form of the Mathieu equation:

$$d^2y/d\xi^2+(b-s\cos^2\xi)y=0,$$

where

$$b=s=(eB_0/2m\omega)^2.$$

From a table of characteristic values[7] one reads that the solutions of (30) remain bounded if $(eB_0/2m\omega)^2$ lies in one of the intervals:

$$0<(eB_0/2m\omega)^2\leq 1.315,$$
$$3.56\leq(eB_0/2m\omega)^2\leq 7.43,$$
$$12.16\leq(eB_0/2m\omega)^2\leq 18.51,\quad\text{etc.}$$

There are infinitely many such intervals, but only the first one appears to be of interest

$$|\omega|\geq 0.433|eB_0/m|. \tag{33}$$

This condition ensures stability of the orbit, that is, a bounded $r(t)$ for any initial conditions, provided $r(t)$ is small enough to justify the linearization (30).

In a sense, the criterion derived by the method in this section is complementary to the one obtained in Sec. A. The condition on the frequency has been relaxed and made precise (33), while it was necessary to impose the somewhat indefinite condition (24).

An exact stability criterion is still lacking, but one might speculate that it takes the form

$$f((eB_0/m\omega)^2, K, p_\varphi^2)\leq 1,$$

where f is an increasing function of its arguments. A number of solutions $r(t)$ have been obtained by numerical integration of (10). Although any such calculations must remain inconclusive since one cannot follow a solution for arbitrarily long times, it was observed that the orbits remained bounded as long as both (23) and (33) were satisfied, indicating that the conditions of either method are far more stringent than necessary.

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[7] *Tables Relating to Mathieu Functions*, by U. S. National Bureau of Standards (Columbia University Press, New York, 1951).

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Landau's Model of Liquid He^3†

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Landau's model of liquid He^3 as a Fermi liquid is studied with regard to its lowest temperature properties. Primary stress is laid on the coefficient of thermal expansion. The spectra used by Abrikosov and Khalatnikov, the perfect-gas type and "bubble" type, are shown to give a positive coefficient of thermal expansion, in contradiction with experiment. An alternative simple spectrum is suggested which can give a negative coefficient of thermal expansion, namely $\Delta+p^2/2m^*$. In addition, the existence of a negative coefficient of thermal expansion is shown to imply a strong temperature dependence of the energy spectrum, which may cause a sharp deviation of the heat capacity curve from the perfect-gas type, similar to the deviations observed in liquid He^3.

1. INTRODUCTION

In 1956 Landau[1] proposed a model with "Fermi-type spectrum" which is not necessarily temperature-independent nor interaction-free as in the case of ideal Fermi gas (hence the qualification "liquid"), and developed a general formalism of some properties of the model. Based on this, Khalatnikov and Abrikosov[2] discussed the thermodynamics of liquid He^3 assuming two particular spectra, perfect-gas type and "bubble" type, and concluded that the latter reproduces the temperature variation of the heat capacity, of the entropy, and of the magnetic susceptibility. This spectrum, however, gives a positive thermal expansion coefficient in contradiction to a recent experimental result,[3] as will be shown presently. It will be shown that Landau's model itself is general enough not to expose any defect in the "Fermi excitation region" below about 0.2°K.

We wish here to develop some formulas which can be derived on the basis of Landau's original idea, in as general a way as possible. Only one point at which we deviate from the idea is that we treat fermions with classical spin, i.e., Ising spin. This restriction allows us to develop unambiguous derivation of the formulas,

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and may be admitted since no one has given a satisfactory treatment that is appropriate to quantum spins.

2. GENERAL FORMALISM

Let us recall the formulation of Landau's Fermi-liquid model in its classical spin version. Denote one-particle states by the index i and the expectation value of the number of occupying particles by n_i. Then the total number of the particles is given by

$$N=\sum_i n_i. \tag{2.1}$$

and its variation by

$$\Delta N=\sum_i \delta n_i. \tag{2.2}$$

As the expression for the entropy of the system we shall use[4]

$$S=-k\sum_i \{n_i \ln n_i+(1-n_i)\ln(1-n_i)\}. \tag{2.3}$$

Hence the variation of the entropy to the second order in δn is

$$\Delta S=-k\sum_i \left\{\left(\ln\frac{n_i}{1-n_i}\right)\delta n_i +\frac{1}{2}\frac{1}{n_i(1-n_i)}(\delta n_i)^2+\cdots\right\}. \tag{2.4}$$

As for the energy, we assume only the expression for the variation

$$\Delta E=\sum_i \epsilon_i\delta n_i+\tfrac{1}{2}\sum_i\sum_j f_{ij}\delta n_i\delta n_j+\cdots. \tag{2.5}$$

Here we can, of course, assume that f is symmetric in its indices. However, ϵ and f will be complicated functionals of $\{n_i\}$, the forms of which we shall leave unspecified for the time being.

