the correlation of density fluctuations over macroscopic intervals should depend on the same properties that determine the character of equilibrium macroscopic inhomogeneities. The reason for this is that the "states" which contribute most prominently to the density fluctuations over large regions must be representable as possible thermodynamic states. It now becomes possible to determine the configuration and free energy of those "states" by the methods of this paper, and so to formulate this problem without resort to microscopic statistical mechanics by use of thermodynamic fluctuation theory.

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Langevin Equation and the ac Conductivity of Non-Maxwellian Plasmas

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The use of the Langevin equation $\{d\mathbf{v}/dt+g\mathbf{v}=(q/m)[\mathbf{E}_0e^{i\omega t}+(\mathbf{v}/c)\times\mathbf{H}]\}$ to describe the electrical conductivity of a non-Maxwellian plasma (a weakly ionized gas in which the average electron collision frequency is temperature dependent) may be in error unless it is understood that the dissipative term, g, is complex. In the limiting cases of either high or low pressures the imaginary part of g is negligible. The real and imaginary parts of g are evaluated for these limiting cases, for four different gases; air, helium, Maxwellian gas, and water. The real part of g is shown to be the average collision frequency multiplied by a numerical factor, the size of which depends on the nature of the gas and the pressure limit.

I. INTRODUCTION

The conventional description of the ac conductivity of a weakly ionized gas (e.g., Mitra[1]) is based upon the assumption that the average drift velocity of electrons in the gas obeys the Langevin equation[2]:

$$\frac{d\mathbf{v}}{dt}+g\mathbf{v}=\frac{q}{m}\mathbf{E}_0e^{i\omega t}, \tag{1}$$

with the steady state solution

$$\mathbf{v}=\frac{1}{i\omega+g}\left(\frac{q}{m}\right)E_0e^{i\omega t}, \tag{2}$$

where $\mathbf{v}$ is the average drift velocity, g is a dissipative term which represents the effect of collisions between electrons and molecules of the gas and the symbols q, m, and E_0 have their usual meaning.

The usual approach has been to equate g with ν, the average collision frequency of the electrons. This particular method is, in general, incorrect for real gases. Indeed, it is only for a Maxwellian plasma[3–5] that g is identically equal to ν.

A Maxwellian plasma is a weakly ionized[6] gas for which the cross section for momentum transfer, $Q(V)$, for electron impact on the neutral molecule varies inversely as V, the relative velocity between the electron and the molecule.[7]

Now, most real plasmas are non-Maxwellian. For example, the cross section, Q, for air[8] or nitrogen[9] is proportional to V, the cross section for helium[10] appears to be constant, and the cross section for water, and other molecules possessing permanent electric dipole moments,[11] varies inversely as V^2.

It is not to be expected that g is simply related to ν for these plasmas except under certain limiting circumstances. We proceed to find what form g must take for the above mentioned plasmas.

II. THE FORM OF g FOR A NON-MAXWELLIAN PLASMA

The average drift velocity for electrons in a gas in the presence of a weak, alternating electric field, may be determined using the relationships presented by Allis[5] and Margenau.[12]

The drift velocity, so determined, is represented, in form, by

$$\mathbf{v}=(q/m)\mathbf{E}_0e^{i\omega t}(B-iD), \tag{3}$$

[1] S. K. Mitra, *The Upper Atmosphere* (The Asiatic Society, Calcutta, 1952), second edition, pp. 623–629.
[2] W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 21, p. 392.
[3] T. Kihara, Revs. Modern Phys. **24**, 49 (1952).
[4] S. Altshuler (unpublished).
[5] Reference 2, p. 413.
[6] By "weakly ionized," we mean the only important interactions are those occurring between electrons and neutral particles.
[7] Or equivalently, the collision frequency independent of energy.
[8] Crompton, Huxley, and Sutton, Proc. Roy. Soc. (London) **A218**, 507 (1953).
[9] Phelps, Fundingsland, and Brown, Phys. Rev. **84**, 559 (1951).
[10] L. Gould and S. C. Brown, Phys. Rev. **95**, 897 (1954).
[11] S. Altshuler, Phys. Rev. **107**, 114 (1957).
[12] H. Margenau, Phys. Rev. **69**, 508 (1946).

TABLE I. Evaluation of g_r and g_i for the low- and high-pressure cases.

