Stable Orbits of Charged Particles in an Oscillating Electromagnetic Field*

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Orbits in a field of the cylindrical wave guide driven in the TE_{01}-mode have been studied. The radial components of the Lorentz force acting on the particle is never positive regardless of its charge; thus, it is attracted toward the axis of the wave guide around which it moves in a complicated path. Sufficient conditions are given under which the orbits are stable, that is, remain bounded for all times.

I. INTRODUCTION

CONSIDER the electromagnetic field in a circular wave guide, which is driven in the TE_{01}-mode at cutoff. The only two field components which do not vanish are[1]

$$E_\varphi = -B_0 J_1(\omega r)\cos\omega t,\qquad B_z = B_0 J_0(\omega r)\sin\omega t. \tag{1}$$

A charged particle placed in this field executes a very complicated orbit, but it is not difficult to see, qualitatively, that this particle experiences a force attracting it towards the axis. The electric field will cause it to oscillate out of phase compared to **E** but in phase with **B** and normal to **B**. The resulting $\mathbf{V}\times\mathbf{B}$ force is on the average directed towards the axis of the wave guide as long as the particle finds itself inside the locus of maximum electric field amplitude ($r<0.293\lambda$). One might expect from this that the particle can be confined to a volume surrounding the axis. This would certainly be true if the attracting force were time independent. However, since the force is oscillating, it is conceivable that energy is imparted to the particle on each cycle, causing it to follow an ever-widening orbit until it escapes. The question then arises under what conditions the particle will be stably bound to the axis; that is, under what conditions the coordinate $r(t)$ of the particle will remain bounded for all times. Sufficient conditions will be derived.

Since the confining force mentioned above is independent of the sign of the charge, one can expect that a neutral plasma consisting of ions and electrons can be confined stably in a field of the type (1). Such a confinement by "radiation pressure" has been considered by Knox[2] for a spherical geometry. The author derived self-consistent solutions for the densities of ions and electrons in equilibrium with the confining electromagnetic field of a TM_{01}-mode.[3] The plasma confinement by a TE_{01}-mode can be treated in an exactly analogous fashion. Boot *et al.*[4] give a similar theory neglecting charge separation.

As long as the particle density is so small that it does not appreciably modify the field, one can expect the plasma to be stably confined since each particle is stably bound. However, as the number of particles increases, local fluctuations in density may so modify the field as to reinforce the fluctuation. The confinement then becomes unstable. It has been shown[5] that in the limit of a very dense, infinitely conducting plasma the confinement by TE_{01} fields is unstable (the same is true for the TM_{01}-mode). Therefore, as the plasma density is gradually increased from zero, the confinement is first stable, then becomes unstable. It seems likely that this transition (stable to unstable confinement) occurs when the ratio of the plasma frequency to the applied frequency, $(ne^2/m\omega^2)^{\frac{1}{2}}$, becomes comparable to unity, because the field begins to be appreciably modified as $n>m\omega^2/e^2$.

The suspension of charged dust particles by oscillating electric fields has recently been investigated both experimentally and theoretically by Wuerker *et al.*[6]

II. ANALYSIS OF THE PARTICLE ORBITS

It is convenient to represent the field (1) by its potentials. The scalar potential, ϕ, is zero and, among the components of the vector potential, only

$$A_\varphi = (B_0/\omega)J_1(\omega r)\sin\omega t \tag{2}$$

does not vanish. The relations curl $\mathbf{A}=\mathbf{B}$ and $-\partial\mathbf{A}/\partial t - \text{grad}\phi = \mathbf{E}$ yield the field (1). The unrelativistic motion of a particle having a mass m and a charge e is governed by its Hamiltonian

$$H=\frac{1}{2m}\left[p_r^2+\frac{1}{r^2}\left(p_\varphi-\frac{eB_0}{\omega}rJ_1(\omega r)\sin\omega t\right)^2+p_z^2\right]. \tag{3}$$

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[1] Rationalized units; velocity of light = 1.
[2] F. B. Knox, Australian J. Phys. **10**, 1, 221 (1957) and **10**, 4, 565 (1957).
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[4] Boot, Self, R-Shersby-Harvie, J. Electronics and Control 4, No. 5 (1958).
[5] Erich S. Weibel, Aeronautical Research Laboratory Report ARL-57-1026, The Ramo-Wooldridge Corporation (1957, unpublished), p. 25.
[6] Wuerker, Shelton, and Langmuir, J. Appl. Phys. (to be published).