The equilibrium distribution at constant volume, V, temperature, T ($\beta\equiv 1/kT$), and chemical potential, ζ, is given by the condition of vanishing first order variation of $(\Delta E-T\Delta S-\zeta\Delta N)$. It gives

$$n_i=\frac{1}{e^{\beta(\epsilon_i-\zeta)}+1}. \tag{2.6}$$

Inserting this into (2.1) and (2.3), we obtain formal expressions for the total average number

$$N=\sum_i\left(\frac{1}{e^{\beta(\epsilon_i-\zeta)}+1}\right), \tag{2.7}$$

and the entropy

$$S=k\beta\sum_i\frac{\epsilon^i-\zeta}{e^{\beta(\epsilon_i-\zeta)}+1}-k\sum_i \ln(1-e^{-\beta(\epsilon_i-\zeta)}). \tag{2.8}$$

Let us assume that ϵ is spin-independent and the number of "orbital" states with "energy" below ϵ is given by $Z\epsilon$. Then these expressions can be written in integral forms (by partial integration)

$$N=\beta\int_{-\infty}^{\infty}\frac{Z}{1+\cosh[\beta(\epsilon-\zeta)]}d\epsilon, \tag{2.9}$$

and

$$S=k\beta^2\int_{-\infty}^{\infty}\frac{(\epsilon-\zeta)Z}{1+\cosh[\beta(\epsilon-\zeta)]}d\epsilon. \tag{2.10}$$

This Z is a certain functional of $\{n_i\}$, which may be specified by the quantum-mechanical analysis of the system. This procedure, however, we agree to avoid in the Fermi-liquid model. Then the question is: Is there anything that can be derived from the formal model above? We shall attempt in this paper to give a partial answer to this question. The argument will necessarily involve a full use of thermodynamics.

First let us notice that $\{n_i\}$ is an equilibrium distribution so that Z may be considered as a function of β, N, and V as well as ϵ. In fact we need only this dependence in dealing with thermodynamic properties. Let us write this in the form $Z=Vz\,(1/\beta, n, \epsilon)$ where $n=N/V$. Putting $\beta(\epsilon-\zeta)\equiv x$ and expanding Z in powers of $1/\beta$ to the second order, we get from (2.9) and (2.10)

$$N=2Z+2\frac{\partial Z}{\partial(1/\beta)}\frac{1}{\beta} +\frac{1}{2}\left[2\frac{\partial^2 Z}{\partial(1/\beta)^2}+\frac{2\pi^2}{3}\frac{\partial^2 Z}{\partial\epsilon^2}\right]\frac{1}{\beta^2}+\cdots, \tag{2.11}$$

and

$$S=k\frac{2\pi^2}{3}\frac{\partial Z}{\partial\epsilon}\frac{1}{\beta}+\cdots, \tag{2.12}$$

where we understand that Z and its derivatives are taken at $\beta=\infty$ and $\epsilon=\zeta$. Thus, from (2.12), the entropy is proportional to temperature in the lowest temperature region. This is characteristic of the Fermi spectrum in general.

