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Possible Superfluidity of a System of Strongly Interacting Fermions*†

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The possible superfluidity of a system of strongly interacting fermions is investigated on the assumption that an adequate description of the system in its "normal" state is given by independent fermions in a momentum-dependent potential. On the basis of this assumption we have investigated whether a correlated wave function of the form used by Bardeen, Cooper, and Schrieffer minimizes the ground-state energy. The nonzero terms in the expectation value of the Hamiltonian contain the modified kinetic energy and the full two-body potential between the fermion pairs. An integral equation is obtained in configuration space for the correlation function between pairs. This integral equation is meaningful even for potentials with hard cores, and a nonzero solution implies the existence of a superfluid state. A variational method is devised which provides a criterion for superfluidity and a lower bound for the transition temperature into the superfluid state. We find that a repulsive hard core does not in principle forbid the existence of a superfluid state, but whereas in the absence of a hard core an attractive two-body potential always leads to a superfluid state at sufficiently low temperatures, in the presence of a repulsive core there appears to be a critical strength of attraction needed to form a superfluid state. When the variational principle is applied to liquid He^3 or to nuclear matter, it is found for a wide class of trial functions that the system does not become a superfluid.

I. INTRODUCTION

THE fact that a system of fermions can become a superfluid is demonstrated by the observed behavior of the electron gas in many metals at low temperatures. It seems natural then to inquire whether or not other systems of fermions might display similar properties, and what the criterion for such behavior would be. This question is of particular interest because of recent conjectures[1] that nuclear matter might be superfluid in the sense that for an infinite medium there would be an energy gap between the ground state and the lowest single particle excitations. It has been further conjectured that this might show up for a finite nucleus as the explanation for the abnormally large single-particle excitation energy of even-even nuclei. Whether or not He^3, the other well-known Fermi fluid, has a superfluid phase at low temperatures has been a matter of concern since the discovery of the λ transition in He^4 and London's conjecture[2] that Bose statistics are crucial to the formation of the superfluid phase.

We have attempted to treat this question by taking over to an arbitrary system of fermions what appears to have worked very well for the electron gas. There the introduction of pair correlations into the wave function and the approximation that only pairs of given total spin and total momentum are strongly correlated was sufficient to account for the observed properties of the superconducting phase. In making this same assumption for an arbitrary system of fermions we have had to assume that in some sense the "normal" fluid could be described in an uncorrelated approximation. The situation in this regard, for a fermion system such as the nucleus, is much less clear than the corresponding situation in a metal where the lattice plays such a dominant rôle.

This basic conjecture of our procedure, the description of the normal fluid as a Fermi gas in a momentum-dependent potential, is discussed in Sec. II. In the third

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[1] Bohr, Mottelson, and Pines, Phys. Rev. **110**, 936 (1958); C. De Dominicis and P. C. Martin, Bull. Am. Phys. Soc. Ser. II, 3, 224 (1958).

[2] F. London, Nature **163**, 694 (1949).

section we derive an integral equation for the pair correlation function which is well defined for arbitrary potentials, and in the fourth, a criterion for superfluidity is presented. It is shown that a hard repulsive core does not prevent, in principle, the formation of a superfluid phase. In Sec. V the method is then applied to He^3 as well as nuclear matter, and it is seen that for a wide class of trial functions the systems are rather far from going into a superfluid state, even at the absolute zero. However, we are unable to exhibit an upper bound to the transition temperature and thus have not proven rigorously (within the assumptions of this paper) the nonsuperfluidity of these systems.

II. FORMULATION

The Hamiltonian for the N-body system of fermions is written as

$$H=\sum_i T_i+\sum_{i>j} v(\mathbf{r}_i-\mathbf{r}_j), \tag{1}$$

where $T_i=-\hbar^2\nabla_i^2/2m$ and $v(\mathbf{r}_i-\mathbf{r}_j)$ is an arbitrary two-body potential.