Gas species	$\nu/\omega<1$ g_r	$\nu/\omega<1$ g_i	$\nu/\omega>1$ g_r	$\nu/\omega>1$ g_i	Form of average collision frequency[a] ν
Q/V constant, air, nitrogen	$(5/3)\nu(1+4.22\nu^2/\omega^2)$	$(10/9)(\nu^2/\omega)\times(1+5\nu^2/\omega^2)$	$\nu[1+4.57(\omega/\nu)^{\frac{3}{2}}]$	$2\omega[1-2.29(\omega/\nu)^{\frac{1}{2}}+4.57(\omega/\nu)^{\frac{3}{2}}]$	$\nu=(3Q/V)\rho KT/m$
Q constant, helium	$\frac{4}{3}\nu(1-0.22\nu^2/\omega^2)$	$0.18(\nu^2/\omega)\times(1+2.14\nu^2/\omega^2)$	$(3\pi\nu/8)(1+0.28\omega^2/\nu^2)$	$0.18\omega(1-7.5\omega^2/\nu^2)$	$\nu=2\rho Q(2KT/\pi m)^{\frac{1}{2}}$
QV constant, Maxwellian gas	ν	0	ν	0	$\nu=\rho QV$
QV^2 constant, H_2O, NH_3; molecules with permanent electric dipole moments	$\frac{2}{3}\nu(1-0.18\nu^2/\omega^2)$	$0.079(\nu^2/\omega)\times(1+0.60\nu^2/\omega^2)$	$(3\pi\nu/16)(1+0.30\omega^2/\nu^2)$	$0.11\omega(1-7.1\omega^2/\nu^2)$	$\nu=2QV^2\rho(m/2\pi KT)^{\frac{1}{2}}$

[a] ν is rigorously defined as follows: $\nu=\int\rho Q(V)Vfd\tau$, where ρ is the neutral particle density, $Q(V)$ is the cross section with its correct functional dependence on V, f is the normalized velocity distribution function (assumed Maxwell-Boltzmann) and integration is over all of velocity space.

where B and D are rather complicated functions of ν, the average collision frequency, and ω the wave angular frequency. Moreover, $(ne^2/m)(B-iD)$ is the complex conductivity of the gas, where n is the electron particle density.

The solution (3) for $\mathbf{v}$, is incompatible with Eqs. (1) and (2) unless g is taken as complex or unless $B=\omega/(\omega^2+g^2)$ and $D=g/(\omega^2+g^2)$. The latter is true only for a Maxwellian gas. Thus, to insure compatibility, $g=g_r+ig_i$, where

$$g_r=B/(B^2+D^2),\qquad g_i=D/(B^2+D^2)-\omega. \tag{4}$$

The functional forms of B and D have been evaluated for the various gases discussed in Part I and are assembled in the Appendix. They lead, in general (except for the Maxwellian plasma), to non-negligible values of g_i.

Now, no simplification results from the use of the Langevin equation to describe non-Maxwellian plasmas unless g is real. Therefore, in order to retain this intuitively satisfactory form we must look for conditions under which g_i may be neglected. This quest is motivated also by the fact of the existence of a tremendous amount of analysis, already in the literature, based upon the arbitrary substitution of ν for g (e.g., the Appleton-Hartree formula). Therefore, under conditions where g_i is negligible, such analysis may be rectified by the formal substitution of g_r for ν, wherever ν appears.

III. LIMITING FORMS OF g_r AND g_i FOR LOW AND HIGH PRESSURES

An inspection of the asymptotic forms of B and D, listed in the Appendix, reveals that g_i may be neglected under the following limiting conditions.

$$\nu/\omega\ll 1,\ \text{the low-pressure case;}\qquad \nu/\omega\gg 1,\ \text{the high-pressure case.} \tag{5}$$

The limiting forms of g_r and g_i for these two cases have been evaluated the second order in ν/ω and are presented in Table I.

It is to be observed that g_r in the low-pressure limit is a simple multiple of $\nu/3$, and g_i is of the order ν^2/ω^2 with respect to ω. Therefore, it is reasonable to ignore g_i in Eq. (2) for $\nu/\omega<1$. On the other hand, g_i in the high-pressure limit, especially for air, may be of the same order as ω and cannot be dropped unless ω itself can be neglected.

Note that according to the last column of the table, g_r is temperature dependent.

IV. CONDUCTIVITY WITH A STATIC MAGNETIC FIELD

With a static magnetic field, H, the average drift velocity becomes[5]

$$\mathbf{v}=\|\mu\|\mathbf{E}_0e^{i\omega t}, \tag{6}$$

where $\|\mu\|$ is the mobility tensor and is given by

$$\|\mu\|=\frac{q}{m}\begin{Vmatrix} L+R & i(R-L) & 0 \\ i(L-R) & L+R & 0 \\ 0 & 0 & 2P \end{Vmatrix}. \tag{7}$$

Here P is just the $B-iD$ that appears in Eq. (3). L is obtained by formally replacing ω, where ever it occurs in $B-iD$, by $\omega+\omega_b$. R is obtained similarly by replacing ω by $\omega-\omega_b$. ω_b is $-q|H|/mc$, the angular cyclotron frequency.