On the basis of this proportionality, one can deduce by thermodynamics that

$$\zeta(T,v)-\zeta(0,v)=\tfrac{1}{2}T\left[v\left(\frac{\partial S}{\partial V}\right)_{TN}-\frac{S}{N}\right]+\cdots \tag{2.13}$$

is valid at lowest temperatures ($nv=1$). This shows that this difference is proportional to T^2. Inserting (2.12) with $\partial Z/\partial\epsilon$ evaluated at $\zeta=\zeta(0,v)\equiv\zeta_0$ into (2.13), we

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[4] We are considering a statistical ensemble, for which the defined probability depends on the set of occupation numbers of quasi-particle states: $p\{n_i'\}$ with $n_i'=0$ or 1 (i.e., the density matrix is diagonal in this representation). This is an appropriate equilibrium ensemble when the energy is diagonal in this representation. If the quasi-particle states can be grouped into such "coarse cells" that the probability p depends only on the sum $\Sigma\, n_i'$ taken in each cell, one obtains the expression (2.3) as the entropy of the ensemble, where n_i=average of n_i'. The average energy will, then, be a function of $\{n_i\}$ and hence we have the definition (2.5).

find

$$\zeta(T,v)-\zeta(0,v)=-\frac{\pi^2}{3}\frac{1}{\beta^2}\left[\frac{mc^2}{n}\frac{\partial^2 z}{\partial\epsilon^2}+\frac{\partial^2 z}{\partial n\partial\epsilon}\right]+\cdots, \tag{2.14}$$

where c is the "velocity of sound" at absolute zero and m is the mass of a particle (not of a quasi-particle). Comparing, then, this result with that deduced by expanding (2.11) in powers of $\zeta-\zeta_0$, we finally get the following two relations:

$$\frac{\partial z}{\partial(1/\beta)}=0, \tag{2.15}$$

and

$$\frac{\partial^2 z}{\partial(1/\beta)^2}=\frac{\pi^2}{3}\left[2\frac{\partial z}{\partial\epsilon}\frac{\partial^2 z}{\partial n\partial\epsilon}+\left(\frac{2mc^2}{n}\frac{\partial z}{\partial\epsilon}-1\right)\frac{\partial^2 z}{\partial\epsilon^2}\right]. \tag{2.16}$$

These are, of course, valid at $\zeta=\zeta_0$. For an ideal gas, $\partial Z/\partial\epsilon$ is a constant times $V\epsilon^{\frac{1}{2}}$ and is equal to $N/2mc^2$ at $\zeta=\zeta_0$. Hence this second derivative also vanishes as it should for this case.

As for the derivatives of z with respect to n, we can utilize the relation

$$n=2z(0,n,\zeta_0). \tag{2.17}$$

For example, the first derivative is given by

$$\frac{\partial z}{\partial n}=\frac{1}{2}-\frac{mc^2}{n}\frac{\partial z}{\partial\epsilon}. \tag{2.18}$$

Finally we will mention some formulas which involve the thermal expansion coefficient. By virtue of the thermodynamic relation

$$\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p=-\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T\left(\frac{\partial S}{\partial V}\right)_T\approx\frac{1}{\rho c^2}\left(\frac{\partial S}{\partial V}\right)_T, \tag{2.19}$$

we can write for the thermal expansion coefficient

$$\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p=\frac{2\pi^2}{3}\frac{k}{\beta}\frac{1}{mc^2}\times\left[\frac{1}{n}\frac{\partial z}{\partial\epsilon}-\frac{mc^2}{n}\frac{\partial^2 z}{\partial\epsilon^2}-\frac{\partial^2 z}{\partial n\partial\epsilon}\right]+\cdots. \tag{2.20}$$

Also the thermodynamic formula (2.13) can be rewritten in the following form

$$\zeta(T,v)-\zeta(0,v)=\tfrac{1}{2}Tmc^2\left[\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p-\frac{s}{mc^2}\right]+\cdots. \tag{2.21}$$

The various functional forms of Z should have been caused by the interaction, which on the other hand is reflected on the second-order term in (2.5). Hence those derivatives of Z appearing above must have some connection with f in (2.5). We shall work out this relation in the next section.

3. FLUCTUATION

First we shall give the relations between macroscopic quantities and the fluctuations which are expected on the grand canonical ensemble without proof.

$$\frac{\chi}{\mu^2}+\frac{N}{m(\partial p/\partial\rho)_T}=4\beta\langle(\Delta N_+)^2\rangle, \tag{3.1}$$

$$-\frac{\chi}{\mu^2}+\frac{N}{m(\partial p/\partial\rho)_T}=4\beta\langle\Delta N_+\Delta N_-\rangle, \tag{3.2}$$

$$C_{\zeta V}=k\beta^2\langle(\Delta E-\zeta\Delta N)^2\rangle, \tag{3.3}$$

and

$$\frac{S}{m(\partial p/\partial\rho)_T}-\frac{N}{v}\left(\frac{\partial v}{\partial T}\right)_p=k\beta^2\langle\Delta N(\Delta E-\zeta\Delta N)\rangle. \tag{3.4}$$