Our basic conjecture concerns the normal fluid: we assume that an adequate description of the system in its normal state is obtained by treating it as a Fermi gas in a momentum-dependent potential. Because the potential between two He^3 atoms (nucleons) is strongly repulsive at distances less than about 2.5 A (0.4×10^{-13} cm), one might argue that the normal fluid should be very different from an ideal Fermi gas, the wave function for the system containing strong correlations between atoms. However, the success of the nuclear shell model and the apparent linear behavior[3] of the low-temperature specific heat of He^3 seem to indicate that to a certain approximation these correlations may be ignored and the fermions regarded as moving freely in the medium with a modified momentum-energy relation. Recent theoretical work of Brueckner[4] and others seems to support this point of view quite strongly.

Our consideration of the system at low temperatures then is based entirely on this conjecture about the nature of the normal fluid. Beginning with this we ask: Will a system such as the one described above enter into correlated states at some very low temperature, similar to the states assumed by electrons in a metal? Again, as in the electron case, we consider only the net energy gain due to two-body correlations, and then approximate the two-body correlation wave function by assuming only pairs with given total momentum and total spin are strongly correlated.[5] (Correlations in which all pairs have the same nonzero total momentum correspond to current carrying states. Pair correlations for pairs with triplet spin are also possible but we consider primarily a ground state with strong singlet spin, zero total momentum correlations.)

In the notation of second quantization the Hamiltonian can be written

$$H=\sum_{\mathbf{k},\sigma}\frac{\hbar^2k^2}{2m}c^\dagger(\mathbf{k},\sigma)c(\mathbf{k},\sigma)+\tfrac{1}{2}\sum_{\mathbf{k}_i\sigma_i}c^\dagger(\mathbf{k}_1,\sigma_1)c^\dagger(\mathbf{k}_2,\sigma_2)\times(1,2|v|3,4)c(\mathbf{k}_4,\sigma_4)c(\mathbf{k}_3,\sigma_3), \tag{2}$$

where $c(\mathbf{k},\sigma)$ and $c^\dagger(\mathbf{k},\sigma)$ are the annihilation and creation operators for a particle (atom or nucleon) with momentum $\hbar\mathbf{k}$ and spin direction σ. Since we expect correlations between pairs of particles with equal and opposite momentum and spin, the trial function will be constructed entirely from combinations of pair creation operators $b_\mathbf{k}^\dagger$ where

$$b_\mathbf{k}=c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}. \tag{3}$$

For such states, the Hamiltonian may be replaced by one involving only the operators $b_\mathbf{k}$ and $b_\mathbf{k}^\dagger$, thus

$$H=2\sum\frac{\hbar^2k^2}{2m}b_\mathbf{k}^\dagger b_\mathbf{k}+\sum b_\mathbf{k}^\dagger b_\mathbf{k}U_{\mathbf{k},\mathbf{k}'}b_{\mathbf{k}'}^\dagger b_{\mathbf{k}'}+\sum b_\mathbf{k}^\dagger V_{\mathbf{k}\mathbf{k}'}b_{\mathbf{k}'}-\sum V_{\mathbf{k}\mathbf{k}}b_\mathbf{k}^\dagger b_\mathbf{k}, \tag{4}$$

where the last term is smaller than the rest by a factor Ω, the volume of the system.

The expression $U_{\mathbf{k},\mathbf{k}'}$ contains the diagonal matrix elements of the interaction v, which are associated with the interactions in the normal fluid, while $V_{\mathbf{k},\mathbf{k}'}$ contains the pair interactions, that is the matrix elements of v between two different pair states.

