Hyperfine Splitting Effects in the Capture of Polarized μ^- Mesons

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We investigate the effect of the hyperfine splitting of μ-mesic atom ground states on the neutron asymmetry from muon capture in hydrogen, in deuterium, and in complex nuclei with spin which are treated by a one-particle model. It is shown that this can provide more information on the capture interaction than the neutron asymmetry from spinless nuclei. Muon polarizations and gyromagnetic ratios in the hyperfine states are also discussed.

I. INTRODUCTION

As Primakoff has pointed out,[1] μ-mesic atoms of nuclei with spin exist in two incoherent hyperfine states. This creates a difference in the lifetimes of the two states, and an estimate of such an effect has been published.[2] Here we deal with those aspects of the hyperfine splitting which concern polarized muons. All such effects will be rather small, however, as the practically 100% polarized μ^- mesons from pion decay suffer an appreciable loss of polarization while cascading down to the lowest Bohr orbit of the mesic atom.[3,4] The hyperfine interaction in nuclei with spin depolarizes the muons further while partly polarizing the nucleons. This will be discussed in the following section,* as well as the gyromagnetic ratios in the hyperfine states. The last section treats the neutron asymmetries from muon capture, which differ even after averaging over the hyperfine states from those in the capture by spinless nuclei, due to the induced nuclear polarization. We study muon capture in hydrogen, deuterium, and in odd A, odd Z nuclei which are described, as in reference 2, by the Schmidt model[5] of a spinless core and "outside" proton. These neutron asymmetries depend on Fermi and Gamow-Teller interference terms, unlike the asymmetries from capture in spinless nuclei.[6]

II. POLARIZATIONS AND MAGNETIC MOMENTS IN HYPERFINE STATES

Negative muons are strongly depolarized while cascading down to the K-shell of the mesic atom. This is caused by the spin-orbit coupling, the interaction of the muon spin with the shell electron magnetic moments, and—for nuclei with spin—the hyperfine interaction of muon and nuclear spin. The measurements for spinless nuclei indicate a remaining polarization of 14 to 20% depending on the target material.[3] In nuclei with spin, the hyperfine coupling will further depolarize the muon. This is certainly the case in the ground state, whose width is $\Delta\nu\sim 10^6-10^9$ sec^{-1} as determined by the muon lifetime; the hyperfine splitting[7]

$$\Delta\nu_{\mathrm{hf}}=\Delta\nu_{\mathrm{H}}(m_\mu/m_e)^2Z^3\sim 6\times 10^{13}Z^3, \tag{1}$$

(using $\Delta\nu_{\mathrm{H}}=1.42\times 10^9$ sec^{-1}, the hyperfine splitting in hydrogen[8]), then assures existence of the mesic atom in two incoherent hyperfine states. For excited states, the widths are in general too large[9] to make this happen.

The hyperfine coupling differs from the other depolarizing mechanisms insofar as a muon spin flip will induce a nuclear spin flip; thus what is lost in muon polarization may be gained in nuclear polarization. This is well known from the Overhauser effect[10]; we shall illustrate it for the ground state of the mesic hydrogen atom as follows. A muon with spin up, α_μ, can combine with an unpolarized proton to form the states $\alpha_\mu\alpha_p$, $\alpha_\mu\beta_p$. Of these, the first one is an eigenstate of the total angular momentum, both spins remain constant in time. The second state is mixed from eigenstates of zero and one total angular momentum, thus muon and proton spins precess around each other and the proton depolarizes. The average over both states results in 50% proton (and also muon) polarization.

If the muon arrives in the K-shell with 100% polarization in the z-direction, the populations of the hyperfine states characterized by $F=J\pm\frac{1}{2}$, M_F (J= nuclear spin) are

$$N_+(M_F)=\frac{J+M_F+\frac{1}{2}}{(2J+1)^2},\quad N_-(M_F)=\frac{J-M_F+\frac{1}{2}}{(2J+1)^2}, \tag{2}$$

normalized in such a way that

$$N_+=\sum_{M_F}N_+(M_F)=\frac{J+1}{2J+1},$$
$$N_-=\sum_{M_F}N_-(M_F)=\frac{J}{2J+1}, \tag{3}$$

