which may arise in connection with the isomeric shift, but confined ourselves only to those which seemed the most specific, especially at the present stage when no experimental data are yet available. See however the note added in proof.‡ We hope that the present study will encourage such experiments.

‡ *Note added in proof.*—Meanwhile new progress along these lines has been reported by the MIT group (Melissinos and Davis, preprint). The authors succeeded in measuring the Hg^{197m}-Hg^{197} shift on the 2537 A line. The value of $\sim 2.10^{-2}$ cm^{-1} found for the shift is probably mainly due to the global quadrupole and compressional effects mentioned above. However a quantitative interpretation of this result will be possible only after the determination of the intrinsic quadrupole moment of the $3p_{\frac{3}{2}}$ state of Hg^{197} (the quadrupole moment of Hg^{197m} has been measured in the work quoted above).

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Nuclear Parameters in the Scattering of Nucleons by Carbon

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Scattering of protons and neutrons by carbon at small angles is discussed. It is shown that it is not possible to obtain a coherent understanding of the scattering at 310 Mev without including an increase of the radius due to nuclear forces. The increase required is in rough agreement with that calculated from nucleon-nucleon scattering.

At 135 Mev, where more complete data are available, it is shown that this increase is not sufficient to obtain a coherent understanding. Either approximations break down, or there is a difference in the scattering of neutrons and protons in addition to the Coulomb forces. Further, small-angle polarization measurements are badly needed at both energies.

INTRODUCTION

NUCLEON–NUCLEUS scattering at high energies has conventionally been analyzed by means of the optical model. At first, this was considered to be only a phenomenological fit, but later many attempts have been made to justify the optical model theoretically, and to derive the parameters on the basis of our knowledge of nucleon-nucleon forces. For example, the treatment of Riesenfeld and Watson[1] shows how the multiple-scattering theory leads directly to the definition of an optical potential, with errors of the order of $1/A$, where A is the atomic number of the nucleus. Unfortunately, when it comes to putting numbers into their formulas it has been necessary to make the approximation that nuclear forces have zero range. This has been recognized, and it has been the practice to allow for this in a highly qualitative way by allowing the nuclear "radius" to be an adjustable constant in any formula.

Our knowledge of nuclear forces has, until recently, been rudimentary. At 313 Mev the proton-proton scattering data are rather complete and allow a phase shift analysis with only a five-fold ambiguity. There has been less success in correlating the neutron-proton scattering data, but enough for Bethe[2] to derive the optical model parameters with little ambiguity.

There has also been great success in correlating all the nucleon-nucleon scattering up to 313 Mev with a many parameter potential.[3,4] If this potential is assumed, one may then discuss optical-model parameters at other energies—for example at 135 Mev where more complete data are available for nucleon-nucleus scattering. This has been begun by McManus and Thaler,[5] who have calculated directly some nucleon-nucleus scattering cross sections and polarizations from this potential, with fair agreement with experiment. This paper discusses some finer details, which are possible if we simultaneously analyze all the neutron and proton data available at one energy.

In the literature there exist several analyses of nucleon-nucleus scattering data that are mutually

[1] W. B. Riesenfeld and K. M. Watson, Phys. Rev. **102**, 1157 (1956).

[2] H. A. Bethe, Ann. Phys. N. Y. **3**, 190 (1958).

[3] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957).

[4] P. S. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958).

[5] H. McManus and R. M. Thaler, Phys. Rev. **110**, 590 (1958); McManus, Thaler, and Kerman, Ann. Phys. (to be published)

contradictory. In particular, we shall discuss those of Bethe[2] and of Brown[6] *et al.* Recently, Bethe's analysis has been extended and improved by Cromer,[7] who covers much of the material of this paper. It will be shown that the contradiction arises from the neglect of the nuclear-force range by both authors. It will be clear that the analysis is not yet capable of the precision assumed in both papers.

In the discussions that follow we shall be considering only data taken at small angles—within the first diffraction maximum. We shall also limit ourselves to a consideration of the scattering of nucleons by carbon, as was considered by Brown and Bethe, but at 135 Mev as well as 310 Mev. This limitation is in order that we may make use of the justification of the optical model which is valid at small angles, for light nuclei, and at high energies.[8] We may then assume that multiple scattering is small and is not dominant in the scattering. The justification for the optical model in this limit is different from that in the limit of lower energies, heavy nuclei, and any angles where multiple scattering plays the dominant role. For this reason, the analysis here, and also those of Bethe and Brown, have no direct connection with those of Fernbach[9] and Glassgold,[10] which depend on many more parameters. All numbers in this paper will be in the laboratory frame of reference; when adding together nucleon-nucleon scattering to form nucleon-nucleus scattering, this avoids the confusion of changing from one center-of-mass system to another.