$$\begin{aligned}U_{\mathbf{k},\mathbf{k}'}&=(\mathbf{k}\uparrow,\mathbf{k}'\uparrow|v|\mathbf{k}\uparrow,\mathbf{k}'\uparrow)-(\mathbf{k}\uparrow,\mathbf{k}'\uparrow|v|\mathbf{k}'\uparrow,\mathbf{k}\uparrow)\\&\quad+(\mathbf{k}\uparrow,-\mathbf{k}'\downarrow|v|\mathbf{k}\uparrow,-\mathbf{k}'\downarrow)\\&\quad-(\mathbf{k}\uparrow,-\mathbf{k}'\downarrow|v|-\mathbf{k}'\downarrow,\mathbf{k}\uparrow);\\V_{\mathbf{k},\mathbf{k}'}&=(\mathbf{k}\uparrow,-\mathbf{k}\downarrow|v|\mathbf{k}'\uparrow,-\mathbf{k}'\downarrow)\\&\quad-(\mathbf{k}\uparrow,-\mathbf{k}\downarrow|v|-\mathbf{k}'\downarrow,\mathbf{k}'\uparrow).\end{aligned} \tag{5}$$

The trial function is taken of the form in BCS and the expectation value of the Hamiltonian evaluated. The diagonal terms correspond to the effective momentum-dependent potential that a particle is subject to in the normal fluid. We make this identification, and assume that the problem is equivalent to a modified Hamiltonian in which the diagonal matrix elements of the potential are replaced by the diagonal elements of the t-matrix of Brueckner, as evaluated for the normal fluid. If no superstate exists, or if the off-diagonal elements associated with the strong correlations of the superstate are neglected, and the expectation value of this modified Hamiltonian is minimized with respect to a BCS type trial function, one obtains in fact the ground-state energy and model-ground-state wave function that Brueckner uses to describe the normal fluid.

As for the expectation value of the term involving

[3] Brewer, Sreedhar, Kramers, and Daunt, Phys. Rev. **110**, 282 (1958).
[4] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1040 (1958).
[5] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957). Hereafter, called BCS.

$U_{\mathbf{k}\mathbf{k}'}$, the trial function will differ only very slightly from the model-ground-state wave function of Brueckner theory; for this reason it is proper to use the diagonal elements of the reaction matrix t in evaluating $U_{\mathbf{k},\mathbf{k}'}$ in order to give as well as possible the proper interaction energy. That is, the expectation value of the true two-body potential with the correct wave function is closely approximated by the expectation value of the reaction matrix with the model wave function.

For the off-diagonal elements, the true two-body potential v must be used rather than the t-matrix, since we are looking for a state which arises from strong two-body interactions, and hence corresponds to iteration of v; t includes this iteration in an incorrect way near the Fermi surface, and certainly cannot be iterated further. It might be thought inconsistent to treat the pair correlations with this Hamiltonian since the strong interparticle potentials will introduce correlations even in the normal state. However, in the normal state any given particle interacts with any other particle with an energy which goes to zero with the volume of the container. The total effect of all of these interactions produces the momentum-dependent potential. Thus, although the normal wave function already contains correlations due to the hard cores, the correlation for any two given particles produces only an infinitesimal energy shift. (They are in a scattering state with respect to one another.) If the superstate is formed, then pairs of fermions of total momentum zero become very strongly correlated and a finite energy is associated with each pair. Compared to this the normal correlation energy of this pair is negligible. Forming the highly correlated state with pairs of fermions of total momentum zero involves changing the correlations between fermions of nonzero total momentum only when either one is near the Fermi surface. The resulting change in the effective potential will be only that due to interactions with the fermions in a thin shell at the Fermi surface and hence appears to be no more serious than in BCS.