[1] H. Primakoff (unpublished).
[2] Bernstein, Lee, Yang, and Primakoff, Phys. Rev. **111**, 313 (1958).
[3] Ignatenko, Egorov, Khalupa, and Chultem, Zhur. Eksptl. i Teoret. Fiz. USSR **35**, 1131 (1958) [Translation: Soviet Phys. JETP **8**, 792 (1959)].
[4] M. E. Rose, *Proceedings of the Gatlinburg Conference on Weak Interactions*, 1958 [Bull. Am. Phys. Soc. Ser. II, **4**, 80 (1959)].
* *Note added in proof.*—Professor V. L. Telegdi kindly pointed out to us that he has investigated these muon depolarizations caused by the hyperfine interaction: International Conference on Mesons and Recently Discovered Particles, Padua-Venice, 1957. Communicazioni, Padova-Venezia, 22–28 Settembre 1957. Padova, 1958.
[5] See, e.g., J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).
[6] H. Überall, Nuovo cimento **6**, 533 (1957).
[7] N. F. Ramsey, *Nuclear Moments* (John Wiley & Sons, Inc., New York, 1953), p. 10.
[8] D. Halliday, *Introductory Nuclear Physics* (John Wiley & Sons, Inc., New York, 1955), second edition, p. 379.
[9] G. R. Burbidge and A. H. de Borde, Phys. Rev. **89**, 189 (1953).
[10] A. W. Overhauser, Phys. Rev. **92**, 411 (1953).

and $N_+ + N_- = 1$. The subscripts $\pm$ refer to $F = J \pm \frac{1}{2}$. The muon polarizations are after averaging over the substates using (2):

$$\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_+ = \frac{1}{3}\frac{2J+3}{2J+1}, \quad \langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_- = \frac{1}{3}\frac{2J-1}{2J+1}, \tag{4}$$

($\mathbf{s}$ is a unit vector in the z-direction); finally the muon polarization averaged over both hyperfine states is

$$\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_0 = N_+\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_+ + N_-\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_- = \frac{1}{3}\left[1+\frac{2}{(2J+1)^2}\right]. \tag{5}$$

This is $\frac{1}{2}$ for $J=\frac{1}{2}$, as mentioned above; it is $\frac{1}{3}$ for $J \gg 1$, as we can see from a simple classical argument: only the component of $\boldsymbol{\sigma}_\mu$ along $\mathbf{F}$, as the time average of $\boldsymbol{\sigma}_\mu$, can be observed, and all directions of $\mathbf{J}$ are possible for an unpolarized nucleus. Thus for $J \rightarrow \infty$, $\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_0 = \langle \cos^2\theta\rangle = \frac{1}{3}$. Only between $\frac{1}{2}$ and $\frac{1}{3}$ of the muon polarization is preserved in the ground state due to the hyperfine coupling, but possibly more if the hyperfine states are taken separately.

In the foregoing, we assumed no appreciable relaxation between the two hyperfine states (which would upset the distribution of population (3), valid at the moment of arrival of the muon in the K-shell). In general, nuclear relaxation times are indeed long compared to the muon lifetime. For hydrogen and deuterium however, transitions to the lowest hyperfine levels should be induced rapidly by collisions of molecules.[11] Our treatment of muon capture by hydrogen and deuterium in the following section, using (3), has therefore to be accepted *cum grano salis*, remembering that in practice all the population may be concentrated in the lowest hyperfine level (except possibly for very rarefied gaseous targets).

The magnetic moments of the two hyperfine states are

$$\mu_+ = \mu_\mu + \mu_N, \quad \mu_- = -\frac{2J-1}{2J+1}\left(\mu_\mu - \frac{J+1}{J}\mu_N\right), \tag{6}$$

or written in terms of gyromagnetic ratios $\gamma = \mu/F$:

$$\gamma_+ = \frac{1}{J+\frac{1}{2}}(\mu_\mu + \mu_N), \quad \gamma_- = -\frac{1}{J+\frac{1}{2}}\left(\mu_\mu - \frac{J+1}{J}\mu_N\right), \tag{7}$$

where μ_μ, μ_N are the magnetic moments of muon and nucleus. The gyromagnetic ratio of a free muon is $\gamma_\mu = 2\mu_\mu$. If the mesic atom is exposed to a magnetic field $\mathbf{H} \perp \mathbf{s}$, as in the experiment of Garwin *et al.*,[12] the two states will precess with different frequencies $\omega_\pm = \gamma_\pm H$. This could be checked by observing the emission asymmetries of the mu-decay electrons, which also precess with these frequencies.

III. HYPERFINE EFFECTS IN NEUTRON ASYMMETRY FROM MUON CAPTURE

Due to parity violation, neutrons from capture of the muon by a proton in the nucleus may show an angular distribution asymmetric around the muon spin direction.[6,13–15] As far as capture in hydrogen, deuterium, and other nuclei with spin is concerned, the calculations have ignored the presence of the hyperfine splitting. In the following, we shall investigate this effect.

Neutron asymmetries can be measured with the muon spin kept fixed, which will give a result corresponding to an average over both hyperfine states. Alternatively, a precession experiment[12] may be used which permits a determination of the asymmetries from the two states separately; this will be of interest too, as the separate asymmetries are quite different from each other and from their average.