Application of the ideas of McManus, Thaler, and Kerman to proton-helium scattering is considered in another paper.[11]

NOTATION

For spin 0 nuclei, it can be shown[12] that the most general scattering amplitude is of the form

$$F = g_n(\theta) + \boldsymbol{\sigma}\cdot\mathbf{n}h_n(\theta), \tag{1}$$

where g_n and h_n are complex. For small angles we may write $h_n(\theta) = i\theta S = i\theta' S'$, where θ' is in degrees, and we may assume that the real and imaginary parts (g_{nR} and g_{nI}, S_R, and S_I) all have their zero-degree values. Thus we have four constants to be determined. In order to use data at small but nonzero angles, it is necessary to use a form factor. If we expand this form factor as a function of q^2, the square of the momentum transfer, then only the second term, giving the mean square radius, $\langle r^2\rangle$, is dominant at small angles.

The scattering is frequently described by a nuclear potential of the form

$$V(r) + \boldsymbol{\sigma}\cdot\mathbf{L}\frac{c}{r}\frac{dV(r)}{dr}. \tag{2}$$

The values of the potential and of the constant c may then be derived from g and h with a knowledge of the nuclear size. This assumes that all the form factors are the same. Bethe[2] has proposed an intermediate step. He derives for this potential the "Born amplitudes" G and H which are the scattering amplitudes in the Born approximation.

In part, the ideas presented in this paper overlap those of Kohler[13] who discusses 150-Mev scattering. Kohler also restricts himself to small angles and makes appropriate approximations, but he analyses proton data alone and neglects neutron data. We prefer, moreover, the elegant computational method introduced by Bethe, which allows us for the first time to proceed in the forward direction from the data to a theoretical deduction, rather than in the reverse direction. In addition, a comparison between neutron and proton scattering is facilitated by Bethe's method, because we can compare directly the scattering amplitudes for neutrons and protons without going to the Born amplitudes or to the nuclear potential, which introduce more uncertainty. We will therefore use Bethe's notation and make frequent references to his paper.

310–350 Mev

The following experimental data may be considered. (i) The total cross section for fast neutrons. This has been measured at many energies and is fairly constant with energy; we quote, for example, $\sigma_T = 28.53 \pm 0.16$ fermi^2 [1 fermi(f) $\equiv 10^{-13}$ cm] at $E = 350$ Mev.[14] (ii) The differential cross section for scattering by 350-Mev neutrons between 1° and 12°.[15] (iii) The differential cross section and polarization for scattering of 313-Mev protons between $3\frac{1}{2}$° and 12°.[16] (iv) Some limited triple scattering measurements of R at 313 Mev.[16] (v) The electron scattering from the carbon nucleus at 187 Mev.[17] [The electron scattering data are our best available data on the size of the carbon nucleus which enters critically into the discussion.] The number of interest is the square of the form factor, $F^2(q)$. It is necessary to know F^2 in the region 0.5 to 1.0, whereas it has been measured in the region 0.001 to 0.5.[17] The extrapolation cannot be made in a model-independent way. The

[6] Brown, Ashmore, and Nordhagen, Proc. Phys. Soc. (London) **71**, 565 (1958).
[7] A. H. Cromer (to be published).
[8] E.g., R. J. Glauber (to be published).
[9] S. Fernbach, Revs. Modern Phys. **30**, 414 (1959); F. Bjorklund and S. Fernbach, University of California Radiation Laboratory Report UCRL-5028 (unpublished).
[10] A. E. Glassgold, Revs. Modern Phys. **30**, 419 (1958).
[11] Cormack, Palmieri, Ramsey, and Wilson (to be published).
[12] L. Wolfenstein, Ann. Rev. Nuclear Sci. **6**, 43 (1956).
[13] H. S. Kohler, Nuclear Phys. **6**, 161 (1958).
[14] Ashmore, Jarvis, Mather, and Sen, Proc. Phys. Soc. (London) **70**, 735 (1957).
[15] Ashmore, Mather, and Sen, Proc. Phys. Soc. (London) **71** 552 (1958).
[16] Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, Phys., Rev. **102**, 1659 (1956).
[17] J. H. Fregeau, Phys. Rev. **104**, 225 (1956). Ehrenberg, Hofstadter, Meyer-Berkhout, Ravenhall, and Sobottka, Phys. Rev. **113**, 666 (1959).

best fit for the nuclear charge density, with even some theoretical justification, is to set $\rho=\rho_0[1+\frac{4}{3}(r^2/a^2)]\times\exp(-r^2/a^2)$ with $a=1.635$ fermi, from which we derive the root mean square radius $\langle r^2\rangle^{\frac{1}{2}}=2.40$ f. The value of $\langle r^2\rangle^{\frac{1}{2}}$ should determine $F^2(q)$ in the region of small q where F^2 is close to unity. Other good fits do not reduce $\langle r^2\rangle^{\frac{1}{2}}$ more than 2%. The electron-scattering data give a measure of the charge distribution only. For the distribution of proton centers we have $\langle r^2\rangle=(2.40)^2-(0.8)^2$ fermi^2, where 0.8 fermi is the rms charge distribution for the proton. Thus we find $\langle r^2\rangle^{\frac{1}{2}}=2.26\pm0.09$ fermi, where the error comes from taking extreme fits to the carbon and proton measurements. We also assume that the neutron distribution in the carbon nucleus is the same as the proton distribution. Bethe[2] has used a Gaussian fit to the electron-scattering data with $\langle r^2\rangle^{\frac{1}{2}}=2.40$ fermi. On the other hand, if we calculate the Coulomb scattering, the charge distribution of both the incident proton and the protons inside the nucleus must be considered to give $(\langle r^2\rangle^{\frac{1}{2}})_{\mathrm{eff}}=2.53$ fermi $[=(2.40^2+0.80^2)^{\frac{1}{2}}]$.