III. INTEGRAL EQUATION FOR THE CORRELATION FUNCTION

Using the trial function

$$\Psi=\prod_{\mathbf{k}}(\alpha_{\mathbf{k}}+\beta_{\mathbf{k}}b_{\mathbf{k}}{}^{\dagger})\Phi_{\text{vacuum}}, \tag{6}$$

where $|\alpha_{\mathbf{k}}|^2+|\beta_{\mathbf{k}}|^2=1$, we can write

$$\begin{aligned}\alpha_{\mathbf{k}}&=(1-h_{\mathbf{k}})^{\frac{1}{2}},\\ \beta_{\mathbf{k}}&=h_{\mathbf{k}}{}^{\frac{1}{2}}e^{i\varphi_{\mathbf{k}}},\end{aligned} \tag{7}$$

where $h_{\mathbf{k}}$ is the expectation value of the number operator for the pair state $\mathbf{k}$:

$$h_{\mathbf{k}}=(\Psi,b_{\mathbf{k}}{}^{\dagger}b_{\mathbf{k}}\Psi), \tag{8}$$

and $0\leq h=h^*\leq 1$. The energy of the system is then given by

$$W=(\Psi,H\Psi)=W_0+W_c, \tag{9}$$

where W_0 is the normal state energy and W_c is the energy due to the pair correlations.

Approximating the dependence of the diagonal part of the Hamiltonian by the use of an effective mass[6] we obtain (in the limit of infinite normalization volume) for the correlation energy:

$$\begin{aligned}W_c=T+V=&\sum_{k<k_F}2|\epsilon_{\mathbf{k}}|(1-h_{\mathbf{k}})+\sum_{k>k_F}2\epsilon_{\mathbf{k}}h_{\mathbf{k}}\\&+\sum_{\mathbf{k},\mathbf{k}'}V_{\mathbf{k}'\mathbf{k}}[h_{\mathbf{k}}(1-h_{\mathbf{k}})h_{\mathbf{k}'}(1-h_{\mathbf{k}'})]^{\frac{1}{2}}\\&\times\exp[i(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})],\end{aligned} \tag{10}$$

where $\epsilon_{\mathbf{k}}=(\hbar^2/2m^*)(k^2-k_F{}^2)$, and $\hbar^2k_F{}^2/2m^*$, which determines k_F, is the Lagrangian multiplier associated with the restriction that the expectation value of the number of particles (that is, $2\langle\sum b_{\mathbf{k}}{}^{\dagger}b_{\mathbf{k}}\rangle$) be held fixed. Minimizing W_c with respect to $h_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ (it is assumed that W_0 remains unaltered by this variation) yields:

$$\begin{aligned}h_{\mathbf{k}}&=\tfrac{1}{2}(1-\epsilon_{\mathbf{k}}/E_{\mathbf{k}}),\\ E_{\mathbf{k}}&=[\epsilon_{\mathbf{k}}{}^2+\mathfrak{F}(\mathbf{k})\mathfrak{F}(\mathbf{k})^*]^{\frac{1}{2}},\end{aligned} \tag{11}$$

together with the nonlinear integral equation which is the criterion for the existence of a minimum, namely,

$$\mathfrak{F}(\mathbf{k})=-\tfrac{1}{2}\sum_{\mathbf{k}'}V_{\mathbf{k},\mathbf{k}'}\frac{\mathfrak{F}(\mathbf{k}')}{E_{\mathbf{k}'}}, \tag{12}$$

and

$$\mathfrak{F}=|\mathfrak{F}|e^{i\varphi}.$$

In order to treat general potentials, in particular potentials with hard repulsive cores, we note that the wave function and Hamiltonian of BCS give the energy due to pair correlations, and their variational method amounts to minimizing this energy. We thus introduce a correlation function between fermions of opposite spin whose total momentum is zero:

$$\rho(\mathbf{r}_1\uparrow;\mathbf{r}_2\downarrow)\equiv\langle\psi_{\uparrow}{}^{\dagger}(\mathbf{r}_1)\psi_{\downarrow}{}^{\dagger}(\mathbf{r}_2)\psi_{\downarrow}(\mathbf{r}_2)\psi_{\uparrow}(\mathbf{r}_1)\rangle. \tag{13}$$

For the wave function of Eq. (5) this becomes (where $\mathbf{r}_1-\mathbf{r}_2=\mathbf{r}$)

$$\rho_{\uparrow\downarrow}=\tfrac{1}{4}n^2+|\Omega^{-1}\sum_{\mathbf{k}}\alpha_{\mathbf{k}}{}^*\beta_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}|^2, \tag{14}$$

where n is the fermion density.