To work out the theoretical values, we use the nonrelativistic form of the Hermitian conjugate of Lee and Yang's Hamiltonian,[16] divided by $\sqrt{2}$ and with e replaced by μ; an effective pseudoscalar coupling[17] is included in first nonrelativistic approximation,[18] assumed to be caused by virtual pion effects. A further hypothetical effect of virtual pions, which was postulated by Gell-Mann[19] to explain the absence of renormalization in vector interactions, will not be considered here.

Further notations to be used are

$$\begin{aligned} G_F &= C_S + C_V, \quad G_F' = C_S' + C_V', \\ G_G &= C_T + C_A, \quad G_G' = C_T' + C_A', \\ G_P &= C_P\bar{\nu}/2M, \quad G_P' = C_P'\bar{\nu}/2M, \end{aligned} \tag{8}$$

with $\bar{\nu}$ the average momentum of the emitted neutrino, and M the nucleon mass.

We shall introduce the density matrix of a muon-proton system for the situation that the muon is in the ground state of a mesic atom consisting of two incoherent hyperfine levels, and the proton having no polarization at the moment of arrival of the muon in the K-shell. Written down generally for any of the

[11] S. S. Gershtein, Zhur. Eksptl. i Teoret. Fiz. U.S.S.R. **34**, 463 and 993 (1958) [translations: Soviet Phys. JETP **7**, 318 and 685 (1958)].

[12] Garwin, Lederman, and Weinrich, Phys. Rev. **105**, 1415 (1957).

[13] Shapiro, Dolinsky, and Blokhintsev, Nuclear Phys. **4**, 273 (1957); L. Wolfenstein, Nuovo cimento **7**, 706 (1958).

[14] E. I. Dolinsky and L. D. Blokhintsev, Zhur. Eksptl. i Teoret. Fiz. U.S.S.R. **34**, 759 (1958) [translation: Soviet Phys. JETP **7**, 521 (1958)]; B. L. Ioffe, Zhur. Eksptl. i Teoret Fiz. U.S.S.R. **33**, 308 (1957) [translation: Soviet Phys. JETP **6**, 240 (1958)].

[15] H. Überall and L. Wolfenstein, Nuovo cimento **10**, 136 (1958).

[16] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956), Eq. (A.1).

[17] L. Wolfenstein, Nuovo cimento **8**, 882 (1958); M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[18] The minus sign given to the pseudoscalar interaction term in Eq. (2) of reference 15 should be changed to plus. Furthermore, the last two figures in column D of Table I in the same paper should read -0.560, -1.000 (instead of -0.628, -1.124), and the last figure in column D_0 should read -1 (instead of $-\frac{4}{3}$).

[19] M. Gell-Mann, Phys. Rev. **111**, 362 (1958).

hyperfine states or their incoherent mixture, we find

$$\rho=\tfrac{1}{4}N[1+\tfrac{1}{3}\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p+\langle\mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle\mathbf{s}\cdot(\boldsymbol{\sigma}_\mu+\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle\boldsymbol{\sigma}_p)], \quad (9)$$

where N is the probability of the state under consideration, [Eq. (3)], and $\langle\mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle$, [Eqs. (4), (5)], and $\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle$ are expectation values of polarization and spin correlation, respectively.

Considering the capture in hydrogen, we use $\langle\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu\rangle_+=1$, $\langle\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu\rangle_-=-3$, $\langle\mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_+=\frac{2}{3}$, $\langle\mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_-=0$, assuming an initial 100% polarization of the muon. The transition probability is then obtained from ρ by the substitution

$$\begin{aligned}
1&\rightarrow a, & a&=\tfrac{1}{2}(a_{FF}+3a_{GG}-2\,\mathrm{Re}\,a_{GP}),\\
\tfrac{1}{3}\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_\mu&\rightarrow b, & b&=\tfrac{1}{2}(2\,\mathrm{Re}\,a_{FG}-2a_{GG}-\tfrac{2}{3}\,\mathrm{Re}\,a_{FP}+\tfrac{4}{3}\,\mathrm{Re}\,a_{GP}), \quad (10)\\
\mathbf{s}\cdot\boldsymbol{\sigma}_\mu&\rightarrow\mathbf{s}\cdot\hat{\nu}c, & c&=\tfrac{1}{2}(-b_{FF}+b_{GG}+2\,\mathrm{Re}\,b_{GP}),\\
\mathbf{s}\cdot\boldsymbol{\sigma}_p&\rightarrow\mathbf{s}\cdot\hat{\nu}d, & d&=\tfrac{1}{2}(-2\,\mathrm{Re}\,b_{FG}-2b_{GG}+2\,\mathrm{Re}\,b_{FP}),
\end{aligned}$$