Here χ is the susceptibility, μ is the magnetic moment of one particle, and N_+ and N_- are the numbers of particles with spin up and spin down, respectively. $C_{\zeta V}$ is the heat capacity at constant ζ and V and is connected with C_{NV}, the heat capacity at constant N and V, by the relation:

$$C_{\zeta V}=C_{NV}+T\frac{(\partial\zeta/\partial T)_{NV}{}^2}{(m/N)(\partial p/\partial\rho)_T}, \tag{3.5}$$

so that the difference is of the order T^3. It is to be noted that the expression on the right-hand side of (2.21) appears in (3.4).

Inserting the equilibrium expression (2.6) into the coefficients in (2.4) and (2.5), we get for the relative probability of the fluctuation the following:

$$e^{-\beta(\Delta E-T\Delta S-\zeta\Delta N)}=\exp\left[-\frac{1}{2}\left(\sum_i\frac{(\delta n_i)^2}{n_i(1-n_i)}+\beta\sum_i\sum_j f_{ij}\delta n_i\delta n_j\right)\right], \tag{3.6}$$

where we understand that the coefficients are evaluated with (2.6). For this ensemble the average value of the product $\delta n_i\delta n_j$ can be shown to be

$$\langle\delta n_i\delta n_j\rangle=(j|(1+\beta\phi)^{-1}|i)\times[n_i(1-n_i)]^{\frac{1}{2}}[n_j(1-n_j)]^{\frac{1}{2}}. \tag{3.7}$$

Here

$$\phi_{ij}\equiv[n_i(1-n_i)]^{\frac{1}{2}}f_{ij}[n_j(1-n_j)]^{\frac{1}{2}}, \tag{3.8}$$

and $(j|(1+\beta\phi)^{-1}|i)$ is the (j,i) element of the inverse matrix to the symmetric matrix $(1+\beta\phi)$. Using (3.7) we can express the right-hand sides of (3.1) to (3.4) more concretely. The results are

$$\frac{N}{m(\partial p/\partial\rho)_T}=\beta\langle(\Delta N)^2\rangle=\beta\sum_i\sum_j(j|(1+\beta\phi)^{-1}|i)\times[n_i(1-n_i)]^{\frac{1}{2}}[n_j(-n_j)]^{\frac{1}{2}},\tag{3.9}$$

$$\frac{\chi}{\mu^2}=\beta\sum_i\sum_j(j|\sigma_z(1+\beta\phi)^{-1}\sigma_z|i)\times[n_i(1-n_i)]^{\frac{1}{2}}[n_j(1-n_j)]^{\frac{1}{2}},\tag{3.10}$$

$$C_{\zeta V}=k\beta^2\sum_i\sum_j(j|(1+\beta\phi)^{-1}|i)(\epsilon_i-\zeta)\times[n_i(1-n_i)]^{\frac{1}{2}}(\epsilon_j-\zeta)[n_j(1-n_j)]^{\frac{1}{2}},\tag{3.11}$$

and

$$\frac{S}{m(\partial p/\partial\rho)_T}-\frac{N}{v}\left(\frac{\partial v}{\partial T}\right)_p=k\beta^2\sum_i\sum_j(j|(1+\beta\phi)^{-1}|i)\times[n_i(1-n_i)]^{\frac{1}{2}}(\epsilon_j-\zeta)[n_j(1-n_j)]^{\frac{1}{2}}.\tag{3.12}$$

Here σ_z is one of the Pauli matrices corresponding to the Z component of the spin. In addition to these, from the condition that $\langle\Delta E-\zeta\Delta N\rangle=0$, we obtain the relation

$$\textstyle\sum_i\sum_j(i|\phi|j)(j|(1+\beta\phi)^{-1}|i)=0,\tag{3.13}$$

which means that ϕ cannot be quite arbitrarily chosen.

