text{(B12)}$$

(2) Final State Admixture, Dissimilar

$j=j'+1$:

$$\beta'=\frac{\sqrt{5}}{16}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+1)}{j_1(j_1-1)(j_1+1)j(j-1)(j+1)}\right]^{\frac{1}{2}}(-3V_0+2jj_1V_1), \quad \text{(B13)}$$

$j=j'-1$:

$$\beta'=\frac{\sqrt{5}}{16}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+1)}{j_1(j_1-1)(j_1+1)j(j-1)(j+1)}\right]^{\frac{1}{2}}[3V_0+2j_1(j+1)V_1]. \quad \text{(B14)}$$

(3) Initial State Admixture, Similar

$j=j'+1$:

$$\beta=\frac{\sqrt{5}}{32}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j-1)}{j_1(j_1-1)(j_1+1)j(j-1)(j+1)}\right]^{\frac{1}{2}}(3-2jj_1)(V_s), \quad \text{(B15)}$$

$j=j'-1$:

$$\beta=\frac{\sqrt{5}}{32}\frac{1}{\Delta E}\left\{\frac{[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}}{(pq)^{\frac{1}{2}}}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+3)}{j_1(j_1-1)(j_1+1)j(j+1)(j+2)}\right]^{\frac{1}{2}}[3+2(j+1)j_1](-V_s). \quad \text{(B16)}$$

(4) Final State Admixture, Similar

$j=j'+1$:

$$\beta'=\frac{\sqrt{5}}{32}\frac{1}{\Delta E}\left\{\begin{matrix}[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}\\ (pq)^{\frac{1}{2}}\end{matrix}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+1)}{j_1(j_1-1)(j_1+1)j(j-1)(j+1)}\right]^{\frac{1}{2}}(3+2jj_1)(-V_s), \quad \text{(B17)}$$

$j=j'-1$:

$$\beta'=\frac{\sqrt{5}}{32}\frac{1}{\Delta E}\left\{\begin{matrix}[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}\\ (pq)^{\frac{1}{2}}\end{matrix}\right\}\left[\frac{n_1(2j_2+1-n_2)}{(2j_2+1)(2j+1)(2j'+1)}\right]^{\frac{1}{2}}$$
$$\times I(j_1j_2jj')\left[\frac{(2j_1-1)(2j+1)}{j_1(j_1-1)(j_1+1)j(j+1)(j+2)}\right]^{\frac{1}{2}}[3-2(j+1)j_1]V_s. \quad \text{(B18)}$$

In Eqs. (B11) through (B18), the curly bracket indicates that the upper quantity, $[(2j+1-p)(2j'+1-q)]^{\frac{1}{2}}$, is used for like-core transitions, while the lower quantity, $(pq)^{\frac{1}{2}}$, is used for unlike-core transitions.

APPENDIX C. NUCLEAR RADIAL PARAMETER

In choosing the nuclear radial parameter, ν, the condition is that the normalized nuclear wave functions give the proper expectation value for the square of the radius; i.e., that the mean square radius for these wave functions is the same as the value for a uniform density. This condition is

$$\int dV_N\, \Psi^* r^2 \Psi = \tfrac{3}{5}R^2. \quad \text{(C1)}$$

This condition is not unique, however, for it is difficult to determine which orbits should be used. It is not correct to use all of the orbits, as can be seen either by examining the nuclear surface generated by this method, or recognizing that the independent particle approximation certainly does not imply that the nucleons deep within the core act as the same nuclear particles near the surface. Therefore various assumptions were made, and the nuclear parameter was calculated; as a result, the range of nuclear parameters obtained should give a good limit to the proper value of the nuclear parameter. The calculations were made for Pb^{208}.

(a) Using the $3s$ protons in the last orbit, $\nu R^2=9.2$.
(b) Using all of the protons in the last orbit, $\nu R^2=9.78$.
(c) Using the neutron last p orbit, $\nu R^2=10.8$.
(d) Using the neutron last g orbit, $\nu R^2=12.5$.
(e) Using all the neutron orbits in the last neutron shell, $\nu R^2=11.4$.