where

$$a_{lm}=G_lG_m^*+G_l'G_m'^*,\quad b_{lm}=G_l'G_m^*+G_lG_m'^*;$$

$\hat{\nu}$ is a unit vector in the direction of the neutrino momentum. Terms quadratic in C_P have been dropped. We introduce a quantity $I_\pm$ for the two hyperfine states, which is proportional to the capture probability, and so defined that I_++I_- coincides with I_0 given by Wolfenstein.[13] The angular distribution is written as $1-A_\pm\mathbf{s}\cdot\hat{p}$, or as $1-\bar{A}\mathbf{s}\cdot\hat{p}$ after averaging over both states ($\mathbf{p}$ being the momentum of the emitted neutron, and $\hat{p}$ a unit vector in its direction). We then obtain:

$$\begin{aligned}
I_+&=\tfrac{3}{8}(a_{FF}+2\,\mathrm{Re}\,a_{FG}+a_{GG}-\tfrac{2}{3}a_{FP}-\tfrac{2}{3}a_{GP}),\\
I_-&=\tfrac{1}{8}(a_{FF}-6\,\mathrm{Re}\,a_{FG}+9a_{GG}+2\,\mathrm{Re}\,a_{FP}-6\,\mathrm{Re}\,a_{GP}),\\
I_0&=\tfrac{1}{2}(a_{FF}+3a_{GG}-2\,\mathrm{Re}\,a_{GP}), \quad (11)\\
(8/3)I_+A_+&=-\tfrac{2}{3}(b_{FF}+2\,\mathrm{Re}\,b_{FG}+b_{GG}-2\,\mathrm{Re}\,b_{FP}-2\,\mathrm{Re}\,b_{GP}),\\
A_-&=0, \quad (12)\\
2I_0\bar{A}&=-\tfrac{1}{2}(b_{FF}+2\,\mathrm{Re}\,b_{FG}+b_{GG}-2\,\mathrm{Re}\,b_{FP}-2\,\mathrm{Re}\,b_{GP}).
\end{aligned}$$

The singlet state asymmetries A_- are zero because of the vanishing muon polarization. Table I gives $I_\pm$, I_0

TABLE I. Capture probabilities and neutron asymmetries from the different hyperfine states in hydrogen, for various coupling types; A_+=asymmetry from triplet state, A_-=asymmetry from singlet state, $\bar{A}$=asymmetry averaged over both hyperfine states, A_0=asymmetry without hyperfine interaction.

Coupling	I_+	I_-	I_0	A_+	A_-	$\bar{A}$	A_0
F	$\frac{3}{4}G^2$	$\frac{1}{4}G^2$	G^2	$\frac{2}{3}$	0	$\frac{1}{2}$	1
G	$\frac{3}{4}G^2$	$(9/4)G^2$	$3G^2$	$\frac{2}{3}$	0	$\frac{1}{6}$	$-\frac{1}{3}$
$F+G$	$3G^2$	G^2	$4G^2$	$\frac{2}{3}$	0	$\frac{1}{2}$	0
$F-G$	0	$4G^2$	$4G^2$	—	0	0	0
$F-(G\pm P)$	0	$2(2-x)G^2$	$2(2-x)G^2$	—	0	0	$-\frac{1}{2}x$

and the asymmetries A_+ (triplet), A_-, $\bar{A}$ (average) and A_0 (without hyperfine effect) for right-handed two-component neutrinos (for left-handed two-component neutrinos, all asymmetries reverse sign) and couplings pure or mixed with equal weights; we call $G_P/G_G=x$, $=\pm0.475$ for $C_P=9G$.[17] Owing to the F, G interference terms present, we now get a difference in the asymmetries for $F+G$ and $F-G$ (in contrast to A_0); but for $F-G$, there is no capture in the triplet state, so that $\bar{A}=0$, a result which is not changed by the effective pseudoscalar interaction (at least if terms quadratic in C_P are neglected). This is again in contrast to A_0.

For capture in deuterium, we neglect again relaxations of the hyperfine states and initial nuclear polarization. The notation of reference 15 is used throughout. The density matrices are explicitly:

$$\begin{aligned}
\rho_+=\tfrac{1}{6}\{&\tfrac{1}{6}(3+\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n)+\tfrac{1}{6}\boldsymbol{\sigma}_\mu\cdot(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n)\\
&+(1/18)\mathbf{s}\cdot[5(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n+\boldsymbol{\sigma}_\mu)+\boldsymbol{\sigma}_p(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_n)+\boldsymbol{\sigma}_n(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p)+\boldsymbol{\sigma}_\mu(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n)]\},\\
\rho_-=\tfrac{1}{6}\{&\tfrac{1}{12}(3+\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n)-\tfrac{1}{6}\boldsymbol{\sigma}_\mu\cdot(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n)\\
&+(1/18)\mathbf{s}\cdot[-(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n-\tfrac{1}{2}\boldsymbol{\sigma}_\mu)+\boldsymbol{\sigma}_p(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_n)+\boldsymbol{\sigma}_n(\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p)-\tfrac{1}{2}\boldsymbol{\sigma}_\mu(\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n)]\},
\end{aligned}$$