Since we have demonstrated the equivalence between the two forms, $B-iD$ and $1/[i(\omega+g_i)+g_r]$, we may rewrite P, L, and R in the following fashion:

$$P=\frac{1}{g_r+i(\omega+g_i)}, \tag{8}$$

$$L=\frac{1}{g_r+i(\omega+\omega_b+g_i)},$$

$$(\omega \text{ in } g_r \text{ and } g_i \text{ replaced by } \omega+\omega_b) \tag{9}$$

$$R=\frac{1}{g_r=i(\omega-\omega_b+g_i)},$$

$$(\omega \text{ in } g_r \text{ and } g_i \text{ replaced by } \omega-\omega_b). \quad (10)$$

The values of g_r and g_i may be obtained from Table I for the low- or high-pressure limits with the following understanding: In Table I, ω is to be formally replaced by $\omega+\omega_b$ when dealing with (9) and ω is to be replaced by $\omega-\omega_b$ when dealing with (10). Thus, should $\omega\approx|\omega_b|$ we may require the high-pressure value of g_r for (8) while simultaneously requiring the low-pressure value of g_r for (9).[13]

In the low-pressure limit, $\|\mu\|$ with the representations (8), (9), and (10) becomes identical with the mobility tensor derived from the Langevin equation with magnetic field:

$$d\mathbf{v}/dt+g_r\mathbf{v}=(q/m)\mathbf{E}_0e^{i\omega t}+(q/m)\mathbf{v}\times\mathbf{H}. \quad (11)$$

Note that the results are identical with those of Allis[14] but with the important exception that g_r replaces ν.

CONCLUSION

We have pointed out how the considerable analysis based upon the cavalier usage of the Langevin equation may be rectified under certain conditions by the formal substitution of g_r for ν. It is certainly the case that for those investigations where ν played an insignificant role, the substitution of g_r for ν will not alter the results. However, in matters relating to say, attenuation of electromagnetic waves in the ionosphere, where the attention (in db) varies as ν or the determination of collision frequencies by electromagnetic interaction with plasmas, the results of analysis may be off by almost a factor of two unless proper precaution is maintained.

We have offered here some simple rules for the use of the Langevin equation to describe several ordinary plasmas. Application of these rules allow the retention of the conventional conductivity theory and the prediction and analysis of attenuation and collision frequency in plasmas in an unequivocal way.

APPENDIX. CONDUCTIVITY $(ne^2/m)(B-iD)$ FOR SEVERAL WEAKLY IONIZED GASES

For the case of weak fields and electrons undergoing only elastic collisions, the following forms of B and D are to be inserted into Eq. (3):

$$B=\frac{4\pi}{3}\int_0^\infty f_1V^4dV, \quad (\text{A-1})$$

$$D=\frac{4\pi}{3}\int_0^\infty \frac{\omega}{\rho QV}V^4f_1dV, \quad (\text{A-2})$$

where

$$f_1=\frac{\rho QV2\beta(\beta/\pi)^{\frac{3}{2}}\exp(-\beta V^2)}{\rho^2Q^2V^2+\omega^2},$$

and $\beta=m/2KT$ and ρ is the neutral particle density. It is assumed that the electrons have a Maxwellian velocity distribution.

B and D may be evaluated for each one of the gases discussed in the text and are herewith presented.

1. Air or N_2 (Q/V constant)

$$\omega B=\tfrac{2}{3}X\{1-4(\pi/2)^{\frac{1}{2}}X^{\frac{3}{2}}\{[\tfrac{1}{2}-C((2X/\pi)^{\frac{1}{2}})]\cos X$$
$$+[\tfrac{1}{2}-S((2X/\pi)^{\frac{1}{2}})]\sin X\}\}, \quad (\text{A-3})$$

$$\omega D=\tfrac{4}{3}X^2\{1-2(\pi/2)^{\frac{1}{2}}X^{\frac{1}{2}}\{[\tfrac{1}{2}-S((2X/\pi)^{\frac{1}{2}})]\cos X$$
$$-[\tfrac{1}{2}-C((2X/\pi)^{\frac{1}{2}})]\sin X\}\}, \quad (\text{A-4})$$

where

$$X=\frac{3\omega}{2\nu}, \quad \nu=4\pi\left(\frac{\beta}{\pi}\right)^{\frac{3}{2}}\int_0^\infty \rho QV^3\exp(-\beta V^2)dV=\frac{3}{2}\frac{\rho Q}{\beta V},$$

and

$$\left.\begin{aligned} C(u)&=\int_0^u \cos(\tfrac{1}{2}\pi z^2)dz \\ S(u)&=\int_0^u \sin(\tfrac{1}{2}\pi z^2)dz \end{aligned}\right\}\text{the Fresnel integrals}$$