We then define

$$\chi(\mathbf{k})=[h_{\mathbf{k}}(1-h_{\mathbf{k}})]^{\frac{1}{2}}e^{i\varphi_{\mathbf{k}}}=\alpha_{\mathbf{k}}{}^*\beta_{\mathbf{k}}, \tag{15}$$

and passing to the limit of very large volume,

$$\chi(\mathbf{r})=\left(\frac{1}{2\pi}\right)^3\int\chi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k},$$

where $\chi(\mathbf{r})$ represents the extra correlation between spin

[6] See reference 3 as well as K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958), for a complete list of references.

zero pairs due to the interaction. Assuming a local inversion-symmetric potential, we can rewrite the potential term of W_c in terms of this correlation function as:

$$V=\sum_{\mathbf{k}\mathbf{k}'}[h_{\mathbf{k}}(1-h_{\mathbf{k}})h_{\mathbf{k}'}(1-h_{\mathbf{k}'})]^{\frac{1}{2}} \times\exp[i(\varphi_{\mathbf{k}}-\varphi_{\mathbf{k}'})]V_{\mathbf{k}',\mathbf{k}},$$

$$=\sum_{\mathbf{k}\mathbf{k}'}\chi^*(\mathbf{k}')\chi(\mathbf{k})V_{\mathbf{k}',\mathbf{k}}, \tag{16}$$

$$=\Omega(2\pi)^{-3}\int d\mathbf{r}v(\mathbf{r})|\chi(\mathbf{r})|^2.$$

It turns out then that the term T of the correlation energy [Eq. (10)] gives the increase in kinetic energy due to two-body correlations while V gives the change in the potential energy due to the correlations. In this last form V can be negative even if $v(\mathbf{r})$ contains a large repulsive core, as $|\chi(\mathbf{r})|^2v(\mathbf{r})=0$ for $v(\mathbf{r})=\infty$. Thus W_c can possibly be made smaller than zero even for potentials with hard cores.

The equation for χ determined by setting the variation of $W_c=0$ can conveniently be expressed as a pair of coupled equations:

$$\chi(\mathbf{k})=\tfrac{1}{2}\frac{\mathfrak{F}(\mathbf{k})}{[\epsilon_{\mathbf{k}}^2+\mathfrak{F}(\mathbf{k})\mathfrak{F}^*(\mathbf{k})]^{\frac{1}{2}}},$$

$$\mathfrak{F}(\mathbf{r})=\frac{1}{(2\pi)^3}\int\mathfrak{F}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}=-v(\mathbf{r})\chi(\mathbf{r}). \tag{17}$$

In this form, singular potentials $v(\mathbf{r})$ can be handled, since χ will be zero in regions where $v(\mathbf{r})$ is infinite; $\mathfrak{F}$, on the other hand, will be finite in such regions, and have a delta function singularity at the core boundary.

It can easily be shown that if a nonzero solution for $\mathfrak{F}(\mathbf{k})$ exists, all of the qualitative conclusions of BCS follow (excluding those explicitly dependent upon the fact that electrons are charged). In fact, if (as usually seems to be the case) $\mathfrak{F}$ is slowly varying in the region of k_F then the explicit calculation of BCS is almost entirely unchanged. Assuming that $\mathfrak{F}$ varies slowly in the vicinity of the Fermi surface, we may replace $|\mathfrak{F}(\mathbf{k})|^2$ by a constant ϵ_0^2, in the square root of Eq. (17). The constant ϵ_0 is now the eigenvalue of the system of equations, and is closely related to the energy gap and the transition temperature. The criterion for existence of a superstate is thus the existence of a nonzero solution to Eqs. (16) which can then be written:

$$\chi(\mathbf{r})=-\tfrac{1}{2}\int G(\mathbf{r}-\mathbf{r}')v(\mathbf{r}')\chi(\mathbf{r}')d\mathbf{r}', \tag{18}$$

where the Green's function $G(\mathbf{r})$ is given by

$$G(\mathbf{r})=\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}}{[\epsilon_k^2+\epsilon_0^2]^{\frac{1}{2}}}. \tag{19}$$