and with their help, we find the following capture probabilities and asymmetries:

$$\begin{aligned}
I_+&=\tfrac{2}{3}[a_{FF}+2\,\mathrm{Re}\,a_{FG}+a_{GG}-\tfrac{2}{3}\,\mathrm{Re}\,a_{FP}-\tfrac{2}{3}\,\mathrm{Re}\,a_{GP}]I_{tt},\\
I_-&=\tfrac{1}{3}[a_{FF}I_{tt}-4\,\mathrm{Re}\,a_{FG}I_{tt}+a_{GG}(4I_{tt}+3I_{ss})+\tfrac{4}{3}\,\mathrm{Re}\,a_{FP}I_{tt}-\tfrac{2}{3}\,\mathrm{Re}\,a_{GP}(4I_{tt}+3I_{ss})], \quad (12)\\
I_0&=a_{FF}I_{tt}+a_{GG}(2I_{tt}+I_{ss})-\tfrac{2}{3}\,\mathrm{Re}\,a_{GP}(2I_{tt}+I_{ss}),\\
\tfrac{3}{2}I_+A_+&=(5/9)[b_{FF}+2\,\mathrm{Re}\,b_{FG}+b_{GG}-2\,\mathrm{Re}\,b_{FP}-2\,\mathrm{Re}\,b_{GP}]I_{tt}',\\
3I_-A_-&=\tfrac{1}{9}[b_{FF}I_{tt}'-4\,\mathrm{Re}\,b_{FG}I_{tt}'+b_{GG}(4I_{tt}'-9I_{ss}')+4\,\mathrm{Re}\,b_{FP}I_{tt}'-2\,\mathrm{Re}\,b_{GP}(4I_{tt}'-3I_{ss}')], \quad (13)\\
I_0\bar{A}&=(1/27)[11b_{FF}I_{tt}'+16\,\mathrm{Re}\,b_{FG}I_{tt}'+b_{GG}(14I_{tt}'-9I_{ss}')+16\,\mathrm{Re}\,b_{FP}I_{tt}'+2\,\mathrm{Re}\,b_{GP}(14I_{tt}'-3I_{ss}')].
\end{aligned}$$

In reference 15, I_{jj} and I_{jj}' were evaluated as functions of the neutron energy, taking into account interactions between the final two neutrons. Their indices $j=s$ or t correspond to singlet or triplet states of the outgoing neutrons. Some general features of the results may be discussed:

(1) For pure Fermi couplings (no spin flip), only final triplet states occur. The I_+, I_- have the Fermi shape (Fig. 1, reference 15) with weights $\frac{4}{3}G^2$, $\frac{2}{3}G^2$. The d of Eq. (10), factor of the proton polarization, vanishes for pure Fermi interactions, thus the asymmetries A_+, A_- and $\bar{A}$ are the Fermi curve (Fig. 2, reference 15) multiplied by just the muon polarizations $\langle\mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle$, which

are 5/9, 1/9, and 11/27, respectively. (Right-handed two-component neutrinos will be assumed always.)

(2) For initial quartet states, $F=J+\frac{1}{2}$, only final triplet states occur. Thus, I_+ and A_+ have the Fermi shape independently of coupling type (except for $F-G$, $F-(G\pm P)$, where $I_+=0$).

(3) The exclusion principle enhances final singlet compared to triplet states. This is the case mostly for neutrons of 1 to 3 Mev,[15] and if in Eqs. (12) and (13), we set $I_{tt}\ll I_{ss}$, we obtain $A_-=\bar{A}=\frac{1}{3}I_{ss}'/I_{tt}'\approx-\frac{1}{4}$ in this energy range, for all except pure Fermi couplings, in good agreement with the calculated values. For energies above $\sim$6 Mev, however, the results are better estimated by setting

$$I_{tt}'\approx I_{ss}'\approx-\tfrac{3}{4}\times(I_{tt}\approx I_{ss}). \qquad (14)$$

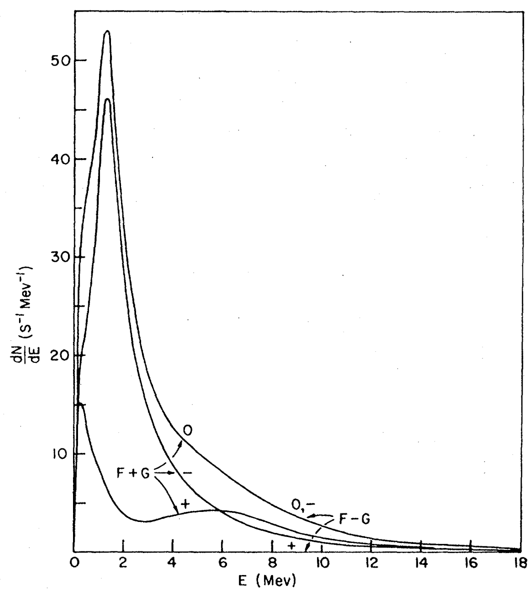

FIG. 1. Neutron spectra for both hyperfine states ($\pm$) and their sum (0) for $F+G$ and $F-G$ coupling, normalized by using the Fermi coupling constant of β decay, $G=1.41\times10^{-49}$ erg cm^3.