The first step in the analysis of Brown is to use the optical theorem,

$$\mathrm{Im}f(0)=(k/4\pi)\sigma_{\mathrm{tot}}=g_{nI}. \qquad (3)$$

This determines g_{nI} with $1\frac{1}{2}$% accuracy at essentially all energies in the range (=10.1 fermi at 350 Mev). We should perhaps point out here that this number differs from that quoted by Brown, the difference arising from our use of the lab system exclusively whereas Brown uses the c.m. system for his calculated cross sections.[18] We should also remember that this is the best determined of all the numbers we shall discuss. The measured differential scattering cross section of 350-Mev neutrons extrapolates to about 1 barn/sterad (=100 fermi^2/sterad) at 0°, though with an error of perhaps 15%. Thus g_{nR} (Bethe's notation) is small and not significantly different from zero. That g_{nR} is small was also deduced by Bethe from the absence of a large interference between the nuclear and the (predominantly real) Coulomb scattering amplitudes. This fact may be used to simplify the subsequent analysis appreciably. Now Brown uses the nuclear size to relate $\sigma_{\mathrm{tot}}=(4\pi/k)g_{nI}$ with the observed neutron-neutron (=proton-proton) and neutron-proton total cross sections. Thus we have

$$\sigma_{\mathrm{tot}}=4\pi\int bdb(1-e^{-K(b)})\cos[kb\delta_1(b)], \qquad (4)$$

where $K(b)$ is the absorption at the impact parameter b due to the nucleons in the nucleus. The free n-p and p-p cross sections are used corresponding to a neglect or cancellation of the effect of the internucleon correlations, and the effect of the Pauli principle.[19] The absorption, however, depends upon the nuclear radius and shape assumed. Brown, using the correct shape for carbon, but a value for $\langle r^2\rangle^{\frac{1}{2}}(=2.40$ f) corresponding to the charge distribution and not the proton distribution, finds that σ_{tot} so derived is too small; a use of the correct $\langle r^2\rangle$ would accentuate the discrepancy. However, Bethe[2] [in his Table IX and his Eq. (8.22)], calculates σ_{tot} directly from nucleon-nucleon scattering and finds 25.9 fermi^2. Herein, then, lies the first disagreement. Bethe finds that the g_{nI} he deduces from the proton data does depend upon the radius; he derives 8.6 fermi at 313 Mev as compared with the value of 9.45 fermi which may be derived from the neutron total cross section at 313 Mev. To obtain exact agreement he would need a radius further increased by 12% [his Eq. (6.6)]. The nuclear absorption corrections cannot be appreciably different for neutrons and protons at this energy, so that agreement can be achieved only with a definite radius. Likewise, Bethe's later comment [his Eq. (7.6)], that the Born amplitude is insensitive to radius, cannot be correct because he neglects the accurate neutron data which determine g_{nI}.

Brown assumes that his failure to predict the total cross section from nucleon-nucleon scattering is due to neglect of the spin dependence implicit in replacing $\cos[bk\delta_1(b)]$ by unity. For δ_1 small, it may clearly be neglected, for $\cos\theta=1$ to order δ_1^2. This was assumed explicity by Bethe, and indeed by most earlier analyses. Brown's procedure is to attribute the whole discrepancy to the second-order effects of the spin-orbit potential. Brown then derives the value for the spin-orbit potential necessary to explain the discrepancy. In terms of the potential discussed above [Eq. (2)], Brown thus finds the value $c=0.24$ fermi^2.

At this stage we should go back and examine Brown's assumption in detail by comparison with other, similar effects, and show that these corrections, neglected by Brown, are as important. The most definite example of a failure of Brown's assumption is the measurement by Cronin *et al.*[20] of the attenuation of π-mesons by carbon and other elements. Here we expect $\sigma_{\mathrm{carbon}}=(6\sigma_{\pi-p}+6\sigma_{\pi-n})X$, where X is the readily calculable attenuation factor; there are no experimental troubles about distinguishing the inelastic scattering. Cronin finds that agreement is obtained only by taking a larger radius for the nucleus than the electron-scattering radius, or by taking a larger surface thickness, corresponding perhaps to a range of force. It is clear that for effects of this sort the effective radius that must be used to calculate X is greater than that derived from electron scattering. Elton[21] has discussed the data by writing $R=\langle r^2\rangle^{\frac{1}{2}}+y$, where y is an adjustable constant found to be 0.3 fermi. As discussed later, it appears that the form $R^2=\langle r^2\rangle+a^2$ is to be preferred. If we derive the

[18] Brown inadvertently compares calculated c.m. cross sections with measured lab cross sections; this makes no difference to the point at issue.