The equations of motion become

$$\dot{r}=p_r/m, \tag{4}$$

$$\dot{\varphi}=\frac{p_\varphi}{mr^2}-\frac{eB_0}{m\omega}\frac{J_1(r\omega)}{r}\sin\omega t, \tag{5}$$

$$\dot{z}=p_z/m, \tag{6}$$

$$\dot{p}_r=-\frac{\partial}{\partial r}\left[\frac{1}{2m}\left(\frac{p_\varphi}{r}-\frac{eB_0}{\omega}J_1(r\omega)\sin\omega t\right)^2\right], \tag{7}$$

and since H is independent of φ and of z

$$p_\varphi=\text{const}, \tag{8}$$

$$p_z=\text{const}. \tag{9}$$

Using (4), one obtains a differential equation which contains only one unknown coordinate, $r(t)$:

$$\ddot{r}=\frac{p_\varphi^2}{m^2r^3}-\frac{\partial}{\partial r}\left[\left(\frac{eB_0}{m\omega}\right)^2\frac{J_1^2(\omega r)}{2}\sin^2\omega t-\frac{p_\varphi}{m}\frac{eB_0}{m\omega}\frac{J_1(\omega r)}{r}\sin\omega t\right]. \tag{10}$$

This equation (10) is exceedingly complicated due to the explicit occurrence of the time, t, and due to the singularity at $r=0$. The solutions of this equation shall be investigated by two methods based on two different approximations, each designed to remove one of the difficulties mentioned.

A. Method of the Average Potential

It is not possible to speak of a potential energy of the particle in the field (1), since the system is not conservative. However, if the oscillations of the right-hand side of (10) are sufficiently rapid, one can, as a first approximation, take only its time average into account. Thus one is led to an approximate equation

$$\ddot{r}=\frac{p_\varphi^2}{m^2r^3}-\frac{\partial}{\partial r}\psi(r), \tag{11}$$

where

$$\psi(r)=\tfrac{1}{4}(eB_0/m\omega)^2J_1^2(\omega r). \tag{12}$$

The potential $\psi(r)$ has a minimum at $r=0$ and a maximum at $r=1.84/\omega$, followed by alternating minima and maxima as r increases. Only the "potential well" $r<1.84/\omega$ shall be considered here. The energy integral of (11) becomes

$$\dot{r}^2+p_\varphi^2/m^2r^2+2\psi(r)=K, \tag{13}$$

where, according to (5),

$$\frac{p_\varphi}{m}=\left[r^2\dot{\varphi}+\frac{eB_0}{m\omega}rJ_1(r\omega)\sin\omega t\right]_{t=t_0}, \tag{14}$$

$$K=\left[\dot{r}^2+\left(\frac{p_\varphi}{mr}\right)^2+2\psi(r)\right]_{t=t_0}. \tag{15}$$

Thus the solution of (13) can be obtained by a quadrature:

$$t-t_0=\int_{r(t_0)}^{r(t)}\frac{dr}{[K-(p_\varphi/mr)^2-2\psi(r)]^{\frac{1}{2}}}. \tag{16}$$

If $r(t_0)$, $\dot{r}(t_0)$, and $\dot{\varphi}(t_0)$ are chosen such that

$$K\leq\left[\frac{p_\varphi^2}{m^2r^2}+2\psi(r)\right]_{\max}, \tag{17}$$

then $r(t)$ is periodic and the particle is trapped. The period becomes

$$T=\int_{r_{\min}}^{r_{\max}}\frac{dr}{[K-(p_\varphi/mr)^2-2\psi(r)]^{\frac{1}{2}}}, \tag{18}$$

where $r_{\min}$ and $r_{\max}$ are the zeros of the denominator. A lower bound for the period can be obtained by replacing $\psi(r)$ by

$$\tilde{\psi}=\tfrac{1}{16}(eB_0/m\omega)^2(\omega r)^2\geq\psi(r). \tag{19}$$

The integral can be evaluated and one obtains

$$T\geq(\sqrt{2}/\pi)(m/eB_0). \tag{20}$$

With this result one is in a position to determine the condition under which (11) is a good approximation to the original Eq. (10). Substituting $r(t)=r_0(t)+r_1(t)$ into (10), where $r_0(t)$ is a solution of (11), one obtains an equation for the perturbation $r_1(t)$:

$$\ddot{r}_1(t)+\left[\frac{3p_\varphi^2}{m^2r_0^4}+\frac{d^2\psi(r_0)}{d^2r_0}\right]r_1(t)=-\frac{\partial}{\partial r}\left[\left(\frac{eB_0}{m\omega}\right)^2\frac{J_1^2(\omega r_0)}{4}\cos(2\omega t)+\frac{p_\varphi}{m}\frac{eB_0}{m\omega}\frac{J_1(\omega r_0)}{r_0}\sin(\omega t)\right]. \tag{21}$$