Of most immediate interest will be, however, to point out the new features brought in by the hyperfine splitting effect, as compared to the treatment[15] where the hyperfine interaction was left out. The first result is again a difference between $F+G$ and $F-G$ couplings, illustrated by Figs. 1 and 2. In the former case, I_+ has the Fermi shape and is smaller than I_-; A_+ is also given by the Fermi curve [factor (5/9)], and A_-, $\bar{A}$ are approximately -25% for $E\sim2$ Mev, and -20%, $+20\%$, respectively for $E\gtrsim6$ Mev. In the $F-G$ case, $I_+=0$ (even if an effective pseudoscalar coupling is added), and $A_-=\bar{A}$ is $\sim-25\%$ for $E\sim2$ Mev, zero for $E\gtrsim6$ Mev. The average spectrum I_0 is the same for both cases and for the situation without hyperfine interaction.

Another striking new result is that an effective pseudoscalar interaction added to $F-G$ does not appreciably

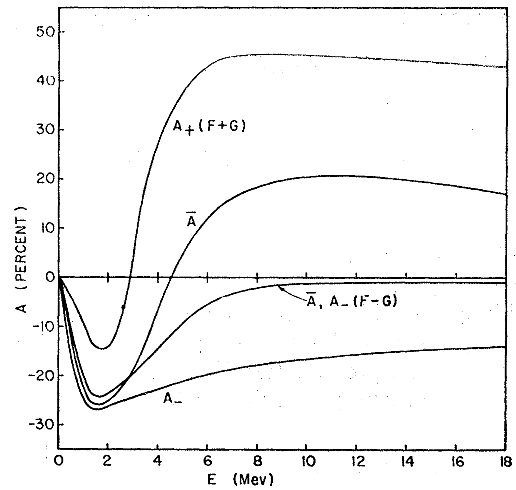

FIG. 2. Asymmetry parameter A for both hyperfine states ($A_\pm$) and its average ($\bar{A}$) *vs* neutron energy E, for $F+G$ and $F-G$ coupling using two-component theory with right-handed neutrinos.

change the asymmetry in Fig. 2, contrary to the old results (reference 15, Fig. 3) where the pseudoscalar was very important, especially in the 1 to 3 Mev region. We note in this connection that A_- and $\bar{A}$ are still $\sim-25\%$ here, in spite of the small muon polarization ($\frac{1}{9}$ for the doublet state!). The proton polarization ($-2/9$ for A_-) adds on to that of the muon, and the relative largeness of the asymmetry is therefore due predominantly to the proton polarization. The unimportance of the pseudoscalar in this energy region is then clear by observing that the coefficient of the proton polarization, d of Eq. (10), has no GP interference term, only an FP one which is unimportant here.

Last, we shall consider muon absorption in odd A, odd Z nuclei, using the same model as in reference 2, namely a spinless core and an odd outside proton which is responsible for all the nuclear spin. Capture by a core proton will then result in a neutron asymmetry similar to that in a zero-spin nucleus,[6] and capture by the odd proton will resemble more the capture in hydrogen discussed above. For an odd A, even Z nucleus, this model would give the same asymmetry as for a spinless nucleus. To utilize the density matrix (9) for the capture by the odd proton, we first calculate the muon-proton spin correlation with the proton in a state of nonzero orbital angular momentum. If $F=J\pm\frac{1}{2}$, $J=$nuclear spin$=L\pm\frac{1}{2}$, $L=$odd proton orbital angular momentum, we find

$$\begin{aligned}\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle_+&=\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J+1},\\ \langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle_-&=-\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J}.\end{aligned} \qquad (15)$$

This vanishes when averaged over the hyperfine states;

however, the last term in the density matrix averages to

$$\begin{aligned}(\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle)_0 &= N_+\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_+\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle_+ \\ &\quad + N_-\langle \mathbf{s}\cdot\boldsymbol{\sigma}_\mu\rangle_-\langle\boldsymbol{\sigma}_\mu\cdot\boldsymbol{\sigma}_p\rangle_- \\ &= -\frac{4}{3}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{(2J+1)^2}; \end{aligned} \tag{16}$$

therefore, a certain proton polarization remains even after averaging.