[19] R. J. Glauber, Physica **22**, 1185 (1956).
[20] Cronin, Cool, and Abashian, Phys. Rev. **107**, 1121 (1956).
[21] L. R. B. Elton, Revs. Modern Phys. **30**, 557 (1958).

value of a^2 from fits to heavier elements in a similar way to Elton, and as is necessary to obtain an answer, we find $a^2 \simeq 2$ fermi^2, giving an increase of radius for carbon of 0.5 fermi. We should note here that this is not a shape-independent determination.

The measured absorption cross sections for high-energy protons and neutrons exclude some nearly-elastic scattering, unlike the more straightforward π-meson data. The early analysis of Williams[22] did not take this into account; he now suggests[23] that a correction for the nearly-elastic scattering will bring the pion absorption and nucleon absorption experiments into agreement. Elton has not made the correction considered here. At lower energies, the data of Voss and Wilson[24] do not experience this difficulty of excluding the nearly-elastic scattering but also need a larger radius as discussed further below.

In a similar way, we should expect that a larger radius than the charge distribution must be used to evaluate g_{nI} from σ_{n-p} and σ_{p-p} to take account of the effects of the finite range of forces. By an overrigid use of a zero-range approximation, Brown neglected this effect, which can—within our ignorance of exactly what to do—be the whole of the discrepancy, though it is also possible that some effect of the spin-orbit force still remains. These uncertainties discouraged others—for example, Voss and Wilson—from analyzing total cross sections in this detail.

We can then propose the question: is there support for a large c from other data? Bethe thinks not, and deduces $c=0.135$ fermi^2—about half Brown's value, contributing only $\frac{1}{4}$ of Brown's amount to the total cross section. On the other hand, Levintov,[25] Heckrotte[26] and Harris[27] have analyzed the polarization of protons scattered by carbon and find $c=0.28$ fermi^2. They all attempted to fit the data in the region of the maximum of the polarization at 13°, and found that in order to do so the spin-orbit scattering amplitude must be large and complex. This angle is still sufficiently small that the polarization is not affected by the inclusion of small amounts of inelastically scattered protons.[28] Thus, at first sight, there is an impressive majority view to support Brown. Bethe fitted data up to 7° only; in this region the effects of the real and imaginary parts of the spin-orbit potential assumed by Levintov cancel to produce the same effect as that of a purely real spin-orbit potential with $c=0.135$ fermi^2. Bethe's view of this is clearly stated: "This is obviously nonsense and would amount to near cancellation of the two terms over most of the measured range."[29] Be this as it may, Bethe's value of c is in agreement with that derived from nucleon-nucleon scattering and the larger value is not.

Bethe discussed two ways which exist of deciding the question of the value of c with small-angle data alone. Firstly, a measurement of R by triple scattering yields[12]

$$1-R=2|h|^2/(|g|^2+|h|^2), \tag{5}$$

which is twice the contribution of the spin-dependent scattering to the total scattering, with no cancellation involved. At 313 Mev and 10°, $(1-R)=0.25$ with, however, a fairly large error. Since $|h|^2/|g|^2$ varies as θ^2, at 7° the spin-dependent scattering is only 6% of the total, in agreement with Bethe and in disagreement with Brown by a factor of 4. Precise measurements of polarization in the Coulomb interference region could also settle the question; for in this region S_I interferes with the predominant real part of the Coulomb potential. No good data exist here, however.

The error in Levintov's treatment presumably lies in considering 15° as if it were a small angle.

The Nuclear Radius

Bethe and Brown have used a nuclear radius derived from electron-scattering data to derive the four nuclear amplitudes. We saw that this led to disagreements. In practice it is not possible to use any data at nonzero angles without considering the variation of these amplitudes with angle.

If we use the impulse approximation and neglect multiple scattering we may sum the nucleon-nucleon scattering amplitudes, following McManus,[5] to yield what he calls the Born amplitude

$$f(q)=\int \bar{M}(q)e^{i\mathbf{q}\cdot\mathbf{r}}\rho(r)d\tau, \tag{6}$$

which decomposes into a product of the nucleon-nucleon amplitude and the form factor for electron scattering from the nucleus (assuming that the neutron density and proton density are equal).

$$f(q)=A\bar{M}(q)F(q). \tag{7}$$

Both of these terms may be expanded in a power series in q^2 where the coefficient of q^2 in the expansion of $F(q)$ is well known as one sixth of the mean square radius. Defining the square of the effective radius as one sixth of the coefficient of q^2 in the expansion of f, we get

$$(R^2)_{\text{eff}}=\langle r^2\rangle+a^2, \tag{8}$$

where a^2 is a constant which has different values for each of the four scattering amplitudes.

This radius correction, it should be realized, is only one of the many corrections to the optical model, but it is a point of this paper to show that the effects are

[22] R. W. Williams, Phys. Rev. **98**, 1387 (1955).
[23] R. W. Williams (private communication).
[24] R. G. P. Voss and R. Wilson, Proc. Roy. Soc. (London) **A236**, 52 (1956).
[25] I. I. Levintov, Doklady Akad. Nauk U.S.S.R. **107**, 240 (1956) [translation: Soviet Phys. Doklady **1**, 175 (1956)].
[26] W. Heckrotte, Phys. Rev. **101**, 1406 (1956).
[27] Harris and R. Jastrow (private communication).
[28] Brown's contrary statement comes from a misreading of the data (private communication).
[29] See reference 2, p. 222.

large, perhaps larger than any other correction, and that to obtain agreement they must be considered.