The asymptotic forms for ωB and ωD are:

$X<1 \quad (X=\tfrac{3}{2}\omega/\nu)$

$$\omega B=\tfrac{2}{3}X[1-2(\pi)^{\frac{1}{2}}X^{\frac{3}{2}}\cos(X-\tfrac{1}{4}\pi)+4X^2], \quad (\text{A-5})$$

$$\omega D=\tfrac{4}{3}X^2[1-(\pi)^{\frac{1}{2}}X^{\frac{1}{2}}\cos(X+\tfrac{1}{4}\pi)]. \quad (\text{A-6})$$

$X>1$

$$\omega B=\frac{5}{2X}\left[1-\frac{7}{4X^2}+\cdots\right], \quad (\text{A-7})$$

$$\omega D=1-\frac{35}{4X^2}. \quad (\text{A-8})$$

2. He (Q, constant)

$$\omega B=\frac{4X^{\frac{1}{2}}}{3\sqrt{\pi}}[1-X-X^2e^XE_i(-X)], \quad (\text{A-9})$$

$$\omega D=\frac{4}{3\sqrt{\pi}}X[(\tfrac{1}{2}-X)\pi^{\frac{1}{2}}+\pi X^{\frac{3}{2}}e^X(1-\phi(\sqrt{X}))], \quad (\text{A-10})$$

where

$$X=4\omega^2/\pi\nu^2, \quad \nu=2\rho Q/(\pi\beta)^{\frac{1}{2}},$$

[13] The computations of g_i and g_r are based on the assumption of an isothermal plasma. In the case of finite electric fields and $\omega=|\omega_b|$, the electron energy may rise considerably above that of the background gas.

[14] Reference 2, p. 394, Eq. (12.6), (12.7).

and

$$-E_i(-X)=\int_X^\infty \frac{e^{-t}}{t}dt \quad \text{(the exponential integral)},$$

$$\phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt \quad \text{(the error integral)}.$$

The asymptotic forms for ωB and ωD are:

$X\ll 1 \quad (X=4\omega^2/\pi\nu^2)$

$$\omega B=\frac{4}{3\sqrt{\pi}}X^{\frac{1}{2}}(1-X), \tag{A-11}$$

$$\omega D=\tfrac{4}{3}X(\tfrac{1}{2}-X). \tag{A-12}$$

$X\gg 1$

$$\omega B=\frac{8}{3\sqrt{\pi}}\frac{1}{\sqrt{X}}\left(1-\frac{3}{X}\right), \tag{A-13}$$

$$\omega D=1-\frac{5}{2X}. \tag{A-14}$$

3. Maxwellian Gas (QV constant)

$$\omega B=X^{\frac{1}{2}}/(1+X), \tag{A-15}$$

$$\omega D=1/(1+X), \tag{A-16}$$

where

$$X=\nu^2/\omega^2, \quad \nu=\rho QV.$$

4. H_2O, NH_3 (QV^2 Constant)

$$\omega B=\frac{4}{3\sqrt{\pi}}X^{\frac{1}{2}}(1-X-X^2e^XE_i(-X)), \tag{A-17}$$

$$\omega D=1-\tfrac{2}{3}X+\tfrac{4}{3}X^2-\tfrac{4}{3}\pi^{\frac{1}{2}}X^{\frac{5}{2}}e^X(1-\phi(\sqrt{X})), \tag{A-18}$$

where

$$X=\pi\nu^2/4\omega^2, \quad \nu=2QV^2\rho(\beta/\pi)^{\frac{1}{2}}.$$

The asymptotic forms for ωB and ωD are:

$X\ll 1 \quad (X=\pi\nu^2/4\omega^2)$

$$\omega B=\tfrac{4}{3}(X/\pi)^{\frac{1}{2}}(1-X),$$

$$\omega D=1-\tfrac{2}{3}X.$$

$X\gg 1 \quad (X=\pi\nu^2/4\omega^2)$

$$\omega B=\frac{8}{3\sqrt{\pi}}\frac{1}{\sqrt{X}}\left(1-\frac{3}{X}\right),$$

$$\omega D=\frac{5}{2X}\left(1-\frac{7}{2X}\right).$$

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