It was assumed at the beginning of this section that the right-hand side of (10) is a rapidly oscillating function. More explicitly, it shall be required that

$$(m\omega/eB_0)^2\gg1. \tag{22}$$

A glance at (20) shows that $r_0(t)$ is changing much more slowly than $r_1(t)$, so that one may set

$$r_1(t)=a\cos(2\omega t)+b\sin(\omega t),$$

and determine a and b from (21) as if r_0 were constant. One finds immediately that $\omega r_1(t)\ll1$ due to (22). To exclude exponentially growing solutions of (21), the coefficient of $r_1(t)$ in that equation must always be positive:

$$\left[\frac{3p_\varphi^2}{m^2r^4}+\frac{d^2\psi(r)}{dr^2}\right]_{r=r_{\max}}>0, \tag{23}$$

where $r_{\max}$ is the larger of the two solutions of

$$K-(p_\varphi/mr)^2-2\psi(r)=0.$$

Hence $r_1(t)$ is indeed small, provided (22) and (23) are satisfied. It is not difficult to see that values of K and p_φ which satisfy (23) also satisfy (17), which ensures that $r_0(t)$ is always bounded.

Thus, two conditions (22) and (23) have been found to ensure bounded orbits of the particle. While condition (23) is definite, condition (22) lacks precision. One wonders just how much larger than unity the quantity $m\omega/eB_0$ must be to justify the approximation. No exact answer has been found so far; however, a different approximation discussed in the next section will shed more light on the question. It will lead to a precise condition for $m\omega/eB_0$ which is much less stringent than (22). But unfortunately a second condition of the type "much smaller than" has to be imposed also.

B. Reduction to a Mathieu Equation

A different method for exploring (10) is open if one is content to know the behavior of only those solutions $r(t)$ which are small,

$$[\omega r(t)]^2\ll 1. \tag{24}$$

Linearization of (10) fails because of the centrifugal term which is singular at $r=0$. A way out of this difficulty is suggested by comparison of the equations of motion (4) through (10) with those of a particle under the influence of a time-dependent *scalar* potential, $m\psi(r,t)$:

$$\ddot{r}=p_\varphi^2/m^2r^3-\partial\psi(r,t)/\partial r, \tag{25}$$

$$\dot{\varphi}=p_\varphi/mr^2, \tag{26}$$

$$\dot{z}=p_z/m. \tag{27}$$

If one identifies

$$\psi(r,t)=\left(\frac{eB_0}{m\omega}\right)^2\frac{J_1^2(\omega r)}{2}\sin^2\omega t-\frac{p_\varphi}{m}\frac{eB_0}{m\omega}\frac{J_1(\omega r)}{r}\sin\omega t, \tag{28}$$

one finds that the equations of motion (6), (8), (9), and (10) are reproduced, but Eq. (5) for the angle is replaced by (26) expressing conservation of the angular momentum. Thus, the orbit defined by (25), (26), and (27), while quite different from the original one, agrees with it insofar as the dependence of r on t is concerned. Since the question of stability depends on $r(t)$ only one can make use of the simpler orbit defined by the scalar potential in $\psi(r,t)$.

The equations of motion for the potential Ψ can now be written in Cartesian coordinates:

$$x=r\cos\varphi,\quad y=r\sin\varphi,$$

$$\ddot{x}=-\frac{\partial}{\partial x}\psi(r,t)=-\frac{x}{r}\frac{\partial}{\partial r}\psi(r,t),$$
$$\ddot{y}=-\frac{\partial}{\partial y}\psi(r,t)=-\frac{y}{r}\frac{\partial}{\partial r}\psi(r,t). \tag{29}$$

These equations are free of any singularity at $r=0$ and can therefore be readily linearized. This yields two completely decoupled equations for x and y, both of the Mathieu type:

$$\{\ddot{x},\ddot{y}\}=-\tfrac{1}{8}(eB_0/m)^2(1-\cos 2\omega t)\{x,y\}, \tag{30}$$

where the term involving p_φ has been omitted. To show that this term is really negligible, it is best to write Eq. (10) in the limit of $\omega r\ll 1$,

$$\ddot{r}=p_\varphi^2/m^2r^3-(eB_0/2m)^2r\sin^2\omega t, \tag{31}$$

where the term in question (involving p_φ linearly) has been dropped provisionally. Let $r(t)$ be a solution of (31) and t_0 a time at which $r(t_0)=r_M$ is a maximum so that $\dot{r}(t_0)=0$, $\ddot{r}(t_0)<0$. Now consider the "comparison equation":

$$\frac{d^2\rho}{dt^2}=\frac{p_\varphi^2}{m^2\rho^3}-\left(\frac{eB_0}{2m}\right)^2\rho,\quad \rho(t_0)=r_M,\quad \dot{\rho}(t_0)=0. \tag{32}$$