Of course, this has long been realized qualitatively, and the difference has been ascribed to the range of nuclear forces, which is another way of stating the same physical fact.[30] For if nuclear forces had a zero range, nucleon-nucleon scattering would have a cross section constant with angle. One feature of the results of McManus *et al.* had already been noted qualitatively: that the radius for the real part of the potential should be larger than the radius for the imaginary part. This had been suggested on both general theoretical grounds[31] and experimental grounds.[32]

It is at least qualitatively reasonable that this larger radius be used to calculate absorption effects even though absorption effects are a manifestation of the (neglected) multiple scattering. A nuclear scattering amplitude smaller at wide angles than assumed corresponds to less multiple scattering and less absorption, which is at least in the same direction as the effect of taking a larger radius. It is even more reasonable that the larger radius be used for studying the form factor derived from small-angle nucleon scattering. Wilson[32] has shown that the effects of multiple scattering and nuclear opacity do not much influence the form factor appropriate for small-angle nucleon scattering, which remains even at 135 Mev only 2% different from that derived in Born approximation. This is confirmed by Riese[33] and by Brown, from which we see that the mean square radius can be well determined from neutron scattering at small angles. Even going to the extreme of a black disk changes the derived radius by only 12%. This extreme is, however, incorrect.

In spite of the residual approximations, it is clear that it is a legitimate and necessary experimental study to determine R_{eff} to compare with theory. The correct theoretical radius is closer to that of McManus than to that of the zero-range approximation.

Since at 310 Mev the scattering amplitude is primarily imaginary, we may take the value of a^2 for the imaginary amplitude with perhaps a 10% admixture of the value for a real amplitude.[5] This yields $R=2.7$ fermi. We can invert Bethe's argument [his Eq. (6.6)] and find the value of R_{eff} necessary to give agreement between the values of g_{nI} deduced from neutron and from proton scattering; we find $R=2.79$ fermi. That we use the larger value for the Coulomb scattering matters little; where the form factor makes an appreciable difference, the Coulomb scattering is small.

Let us turn to the 350-Mev neutron scattering, hardly used so far in our discussion. The ratio $I(\theta)/I(0)$ has been measured by Ashmore[15]; we obtain the form factor directly and can determine R. We find $R_{\text{eff}}=2.5\pm0.2$ fermi, in agreement, so far as it goes, with the value suggested above, though from this alone we could not rule out Brown's procedure of putting $a^2=0$. This fit is a little different from that of Ashmore,[15] who used an opaque nucleus model.

In conclusion then, at 310 Mev we can determine g_{nI} and g_{nR} if the Coulomb interference disagreement is resolved by precise computation. The spin-dependent amplitudes S_R and S_I are not independently determined; for if S_I is assumed large, this determination also affects everything else. R_{eff} is determined mainly by indirect evidence and is dominated only by g_{nI}. R_{eff} is larger than the charge radius, in agreement with theory. The parameters g_{nI} and g_{nR} can be easily made to agree with nucleon-nucleon parameters, as Bethe did. The fits of Brown *et al.*[6] are probably wrong. The moral of the story is unfortunately clear: the terms—principally the nuclear radius correction—that are normally neglected in the derivation of the optical model are very important if better than 30% agreement is to be obtained.

135-Mev Data

At 135 Mev the analysis becomes more difficult in some ways and easier in others. The WKB approximation, used to make analyses analytically tractable, is not so good as at 310 Mev; this is particularly unfortunate, for the Coulomb interference dominates the proton scattering so that exact calculations of the Coulomb effects are vital. However, better data are available: (i) Neutron total cross section may be taken as 36 ± 1 fermi^2, where a large part of the uncertainty comes from the uncertainty in the measurement of the energy.[34] This yields $g_{nI}=7.64\pm0.22$ fermi. (ii) Neutron differential cross sections are available,[35] yielding $g_{nR}^2+g_{nI}^2=118\pm8$ fermi^2, whence $g_{nR}=7.7\pm0.6$ fermi. These data are accurate enough, also, to give a value for $R_{\text{eff}}=3.0\pm0.1$ fermi. (iii) Proton scattering and polarization data are available at 135 Mev[36] and 155 Mev.[37] (iv) The neutron absorption cross section[38] is 22 ± 1 fermi^2 which is less dependent on special assumptions than are similar data at higher energies. (v) Neutron differential cross sections and polarizations are available at 155 Mev[39] in good agreement with the 135-Mev data.

We again start with the neutron data. These already define g_{nI}, g_{nR}, R_{eff}; and an average of S_R, and S_I which we may call, following Bethe, S_{eff}. We note that the cross sections are three times the proton cross sections at angles of 5° to 7° so that the Coulomb interference certainly has a dominant role.