Let us assume that ϕ has the form

$$\phi\equiv(1+\sigma_x)\varphi+(1-\sigma_x)\psi,\tag{3.14}$$

with the understanding that φ and ψ operate only on "orbital" states. σ_x is one of the Pauli matrices, which is given by

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}.\tag{3.15}$$

Or, expressed in another way, we assume that

$$f\equiv(1+\sigma_x)h+(1-\sigma_x)g,\tag{3.14$'$}$$

where, of course, h and g are connected with φ and ψ, respectively, through relations similar to (3.8). Therefore, for parallel and antiparallel spin pairs, f takes the value $h+g$ and $h-g$, respectively. This form makes it possible to write down the inverse matrix as follows:

$$(1+\beta\phi)^{-1}=1-\beta(1+\sigma_x)\varphi(1+2\beta\varphi)^{-1}-\beta(1-\sigma_x)\psi(1+2\beta\psi)^{-1}.\tag{3.16}$$

Inserting this and restricting ourselves to the lowest temperature expressions, we get the following formulas from (3.9) to (3.12)

$$\frac{N}{mc^2}\approx2\left(\frac{\partial Z}{\partial\epsilon}\right)_0\left\{1-2\left(\frac{\partial Z}{\partial\epsilon}\right)_0(\zeta_0|Q|\zeta_0)\right\},\tag{3.17}$$

$$\frac{\chi}{\mu^2}\approx2\left(\frac{\partial Z}{\partial\epsilon}\right)_0\left\{1-2\left(\frac{\partial Z}{\partial\epsilon}\right)_0(\zeta_0|P|\zeta_0)\right\},\tag{3.18}$$

$$C_{\zeta V}\approx\frac{2\pi^2}{3}\left(\frac{\partial Z}{\partial\epsilon}\right)_0\frac{k}{\beta},\tag{3.19}$$

in agreement with (2.12), and

$$\frac{N}{v}\left(\frac{\partial v}{\partial T}\right)_p\approx\frac{2\pi^2}{3}\frac{k}{\beta}2\left(\frac{\partial Z}{\partial\epsilon}\right)_0\times\left\{\left[1-2\left(\frac{\partial Z}{\partial\epsilon}\right)_0(\zeta_0|Q|\zeta_0)\right]\times\left[\frac{1}{N}\left(\frac{\partial Z}{\partial\epsilon}\right)_0-\frac{1}{2}\frac{(\partial^2Z/\partial\epsilon^2)_0}{(\partial Z/\partial\epsilon)_0}\right]+\left(\frac{\partial Z}{\partial\epsilon}\right)_0\left[\frac{\partial}{\partial\epsilon}(\epsilon|Q|\zeta_0)\right]\right\}.\tag{3.20}$$

Here we wrote for the average of the element of $\varphi(1+2\beta\varphi)^{-1}$ taken over the states with energy ϵ and ϵ' respectively,

$$[n_\epsilon(1-n_\epsilon)]^{\frac{1}{2}}[n_{\epsilon'}(1-n_{\epsilon'})]^{\frac{1}{2}}(\epsilon|Q|\epsilon'),$$

and similarly for the average of the element $\psi(1+2\beta\psi)^{-1}$,

$$[n_\epsilon(1-n_\epsilon)]^{\frac{1}{2}}[n_{\epsilon'}(1-n_{\epsilon'})]^{\frac{1}{2}}(\epsilon|P|\epsilon').$$

From the relation (2.21), we can see that

$$\zeta(v,T)-\zeta(v,0)\approx-\frac{1}{2}\left(\frac{2\pi^2}{3}\right)\frac{1}{\beta^2}\left\{\frac{(\partial^2Z/\partial\epsilon^2)_0}{2(\partial Z/\partial\epsilon)_0}-\frac{(\partial Z/\partial\epsilon)_0[(\partial/\partial\epsilon)(\epsilon|Q|\zeta_0)]_0}{1-2(\partial Z/\partial\epsilon)_0(\zeta_0|Q|\zeta_0)}\right\}.\tag{3.21}$$