In this last form we see that the criterion for superfluidity is equivalent to the existence of a solution for an integral equation in the two-body correlation function, where ϵ_0^2 plays the role of an eigenvalue. It is instructive to compare Eq. (19) with the Schrödinger integral equation for the relative coordinate wavefunction of a pair with zero total momentum:

$$\psi(\mathbf{r})=-\tfrac{1}{2}\int G_S(\mathbf{r}-\mathbf{r}')v(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}',$$

$$G_S(\mathbf{r})=\frac{1}{(2\pi)^3}\int\frac{d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}}{(\hbar^2/2m)(k^2-k_0^2)}. \tag{20}$$

The entire difference, we see, occurs in the denominator of the Green's function. Instead of favoring states of $k=0$, the Green's function of Eq. (19) picks out states near the Fermi surface $k=k_F$. Since these states already oscillate rapidly it is often possible to construct "bound solutions" to Eq. (19) where no bound solution to Eq. (20) exists.

If we chose to construct states with triplet spin our results would remain essentially unaltered except that

$$\chi(\mathbf{r})=-\chi(-\mathbf{r});$$

the correlation function being antisymmetric in its coordinate. Solutions of the integral equation then must have odd parity.

IV. CRITERION FOR SUPERFLUIDITY

We can now write down a criterion for superfluidity in a particularly simple form. If we define

$$\lambda=2\int d\mathbf{k}|\chi(\mathbf{k})|^2E_{\mathbf{k}}+\int d\mathbf{r}|\chi(\mathbf{r})|^2v(\mathbf{r}), \tag{21}$$

it is easy to show that for any χ

$$\lambda\geqslant(2\pi)^3W_c/\Omega, \tag{22}$$

so that if we can find a χ which makes $\lambda<0$, W_c will also be smaller than zero. Replacing $|\mathfrak{F}|^2$ by ϵ_0^2 in $E_{\mathbf{k}}$, where $\epsilon_0^2=|\mathfrak{F}(k_F)\mathfrak{F}^*(k_F)|$ (which is equivalent to the approximation which linearizes the integral equation in the previous section) we have

$$\lambda(\epsilon_0)=2\int d\mathbf{k}|\chi(\mathbf{k})|^2(\epsilon_{\mathbf{k}}^2+\epsilon_0^2)^{\frac{1}{2}} +\int d\mathbf{r}|\chi(\mathbf{r})|^2v(\mathbf{r}), \tag{23}$$

where $\lambda(\epsilon_0)$ is to be evaluated for a trial function χ, and then the equation $\lambda(\epsilon_0)=0$ solved for ϵ_0. The resulting value of ϵ_0 will always be less than the maximum value of ϵ_0, attaining the correct value when the trial

function χ satisfies Eq. (18). We have written the variational equation partly in momentum space and partly in configuration space because the kinetic energy would be nonlocal and quite complicated in configuration space, while the potential energy is easy to evaluate even for infinitely repulsive cores in configuration space [$|\chi(\mathbf{r})|^2 v(\mathbf{r})$ is zero in such regions.] It should be observed that if $\lambda(0)$ is negative then there exists a real ϵ_0 for which $\lambda(\epsilon_0)=0$. Thus the criterion for a superfluid phase at some sufficiently low temperature is simply that for some trial function[7]:

$$\lambda(0)=2\int d\mathbf{k}|\chi(\mathbf{k})|^2|\epsilon_k|+\int d\mathbf{r}|\chi(\mathbf{r})|^2 v(\mathbf{r})<0. \quad (24)$$

Inspection of the criterion Eq. (24) shows immediately that (a) a potential which is everywhere repulsive cannot satisfy the criterion; (b) a hard core is not in principle a deterrent to superfluidity. It is seen that what is required is that the potential have an attractive region. The correlations then tend to increase the probability that particles reside in the attractive region. This is done by building the correlation out of Fourier components with wave numbers as close to k_F as possible.