[30] S. D. Drell, Phys. Rev. **100**, 97 (1955).
[31] R. J. Glauber (private communication).
[32] Richard Wilson, Phil Mag. **47**, 1013 (1956).
[33] J. Riese, thesis, Massachusetts Institute of Technology, 1958 (unpublished).
[34] A. E. Taylor and E. Wood, Phil. Mag. **44**, 95 (1953).
[35] Van Zyl, Voss, and Wilson, Phil. Mag. **47**, 1003 (1956).
[36] J. Dickson and D. C. Salter (private communication).
[37] Alphonce, Johansson, and Tibell, Nuclear Phys. **4**, 672 (1957).
[38] R. G. P. Voss and Richard Wilson, Proc. Roy. Soc. (London) **A236**, 46 (1958).
[39] R. S. Harding, Phys. Rev. **111**, 1164 (1958).

When g_R and g_I are approximately equal it is difficult to follow Bethe's treatment to obtain g_R and g_I without assumptions about R_{eff}. We have made many such calculations and find that the value for g_R becomes much smaller than the neutron value. This has been found by other authors.[39] The problem is to bridge the factor of 3 between the neutron and proton cross sections. That this is a problem, we demonstrate explicitly by stretching the experimental data to the limits of their errors. Thus, the proton data will be normalized upwards by 20% corresponding to an error in beam measurement. The value of g_{nI} is so accurate that it can be regarded as constant, but g_{nR} will be reduced by twice its error. We will make the same assumption that Bethe did: that c is small and the spin-dependent amplitudes are correspondingly so; any other assumption intensifies the discrepancy. The form factor will be taken from a smoothed curve fitting the neutron scattering. This is equivalent to a value of $R_{eff}=3.0$ fermi.[32] The use of the experimental neutron form factor directly avoids, somewhat, the error of treating the form factor as separable, as is strictly true only in Born approximations.

Table I shows the calculation from the proton scattering, following Table VI of Bethe. In distinction to Bethe's analysis, however, we try to fit g_{nI} and g_{nR} with values "reasonable" from neutron data. In Table I, angles up to 15° are included, though only angles up to 10° are taken seriously. Line 12 is a residual which should be constant and equal to g_{nR}^2. A fit to $g_{nR}=6.5$ is not unreasonable, though too small to be a really good fit to the neutron data. If the radius is taken to be *larger*, residuals in line 12 will increase at large angles and g_{nR}^2 is seen to be larger, as desired. The value obtained from small angles could also be increased by the consequent larger interference. It follows, then, that the radius must be at least as large as assumed.

In fitting the polarization data, we have assumed that the spin-dependent amplitude h is almost a pure imaginary (arising from a purely real potential at 313 Mev), giving $S_I=0$. While there are no triple scattering data to decide the question, the small-angle data are seen in Table I to be consistent with this. The data from protons on oxygen[37] are similar.

The data from protons on helium[11] are confusing. The polarization at 2° is negative, suggesting an imaginary spin-orbit term, but the data at 3° to 4° do not agree with a large imaginary term; possibly the discrepancy is experimental. The data of Feld and Maglić[40] apply, at the moment, only to heavy elements.

Inclusion of a large S_I would also necessitate a larger contribution to $|h|^2$ (line 5), rendering the discrepancy in g_{nR} worse.

We have not yet discussed the neutron absorption cross-section data. In contradistinction to the small-angle scattering, this is dependent upon more details

[40] B. T. Feld and B. C. Maglić, Phys. Rev. Letters 1, 375 (1958).

TABLE I. Calculations from proton scattering data.[a]

1. Angle θ (degrees)	2	3	3½	4	5	6	7	8	9	10	11	12	15
2. Observed $d\sigma/d\omega$ (fermi²/sterad) (×1.20)	...	...	124	67	44.7	45.5	45.4	45.0	41.5	39.5	37.4	30.6	16.7
3. Form factor $F^2(q)$	...	...	0.92	0.90	0.85	0.80	0.73	0.66	0.59	0.53	0.46	0.40	0.21
4. $(d\sigma/d\omega)_{point}$=line 2/line 3	...	...	134	74.5	52.3	56.9	62.1	68.1	69.6	74.5	81	76.5	77.5
5. Spin-dependent contribution $\|h\|^2$	...	...	1	1.1	1.3	1.65	2.1	2.5	3.0	3.6	4.2	4.8	8.0
6. Pure Coulomb scattering $50400/\theta^4$	3150	592	338	197	80	37	21	12.3	7.7	5.0	3.5	2.1	1.0
7. Nuclear and interference	...	...	−205	−124	−29.0	18.2	39.0	53.3	58.9	65.9	73.3	69.6	68.5
8. Interference with g_{nI} (=7.64 f)	−270	−94	−59	−35	−20.1	−10.8	−6.0	−3.5	−1.8	−0.7	0	+0.6	+1.0
9. Remainder line 7—line 8	...	...	−146	−88	−8.9	29.0	45.0	56.8	60.7	66.6	73.3	69.0	67.5
10. Interference with g_{nR} (=6.5 f)	−730	−324	−240	−183	−117	−81	−59.5	−45.6	−36	−29.2	−24.0	−20.1	−12.9
11. Nuclear scattering alone	...	...	94	96	108.1	110	104.5	102.4	96.7	95.8	97.3	89.2	90.4
12. Subtract $g_{nI}^2=58.5$ f²	...	...	35.5	37	49.6	51.5	46.0	43.9	38.2	37.3	38.8	30.7	31.9
13. $g_{nR}=(\text{line }12)^{1/2}$	...	...	5.95	6.1	7.0	7	6	6.6	6	6.1	6.2	5.5	5.6
14. Calculated point from $g_{nR}=6.5$ f	2250	274	141	79.5	44.9	47.6	58.3	66.4	73.6	79.4	84.4	88	71
15. Observed polarization %	(−4.4±6.9)	−0.6±5	6±2	25	42	45	45	47	46	51	50	55	63
16. Polarized cross section (line 14×line 15) fermi²/ster	−100	−1.6±15	8.4±3	19.9	18.9	21.4	26.3	31.2	33.8	40.5	42.2	47.5	57.5
17. Magnetic moment term: $(2.78/\theta)g_{nI}$ $-0.18g_{nR}\ln(11.7/\theta^2)$	7.8	5.6	5.2	4.8	3.9	3.3	2.9	2.5	2.4	2.2	2.0	1.8	1.4
18. Line 16—line 17	−108	−7.2±15	3.3±3	15.1	15.0	18.1	23.4	28.7	31.4	38.3	40.2	45.7	56.1
19. $g_{nI}\theta-(40.4/\theta)\ln(11.7/\theta)$	−20.1	4.5	13.3	21.3	31.6	41.5	50.7	59.3	67.8	75.9	84	92.3	116
20. Line 18/line 19=$2S_{eff}'$	...	...	0.25±0.2	0.71	0.48	0.44	0.46	0.48	0.46	0.50	0.48	0.50	0.48