Adding the odd-proton and the core contributions together, we find the following expressions for the relative capture probabilities (defined such as to coincide with the hydrogen results for $L=0$, $Z=1$):

$$\begin{aligned} I_+ &= \frac{J+1}{2J+1}\left[a+\frac{1}{Z'}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J+1}b\right], \\ I_- &= \frac{J}{2J+1}\left[a-\frac{1}{Z'}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J}b\right], \\ I_0 &= a; \end{aligned} \tag{17}$$

and for the asymmetries:

$$\begin{aligned} \frac{2J+1}{J+1}I_+A_+ &= P_\mu\frac{\epsilon}{3}\frac{2J+3}{2J+1}\left[c+\frac{1}{Z'}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J+1}d\right], \\ \frac{2J+1}{J}I_-A_- &= P_\mu\frac{\epsilon}{3}\frac{2J-1}{2J+1}\left[c-\frac{1}{Z'}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{J}d\right], \\ I_0\bar{A} &= P_\mu\frac{\epsilon}{3}\left[1+\frac{2}{(2J+1)^2}\right] \\ &\quad\times\left[c+\frac{4}{Z'}\frac{J(J+1)-L(L+1)+\frac{3}{4}}{2+(2J+1)^2}d\right]. \end{aligned} \tag{18}$$

Here, P_μ is the muon polarization at the moment of arrival in the K shell, $Z'=(Z-1)\xi+1$, and we introduced two quantities ξ, ϵ, which are functions of the energy E of the emitted neutrons: $\xi(E)$ is the ratio of muon capture probability by a core proton to the capture probability by the outside proton, the difference being caused by a difference in phase space and mainly by the exclusion principle. It can be estimated by treating the problem with a Fermi gas model,[6] or from Primakoff's formula for muon capture,[20] and is found to be $\xi\sim 0.20$ to 0.25 if the odd proton can be treated as free, somewhat larger if the odd proton is also appreciably influenced by the exclusion principle. The function $\epsilon(E)$ is defined for capture in nuclei by

$$1+A\mathbf{s}\cdot\hat{\nu}=1-\epsilon A\mathbf{s}\cdot\hat{p},$$

and represents a "smearing-out" of the neutrino asymmetry as reflected by the neutron asymmetry, this being caused by the proton motion in the nucleus. It is estimated from the Fermi gas model[6] as $0.7\leqslant\epsilon\leqslant 1.0$, going up for the more energetic neutrons which reproduce the neutrino asymmetry more exactly.

Formulas (17) express the hyperfine effect on the lifetime as discussed by Bernstein *et al.*,[2] and (18) the effect on the neutron asymmetry. They show essentially a mixing of the asymmetries caused by polarized muons only (determined by c/a), and by polarized protons only (determined by d/b), with "weights" Z' and 1. The interesting fact is again a distinction between the cases $F+G$ and $F-G$; the asymmetries vanish for the latter interaction type, whereas for the former case, $c=0$ (and $b=0$), so that the asymmetry is essentially determined by (d/aZ'), $=1/Z'$ for right-handed two-component neutrinos.

IV. DISCUSSION

We pointed out the existence of the hyperfine splitting effect on the neutron asymmetry, as it had been left out of consideration in all earlier papers on this subject. An obstacle for any experimental verification is unfortunately presented by the small remaining polarization P_μ of the muon as it reaches the K-shell. Our discussion of muon capture in hydrogen and deuterium wants only to show the intrinsic size of the effect rather than to suggest an experiment, for which the small capture rate would at present be prohibitive. The place where measurements could be attempted are light nuclei (because of the factor Z'^{-1} of d), with an aim to see the d-term. This can be done in two ways. First, one might compare the asymmetry A_0 from a spinless nucleus with $\bar{A}$ from a nucleus which differs from the former one by possessing an additional proton. An example would be O^{18} and F^{19}, the latter nucleus having $J=\frac{1}{2}$ and lying almost exactly on the $L=J-\frac{1}{2}$ Schmidt line. One should first check the reduction of muon polarization in the fluorine (expected $\frac{1}{2}$ of that in oxygen) by measuring the decay electron asymmetries, then compare the neutron asymmetries. For $F+G$ coupling, e.g., A_0 should be zero, but $\bar{A}/\frac{1}{2}P_\mu\epsilon\sim 30\%$. The second way is to compare A_+ and A_- in the same nucleus, using a precession experiment. If we take, e.g., B^{11} with $J=\frac{3}{2}$, $L=J-\frac{1}{2}$, the muon polarization ratio (3 : 1) in the $+$, $-$ states could be checked using the decay electrons, then the A_+, A_- compared. These should show opposite signs if $F+G$

[20] H. Primakoff, Revs. Modern Phys. (to be published); quoted, e.g., by J. C. Sens, Phys. Rev. **113**, 679 (1959), Eq. (5), or by Astbury, Kemp, Lipman, Muirhead, Voss, Zangger, and Kirk, Proc. Phys. Soc. (London) **72**, 494 (1958), Eq. (2).

coupling applies, and for the B^{11} case would be of the order $A_+/P_\mu\epsilon \sim 22\%$, $A_-/P_\mu\epsilon \sim -12\%$.