At the lowest temperatures, Q and P can be evaluated in terms of h and g, expanding formally $(1+2\beta\varphi)^{-1}$, for example, in powers of φ. The results are

$$(\epsilon|Q|\epsilon')=h_{\epsilon\epsilon'}-\frac{2(\partial Z/\partial\epsilon)_0h_{\epsilon\zeta_0}h_{\epsilon'\zeta_0}}{1+2(\partial Z/\partial\epsilon)_0h_{\zeta_0\zeta_0}},\tag{3.22}$$

and a similar expression for P in terms of g instead of h. Here $h_{\epsilon\epsilon'}$ is the average of the element of h taken over the states with energy ϵ and ϵ', respectively. In terms of these, we can rewrite (3.17), (3.18), and (3.20) as follows:

$$mc^2\approx N\left\{\frac{1}{2(\partial Z/\partial\epsilon)_0}+\bar{h}\right\},\tag{3.23}$$

$$\frac{\mu^2}{\chi}\approx\frac{1}{2(\partial Z/\partial\epsilon)_0}+\bar{g},\tag{3.24}$$

and

$$\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p\approx\frac{2\pi^2}{3}\left(\frac{k}{\beta}\right)\frac{1}{mc^2}\left[\frac{1}{N}\left(\frac{\partial Z}{\partial\epsilon}\right)_0-\frac{1}{2}\frac{(\partial^2Z/\partial\epsilon^2)_0}{(\partial Z/\partial\epsilon)_0}+\left(\frac{\partial Z}{\partial\epsilon}\right)_0\left(\frac{\partial h_{\epsilon\zeta_0}}{\partial\epsilon}\right)_{\zeta_0}\right].\tag{3.25}$$

Here $\bar{h}$ and $\bar{g}$ represent $h_{\zeta_0\zeta_0}$ and $g_{\zeta_0\zeta_0}$, respectively. The results (3.23) and (3.24) are essentially those which were given by Landau, except that we derived these here without referring to the momentum of the "quasi-particle," which method is considered better in view of the next step where various types of spectra will be compared. The formula (3.25) is new and will be discussed with primary emphasis in this report.

Now referring to Sec. 2 we can find relations which connect the various derivatives of Z with f. Insert (3.23) into (2.18), one then obtains

$$\frac{\partial z}{\partial n}=-\frac{N}{n}\bar{h}\frac{\partial z}{\partial \epsilon}. \tag{3.26}$$

Comparing (3.25) and (3.23) with (2.20) we also obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial n\partial \epsilon}&=-\frac{N}{n}\left(\bar{h}\frac{\partial^2 z}{\partial \epsilon^2}+\frac{\partial h}{\partial \epsilon}\frac{\partial z}{\partial \epsilon}\right)\\&=-\frac{N}{n}\left[\frac{\partial}{\partial \epsilon}\left(h_{\epsilon\zeta_0}\frac{\partial z}{\partial \epsilon}\right)\right]_{\epsilon=\zeta_0},\end{aligned} \tag{3.27}$$

and hence

$$\frac{\partial^2 z}{\partial (1/\beta)^2}=-\frac{2\pi^2}{3}\left(\frac{\partial z}{\partial \epsilon}\right)^2\frac{N}{n}\left(\frac{\partial h_{\epsilon\zeta_0}}{\partial \epsilon}\right)_{\zeta_0}. \tag{3.28}$$

According to (3.26) and (3.27), we can consider that

$$\frac{\partial Z(0,n,\epsilon)}{\partial n}=-\frac{N}{n}h_{\epsilon\zeta_0}\frac{\partial Z(0,n,\epsilon)}{\partial \epsilon} \tag{3.29}$$

is valid in the neighborhood of ζ_0. All these formulas show clearly that these derivatives vanish in the case of an ideal gas.

4. DISCUSSION

From the recent experimental evidence[3] it can be expected that the thermal expansion coefficient of liquid helium-3 at lowest temperatures is negative. Referring to Eqs. (2.20) and (3.25), we shall discuss this point. The density of states at the Fermi surface at absolute zero is of course positive, i.e., $(\partial Z/\partial \epsilon)_0$. On the other hand, $(\partial^2 Z/\partial \epsilon^2)_0$ can be positive or negative according to the spectrum. For example, both the perfect-gas type,

$$\epsilon=p^2/2m^*, \tag{4.1}$$

and the "bubble" type, which was used in the calculation of Khalatnikov and Abrikosov,