Since $d^2\rho/dt^2\leq\ddot{r}<0$, the value r_M is also a maximum of ρ:

$$\rho\leq r_M.$$

However, as a consequence of (32),

$$\dot{\rho}^2=-(p_\varphi^2/m^2\rho^2)-(eB_0/2m)^2\rho^2+\text{const},$$

so that one obtains for the maximum and minimum of ρ

$$\rho_M\rho_m=|2p_\varphi/eB_0|.$$

Hence

$$|p_\varphi/m|\omega^2\leq|eB_0/2m|(\omega r_M)^2\ll|eB_0/2m|.$$

Thus the last term of (10) and the last term of (28) are indeed negligible if $(\omega r)^2\ll 1$. The angular momentum p_φ still plays a role in the centrifugal term of (31) which is equivalent to (30).

One might have tried to analyze the stability by resolving the equations of the *actual* motion into Cartesian coordinates. Thereby one would certainly avoid the difficulty of the singular centrifugal force and the equations could be linearized. It turns out, however, that the linearized equations of the actual motion are coupled and contain the first derivatives of x and y as well as the second. Equations (30) are far simpler, and while they do not describe the actual motion, the $r(t)$ dependence is the same.

A simple transformation brings (30) into a standard form of the Mathieu equation:

$$d^2y/d\xi^2+(b-s\cos^2\xi)y=0,$$

where

$$b=s=(eB_0/2m\omega)^2.$$

From a table of characteristic values[7] one reads that the solutions of (30) remain bounded if $(eB_0/2m\omega)^2$ lies in one of the intervals:

$$0<(eB_0/2m\omega)^2\leq 1.315,$$
$$3.56\leq(eB_0/2m\omega)^2\leq 7.43,$$
$$12.16\leq(eB_0/2m\omega)^2\leq 18.51,\quad \text{etc.}$$

There are infinitely many such intervals, but only the first one appears to be of interest

$$|\omega|\geq 0.433|eB_0/m|. \tag{33}$$

This condition ensures stability of the orbit, that is, a bounded $r(t)$ for any initial conditions, provided $r(t)$ is small enough to justify the linearization (30).

In a sense, the criterion derived by the method in this section is complementary to the one obtained in Sec. A. The condition on the frequency has been relaxed and made precise (33), while it was necessary to impose the somewhat indefinite condition (24).

An exact stability criterion is still lacking, but one might speculate that it takes the form

$$f((eB_0/m\omega)^2, K, p_\varphi^2)\leq 1,$$

where f is an increasing function of its arguments. A number of solutions $r(t)$ have been obtained by numerical integration of (10). Although any such calculations must remain inconclusive since one cannot follow a solution for arbitrarily long times, it was observed that the orbits remained bounded as long as both (23) and (33) were satisfied, indicating that the conditions of either method are far more stringent than necessary.

[7] *Tables Relating to Mathieu Functions*, by U. S. National Bureau of Standards (Columbia University Press, New York, 1951).

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Landau's Model of Liquid He^3†

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Landau's model of liquid He^3 as a Fermi liquid is studied with regard to its lowest temperature properties. Primary stress is laid on the coefficient of thermal expansion. The spectra used by Abrikosov and Khalatnikov, the perfect-gas type and "bubble" type, are shown to give a positive coefficient of thermal expansion, in contradiction with experiment. An alternative simple spectrum is suggested which can give a negative coefficient of thermal expansion, namely $\Delta+p^2/2m^*$. In addition, the existence of a negative coefficient of thermal expansion is shown to imply a strong temperature dependence of the energy spectrum, which may cause a sharp deviation of the heat capacity curve from the perfect-gas type, similar to the deviations observed in liquid He^3.

1. INTRODUCTION

IN 1956 Landau[1] proposed a model with "Fermi-type spectrum" which is not necessarily temperature-independent nor interaction-free as in the case of ideal Fermi gas (hence the qualification "liquid"), and developed a general formalism of some properties of the model. Based on this, Khalatnikov and Abrikosov[2] discussed the thermodynamics of liquid He^3 assuming two particular spectra, perfect-gas type and "bubble" type, and concluded that the latter reproduces the temperature variation of the heat capacity, of the entropy, and of the magnetic susceptibility. This spectrum, however, gives a positive thermal expansion coefficient in contradiction to a recent experimental result,[3] as will be shown presently. It will be shown that Landau's model itself is general enough not to expose any defect in the "Fermi excitation region" below about 0.2°K.

We wish here to develop some formulas which can be derived on the basis of Landau's original idea, in as general a way as possible. Only one point at which we deviate from the idea is that we treat fermions with classical spin, i.e., Ising spin. This restriction allows us to develop unambiguous derivation of the formulas,

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