[a] Note that the terms in parentheses are from Upsala data at 155 Mev.

of the nuclear size than the mean square radius; it also depends on the imaginary part of the "Born amplitude," which is not well determined. There is, however, slight evidence[24] that the radius is less than that given by the scattering. It is in fact about 0.1 fermi less than given by Voss when correction is made for refraction of the nucleons by the attractive nuclear potential. We quote a value of $R_1=2.7\pm0.2$ fermi from the shape given by the charge distribution.

We may also derive the polarized cross section directly from the neutron data of Harding,[39] while making allowance for the change in energy. A direct comparison is made in Table II between the polarized cross section (corrected for form factor and magnetic moment interaction) for protons (line 18, Table I) and that derived directly from neutron data, also with a small allowance for the magnetic moment interaction. Since Table I lists data for 135 Mev, the values have been multiplied by 1.2 in line 5 of Table II. These are, within statistics, the values from the proton data of Alphonce[37] at 155 Mev. We see that the neutron data give consistently larger numbers. This is partly because of the discrepancy in g_{nR} already discussed, but partly because of the polarization itself being about 10% greater for neutrons. A part of this difference can be ascribed to the measurement of the polarization of the neutron beam used. Harding measured the asymmetry in the two successive reactions $C^{12}(p,n)$ and $C^{12}(n,p)$, and in spite of an energy change of 30% used the equation $e=P^2$. Voss and Wilson[41] suggest that it might be more accurate to set P proportional to E; in which case the neutron beam had a higher polarization than supposed, and the polarizations of Harding should be divided by $(220/155)^{\frac{1}{2}}$ or 1.2; in column 3 of Table II this correction is made and the lower value (100 fermi^2/sterad) used for $d\sigma/d\Omega$. The agreement is more satisfactory.

Averaging over the slight discrepancy between neutron and proton data, we derive a reasonable set of parameters with an estimate of the errors:

$g_{nI}=7.6\pm0.2$ fermi;
$g_{nR}=6.5\pm1.0$ fermi;
$S_R'=0.23\pm0.02$ fermi/degree; $S_R=12$ fermi/rad;
$S_I'=0$;
$R_{\text{eff}}=3.0\pm0.1$ fermi (averaged in some way over real and imaginary parts);
$a_{\text{eff}}^2=3.9\pm0.6$ fermi^2;
$R_{\text{imag}}\simeq2.7\pm0.2$ fermi;
$a_I^2=2.2\pm1$ fermi^2.

The transition from the scattering amplitudes to the Born amplitudes depends critically upon the radius and indeed on more details than only the mean square radius. The transition is therefore not so accurate as the derivation of the real nuclear scattering amplitudes and is particularly bad for the derivation of ImG. With this further uncertainty in mind, we derive, following Bethe, the components for the Born amplitude $G+\boldsymbol{\sigma}\cdot\mathbf{n}H$ which we compare with the predictions of McManus and Thaler.[5] For convenience we use a Woods-Saxon potential with the root mean square radius 3 fermi.

	Experiment	Theory
ImG=	8.6 ± 1 fermi	10 fermi
ReG=	13 ± 3 fermi	11 fermi
ReH=	-5.5 fermi/rad	-5 fermi/rad
ImH=	19.5 ± 4 fermi/rad	21 fermi/rad
R_{eff}=	3.0 ± 0.1 fermi	3.0 (average of R_{Real} and R_{imag})
R_{imag}=	2.7 ± 0.2 fermi	2.7 fermi

The existence (or otherwise) of the term ReH cannot be derived from these data.