$$\epsilon=(p-p_0)^2/2m^*, \tag{4.2}$$

give positive values for $(\partial^2 Z/\partial \epsilon^2)$, but only to the extent that the sum of the first and second terms in (3.25) comes out to be positive. Therefore, if the spectrum (4.2), for example, should give the correct description, as suggested in Khalatnikov and Abrikosov's paper, the last term involving the derivative of h must be negative and large enough to reverse the sign of the total expression. It is apparently expected that this derivative is negative for the following reasons: h is positive, according to the fact that the sound velocity is greater than the value of the first term in (3.23) with the value of the state density derived from the heat capacity according to (3.19); and the value of $h_{\epsilon\epsilon'}$ must decrease as the difference of the energies of the two states involved increases. But (3.25) tells us that this derivative must be large enough to make the whole expression negative.

Now, as mentioned in relation to (3.21), this derivative gives Z_ϵ a temperature dependence according to (3.28). It seems obvious that Khalatnikov and Abrikosov have neglected this extra temperature dependence in their calculation, because they calculated the heat capacity with Z_ϵ which is derived from (4.2) treating p_0 and m^* as constant. Hence, from the manner of their calculation, there will result only a positive thermal expansion coefficient.

Then the question is: is there any possibility remaining for the spectra, when revised in such a way that m^* and p_0 are not constants any more, but depends on β and n, to give a negative thermal expansion coefficient? The following discussions are based on Eq. (2.20) which is valid also in the case of quantum spins insofar as we can assume ϵ to be spin-independent.

For the perfect-gas type (4.1), we get according to (2.18) the following relation which is valid at $\beta\rightarrow\infty$:

$$\frac{n}{m^*}\frac{\partial m^*}{\partial n}=\frac{2}{3}\frac{mc^2}{\epsilon_0}, \tag{4.3}$$

where ϵ_0 is given by

$$\frac{n}{2}=\frac{4\pi}{3h^3}(2m^*\epsilon_0)^{\frac{3}{2}}. \tag{4.4}$$

Or, put in another way, we are assuming according to (3.29) that

$$Nh_{\epsilon\epsilon_0}=-\epsilon\frac{n}{m^*}\frac{\partial m^*}{\partial n} \tag{4.5}$$

in the neighborhood of the Fermi surface. When we use (4.3) or (4.5), we get for the thermal expansion coefficient from (2.20) or (3.25)

$$\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p=2^{\frac{1}{3}}\pi^2\left(\frac{4\pi}{3}\right)^{\frac{4}{3}}\left[k\bigg/\frac{h^2n^{\frac{2}{3}}}{2m^*}\right]^2T, \tag{4.6}$$

which is definitely positive.

As for the "bubble" type spectrum (4.2), we get, corresponding to (4.3),

$$\frac{n}{m^*}\frac{\partial m^*}{\partial n}=\left[1+\frac{p_0^2}{2m^*}\frac{1}{\epsilon_0}\right]^{-1}\left[\frac{2}{3}\frac{mc^2}{\epsilon_0}+\frac{1}{\epsilon_0}\frac{p_0^2}{m^*}-\frac{p_0^2}{2m^*}\frac{mc^2}{\epsilon_0^2}-\frac{1}{\epsilon_0}\frac{n}{m^*}\frac{\partial p_0^2}{\partial n}\right], \tag{4.7}$$

where ϵ_0 is given by

$$\frac{n}{2}=\frac{4\pi}{3h^3}\{6p_0^2(2m^*\epsilon_0)^{\frac{1}{2}}+2(2m^*\epsilon_0)^{\frac{3}{2}}\}. \tag{4.8}$$

This gives for the thermal expansion coefficient

$$\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p=\frac{8\pi^3}{3}T\frac{k^2\epsilon_0^{\frac{1}{2}}}{mc^2(h^2n^{\frac{2}{3}}/2m^*)^{\frac{3}{2}}}\left(1+\frac{2m^*\epsilon_0}{p_0^2}\right)^{-1}$$
$$\times\left[\frac{4n}{p_0}\frac{\partial p_0}{\partial n}+\frac{p_0^2}{2m^*}\frac{mc^2}{\epsilon_0^2}+2\frac{mc^2}{\epsilon_0}+\frac{2m^*mc^2}{p_0^2}-\frac{4}{3}\right]. \tag{4.9}$$