If we assume that $\chi(\mathbf{r})$ is spherically symmetric (an S-state solution), and let

$$g(k)=k\chi(\mathbf{k}),$$
$$g(r)=r\chi(\mathbf{r}),$$
$$g(r)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty g(k)\sin krdk, \quad (25)$$

adopting as the unit of length $k_F^{-1}(x=k_Fr)$, and as the unit of energy $\hbar^2k_F^2/2m^*$, the criterion for superfluidity may be written

$$\frac{1}{8\pi}\lambda(0)=\int_0^\infty |k^2-1||g(k)|^2dk + \tfrac{1}{2}\int_0^\infty v(r)|g(r)|^2dr<0. \quad (26)$$

V. APPLICATION TO He^3 AND NUCLEAR MATTER

In this section the criterion for superconductivity [Eq. (26)] is used to study liquid He^3 and nuclear matter. For this purpose we have employed many trial functions. The calibre of a trial function can readily be expressed by indicating the factor by which the potential must be multiplied (for fixed density, extent of potential, etc.) in order to make the system just go into a superfluid state at the absolute zero. The smallest such factor we have found is approximately 6 for He^3, while for nuclear matter it is about 5.

In the absence of a repulsive core the trial functions, $g_1(x)=\mathrm{Ci}(\beta x)\sin x$, $g_3(x)=\ln(\beta x)\sin x$

$$\text{for } x<1/\beta,$$
$$=0 \quad \text{for } x\gtrsim 1/\beta,$$
$$g_2(x)=K_0(\beta x)\sin x,\quad g_4(x)=\ln[\beta(x+1)]\sin x \quad (27)$$
$$\text{for } x+1<1/\beta,$$
$$=0 \quad \text{for } x+1\gtrsim 1/\beta,$$

all have the property that for small β the kinetic energy term in $\lambda(0)$ goes as $\ln(1/\beta)$, while the potential energy term goes as $\ln^2(1/\beta)$. Thus, for arbitrarily weak attractive potential by choosing a trial function with β sufficiently small $\lambda(0)$ can be made negative. Thus such systems are always in a superfluid state at the absolute zero.

In regions of strong repulsion the function $g(x)$ must be very small so as not to contribute a large positive term to $\lambda(0)$. We have approximated the true two-body interaction in He^3 [8] and nuclear matter[9] with a central potential which is infinite for $r<r_c$ $(x<c)$ and attractive beyond r_c.[10] Thus $g(x)$ is identically zero for $x<c$. Since the kinetic energy operator is $|k^2-1|$, the function $g(x)$ must be a function with wave number predominantly near 1 (k_F in these units). It may be expected also that for large x the function $g(x)$ will be in phase with $\sin x$, or else the kinetic energy will tend to be very large. It is also important for the no-core situation that $g(x)$ be very large for small x. Guided by this we choose

$$g_5(x)=\ln[\beta(x-c)][1-e^{-\eta(x-c)}]\sin x, \quad c<x<1/\beta,$$
$$=0, \quad x<c, \quad x>1/\beta. \quad (28)$$

In this case the kinetic energy term in $(1/8\pi)\lambda(0)$ may be evaluated in the limit as $\beta\to 0$. For large η this takes

[7] A similar criterion has been obtained by Bogoliubov, Tolmachov, and Shirkov (to be published): Joint Institute for Nuclear Research, Steklov Mathematical Institute of the U.S.S.R. Academy of Sciences, Dubna, June, 1958. The above preprint was received while the present manuscript was in preparation. One of the authors (L.N.C.) wishes to thank the above authors and the Joint Institute for Nuclear Research for this and other preprints.