Conclusions from 135-Mev Data

We may ask if there is any way around the disagreement between the neutron and proton data in the last section. We might stretch the errors on R_{eff} to 3.3 fermi; this would raise the values in Table I line 4 for the larger angles. The Coulomb interference would also be larger, both involving a larger g_{nR}. The discrepancy is not yet removed. Alternatively we may say that the constants differ for neutron and proton bombardment, due perhaps to the exclusion principle working differently in the two cases. Until such effects are calculated, we can only regard such a procedure as an exercise in phenomenology. g_{nI} as derived from the proton data would remain essentially the same as that derived from the neutron data, but there is a difference in g_{nR}. We easily derive:

$$g_{nR}=7.7\pm0.3 \text{ fermi for neutrons,}$$
$$g_{nR}=5.9\pm0.5 \text{ fermi for protons.}$$

It would be wrong, however, to attach too much detailed significance to these numbers at this stage for the following reasons: (i) We have assumed that the spin-orbit potential is small, which though in agreement with Thaler's predictions may not be true. (ii) We are not sure how to derive the Born amplitudes from g_{nI} and g_{nR} once the zero-range approximation is dropped. (iii) The interference calculations—following Bethe—assume that the Coulomb itself is not affected by the nuclear potential which becomes increasingly inaccurate as the energy is reduced. (iv) The WKB approximation

Table II. Neutron polarization measurements.

θ_{lab} 155 Mev	$P(d\sigma/d\Omega)$ (% fermi^2/sterad)	$P(d\sigma/d\Omega)$ (corrected)	$P(d\sigma/d\Omega)$ protons line 18 Table I	$P(d\sigma/d\Omega)$ corrected to 155 Mev
$3\frac{1}{2}°$	33	26 ± 8	3	4
$5°$	35	27 ± 6	15	18
$7\frac{1}{2}°$	41	32 ± 6	26	31
$10°$	65	50 ± 8	38	42
$12\frac{1}{2}°$	71	54 ± 8	48	57

[41] R. G. P. Voss and Richard Wilson, Phil. Mag. 1, 175 (1956).

is not too good for it assumes, *inter alia*, that $V/E \ll 1$, whereas we here derive $V/E=0.1$. (v) We have taken the same form factor for real and imaginary parts of the potential.

The value of the radius in nucleon-nucleus scattering derived by McManus and Thaler[5] is also in agreement with proton-helium scattering at energies from 95 to 300 Mev.[11] It is interesting to note that the radius increase from 310 Mev to 135 Mev is quite definite; since $\beta(=v/c)$ for the incoming proton only varies from 0.69 to 0.49, the variation cannot be ascribed to the variation in β. The only change in this energy region is from a nuclear scattering with a primarily imaginary amplitude at 310 Mev to one with equal real and imaginary parts at 135 Mev. This is the explanation given here. For heavy elements, however, agreement is not so good. The data of Ashmore *et al.*[15] show that at 350 Mev the radius is essentially that of the charge distribution. This agrees with the prediction of McManus and Thaler, for the extra term adds according to the equation

$$R_{\mathrm{eff}}^2=\langle r^2\rangle+a^2, \tag{9}$$

and for large r, the addition is negligible. This is not the case for lead at 135 Mev,[32] where a sizable increase was found. It is clear that this radius increase must have another origin; thus a transition is beginning to another region of the optical model where the effects of multiple scattering play a role. The effects of the exclusion principle and internal scattering off the energy shell become important at lower energies, though they should not be a function of A. We note, however, that even for lead, a smaller radius is determined for the imaginary potential (from the inelastic cross section) than for the average of the real and imaginary (from the forward scattering). The question of the nonlinearity of the potential with the density has been discussed by Brueckner[42] who finds the desired increase in radius. However, this region is outside the scope of the discussion of this paper.

[42] K. A. Brueckner, Phys. Rev. **103**, 1121 (1956).

Outstanding Questions

In this paper we attempted to discuss, in a very limited region of small angles, high energies, and light elements, how well the optical-model parameters may be determined from the data. We ran into several difficulties. It is necessary to add corrections to the simple optical model for the range of nuclear forces, and others may also be important. Some of the difficulty may be purely computational. Although it is possible to understand simple analytic procedures, particularly that of Bethe, it is hoped to check and extend the calculation by exact numerical computation. Some of this extension has already been completed by Cromer.[7]

At the same time some indications are given of desirable experiments to give a lead to the theory and define parameters uniquely. At 135 Mev, the data are already moderately complete, and the analysis runs into inconsistencies. There is little evidence about the existence of an imaginary part of the spin-orbit potential. Two experiments could throw light on this question; accurate polarization data in the Coulomb region of 2° to 3° lab, and one or two measurements of the triple-scattering parameter R around 10°. Both of these are planned in the near future.

At 310 Mev the data are not complete. In addition to the experiments needed at 135 Mev, more accurate small-angle neutron scattering data are needed to confirm the effect of the range of nuclear forces. Neutron polarization data are also absent.

Meanwhile we can say that we can describe nucleon scattering from carbon at small angles and high energies by summing nucleon-nucleon scattering. The agreement in the amplitudes is about 20%. We can neither derive this agreement uniquely, nor can we say that the residual 20% is a serious discrepancy.

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