According to Khalatnikov and Abrikosov, $p_0^2\gg m^*\epsilon_0$, so that the factor in the square brackets in (4.9), which determines the sign of the thermal expansion, can be approximated by

$$\frac{4n}{p_0}\frac{\partial p_0}{\partial n}+(16\pi)^4mc^2\frac{2m^*p_0^{10}}{(h^3n)^4},$$

which we may reasonably expect to be positive. But even when we do not suppose this, the condition that the thermal expansion coefficient should be negative requires that

$$-\frac{n}{p_0}\frac{\partial p_0}{\partial n}>32,$$

if we put in Khalatnikov and Abrikosov's values of the parameters. This dependence is highly unreasonable. Thus we can conclude that both the perfect-gas and the "bubble" spectra give positive thermal expansion in contradiction to experiment.

Let us, then, try another type of spectrum. One of the simplest spectra will be

$$\epsilon=\Delta+p^2/2m^*, \tag{4.10}$$

in which Δ and m^* are supposed to depend on β and n. For this, the relation corresponding to (4.3) is given by

$$\frac{n}{m^*}\frac{\partial m^*}{\partial n}-4\left(\frac{4\pi}{3h^3}\right)^{\frac{2}{3}}m^*\left(\frac{n}{2}\right)^{\frac{1}{3}}\left\{\frac{\partial\Delta}{\partial n}-\frac{mc^2}{n}\right\}=\frac{2}{3}, \tag{4.11}$$

and the Fermi energy by

$$\frac{n}{2}=\frac{4\pi}{3h^3}[2m^*(\epsilon_0-\Delta)]^{\frac{3}{2}}. \tag{4.12}$$

Correspondingly, we are assuming an "interaction" which has

$$Nh_{\epsilon\epsilon_0}=n\frac{\partial\Delta}{\partial n}-(\epsilon-\Delta)\frac{n}{m^*}\frac{\partial m^*}{\partial n}, \tag{4.13}$$

as an expression valid in the neighborhood of ϵ_0. The expression for the thermal expansion coefficient in this case can be expressed in either of the following forms:

$$\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p=2\pi^2k^2T\left(\frac{4\pi}{3h^3}\frac{2}{n}\right)^{\frac{4}{3}}m^{*2}\left[1-\frac{n}{mc^2}\frac{\partial\Delta}{\partial n}\right]$$
$$=\pi^2k^2T\left(\frac{4\pi}{3h^3}\frac{2}{n}\right)^{\frac{2}{3}}\frac{m^*}{mc^2}\left[\frac{2}{3}-\frac{n}{m^*}\frac{\partial m^*}{\partial n}\right]. \tag{4.14}$$

This shows that this spectrum can give a negative thermal expansion coefficient if we assume that

$$\partial\Delta/\partial n>mc^2/n, \tag{4.15}$$

or, equivalently,

$$\partial m^*/\partial n>\tfrac{2}{3}m^*/n. \tag{4.16}$$

As for the heat capacity or the entropy in this case, we must take into account the temperature dependence of m^* and Δ. This problem is beyond the scope of the present development of the formalism. However, according to (3.28) and (4.13) we have

$$\frac{\partial^2 z}{\partial(1/\beta)^2}=\frac{2\pi^2}{3}\left(\frac{\partial z}{\partial\epsilon}\right)^2\frac{1}{m^*}\frac{\partial m^*}{\partial n}, \tag{4.17}$$

which indicates some of the character of this dependence. Thus we can see that there may be a strong deviation of the temperature dependence of the heat capacity or of the entropy from that of the perfect gas in spite of the same dependence of the spectrum upon the momentum. Thus it may be possible that the sharp change of the specific heat to a fairly constant value, which occurs at about 0.25°K, might be closely connected with the negative thermal expansion.

Further analysis along this line will be quite complicated and we would like to reserve it for a later work. Further, it should also be noted that the expressions for the deviation of entropy and energy to the second order may also be insufficient.

As far as the expressions for the properties at lowest temperatures are concerned, the unknown parameters are five in number, so that the data on sound velocity, susceptibility, specific heat, and thermal expansion coefficient are insufficient to check Landau's idea. This, then, is the point we wished to make when we stated that Landau's model is general enough not to expose any defect in the "Fermi excitation region."

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