The actual coupling, even if all weak interactions are universal, will be neither $F-G$ nor $F+G$, but the Fermi and Gamow-Teller couplings are expected to be different owing to pion renormalization effects, similarly as in β decay. Moreover, relativistic effects can cause appreciable deviations from the nonrelativistic formulas, especially in complex nuclei, and of course the applicability of the Schmidt model may be questioned.

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Model of a Linear Harmonic Oscillator in the General Theory of Relativity

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A model of a linear harmonic oscillator in the general theory of relativity is examined. It is based on a classical model of a point mass vibrating harmonically about the center of an ideal liquid sphere. The motion is no longer harmonic. However, to a first approximation, it is still harmonic, but the period of vibration depends both on the amplitude and on the curvature of the space. Both quantities tend to slow down the motion. In the limiting case of flat space, the classical frequency is again restored.

I. INTRODUCTION

THERE is no unique way to define a harmonic oscillator in the general theory of relativity. The definition depends upon the physical properties of the classical oscillator we wish to preserve in the general theory. One possibility is to define a suitable potential energy which in the limiting case of flat space becomes identical with the usual harmonic potential. In any case, the equations of motion must become identical with the classical equations in the limiting case. Our approach is based upon the following observation. It is well known that, in Newtonian mechanics, a small mass vibrating under gravity about the center of a sphere composed of an ideal liquid will execute a simple harmonic motion. We are thus led to investigate how a mass point will move in the same medium if the curvature of the space is taken into account.

II. THE EQUATIONS OF MOTION

In a stationary system of spherical symmetry, the world line-element can be written in the form[1]

$$ds^2 = g_0 dt^2 - g_1 dr^2 - r^2(d\psi^2 + \sin^2\psi d\varphi^2), \tag{1}$$

where g_0 and g_1 are functions of r only. We shall put:

$$x_0 = t, \quad x_2 = \psi, \qquad x_1 = r, \quad x_3 = \varphi. \tag{2}$$

The nonvanishing Christoffel-symbols are

$$\Gamma_{01}{}^0 = g_0'/(2g_0), \tag{3a}$$

$$\Gamma_{00}{}^1 = g_0'/(2g_1), \tag{3b}$$

$$\Gamma_{11}{}^1 = g_1'/(2g_1), \tag{3c}$$

$$\Gamma_{22}{}^1 = -r/g_1, \tag{3d}$$

$$\Gamma_{33}{}^1 = -r\sin^2\psi/g_1, \tag{3e}$$

$$\Gamma_{12}{}^2 = 1/r, \tag{3f}$$

$$\Gamma_{33}{}^2 = -\sin\psi\cos\psi, \tag{3g}$$

$$\Gamma_{13}{}^3 = 1/r, \tag{3h}$$

$$\Gamma_{23}{}^3 = \cot\psi. \tag{3i}$$

(Primes indicate derivative with respect to r.)

The equations of motion of a point mass are

$$\frac{d^2x^i}{ds^2} + \Gamma_{kl}{}^i \frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \tag{4}$$

In our case, Eqs. (4) reduce to

$$\frac{d^2t}{ds^2} + 2\Gamma_{01}{}^0 \frac{dt}{ds}\frac{dr}{ds} = 0, \tag{5a}$$

$$\frac{d^2r}{ds^2} + \Gamma_{00}{}^1\left(\frac{dt}{ds}\right)^2 + \Gamma_{11}{}^1\left(\frac{dr}{ds}\right)^2 + \Gamma_{22}{}^1\left(\frac{d\psi}{ds}\right)^2 + \Gamma_{33}{}^1\left(\frac{d\varphi}{ds}\right)^2 = 0, \tag{5b}$$

$$\frac{d^2\psi}{ds^2} + 2\Gamma_{12}{}^2 \frac{dr}{ds}\frac{d\psi}{ds} + \Gamma_{33}{}^2\left(\frac{d\varphi}{ds}\right)^2 = 0, \tag{5c}$$

$$\frac{d^2\varphi}{ds^2} + 2\Gamma_{13}{}^3 \frac{dr}{ds}\frac{d\varphi}{ds} + 2\Gamma_{23}{}^3 \frac{d\psi}{ds}\frac{d\varphi}{ds} = 0. \tag{5d}$$

[1] Hans Bauer, Anz. Akad. Wiss. Wien, Math.-naturw. Kl. Sitzber. Abt. IIa, 127, 10 (1918).