[8] J. L. Yntema and W. G. Schneider, J. Chem. Phys. **18**, 641 (1950).

[9] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 1337 (1957).

[10] For both nuclear matter and liquid He^3 the calculations seem quite insensitive to the features of the two-body potential other than the gross characteristics of the strongly repulsive internal region and the weakly attractive outer region. In particular, $\chi^2(r)v(r)$ becomes very small in a region where the potential is large and positive, so that the use of an infinite repulsive core for He^3 is a good approximation to the repulsive part of the Yntema-Schneider potential. We have consequently used in our calculation the following simple analytic S state potential:

$$V=\infty \quad \text{for } r<r_c$$
$$=-V_0e^{-\mu(r-r_c)} \quad \text{for } r\gtrsim r_c,$$

where $r_c=0.4\times10^{-13}$ cm (2.5 A), $V_0=26$ Mev (10°K), $\mu=0.544\times10^{13}$ cm^{-1} (1.85 A^{-1}).

the following relatively simple form:

$$\text{K.E.}=\frac{\ln^2(1/\beta)}{4\pi}\Big\{\frac{\pi}{2}(1-\cos 2c)\eta+\Big[2\ln 4\gamma c-2\,\text{Ci}4c-2+\tfrac{1}{2}\pi\sin 2c+2\cos 2c+\frac{1}{2c}\sin 4c-\frac{1}{c}\sin 2c\Big]+\Big[\frac{2}{c}+\Big(\frac{1}{c^2}-4\Big)\sin 2c+\Big(\frac{\pi}{2}-\frac{2}{c}\Big)\cos 2c-\frac{2}{c}\cos 4c-4\,\text{sinc}\Big]_{\eta}^{1}+\cdots\Big\},$$

where γ is the Euler-Mascheroni constant, equal to $1.781\cdots$.

For small core radii the dominant term in the kinetic energy as $\beta\to 0$, for large η may be written as

$$\text{K.E.}=\frac{\ln^2(1/\beta)}{4\pi}\Big\{[\tfrac{1}{4}\pi c^2+\mathcal{O}(c^3)]\eta+[\pi c+\mathcal{O}(c^2)]+[\tfrac{1}{2}\pi+\mathcal{O}(c)]\frac{1}{\eta}+\mathcal{O}\Big(\frac{1}{\eta^2}\Big)\Big\}. \quad (29)$$

Thus the kinetic energy has a minimum value for $\eta\approx\sqrt{2}/c$ which is $\text{K.E.}\simeq 0.42c\ln^2(1/\beta)$. As $c\to 0$ and $\eta\to\infty$, therefore, the term in the kinetic energy proportional to $\ln^2(1/\beta)$ vanishes. Since the P.E. varies as $\ln^2(1/\beta)$ for this trial function one can see that as $c\to 0$, and for β sufficiently small, the system will certainly become superfluid for any attractive potential outside the repulsive core.

We have evaluated the $\ln^2(1/\beta)$ term in $\lambda(0)$ for He^3 and nuclear matter, using this trial function. We have chosen the value of η which minimized the $\ln^2(1/\beta)$ term in $\lambda(0)$, and find that unless the attractive part of the potential is 15 times larger than it is in He^3 the system does not go superfluid with this trial function. For nuclear matter ($c=0.59$, compared to $c\simeq 1.9$ for He^3) the factor is 5.

Most trial functions do not go continuously as $c\to 0$ into trial functions which indicate superfluidity for arbitrarily weak attractive potentials. Nevertheless since c is so large for He^3 it was thought profitable to use some of these functions. We have tried

$$g_6(x)=\sin x[e^{-\beta_1(x-c)}-e^{-\beta_2(x-c)}];\quad x>c,\qquad =0;\quad x\le c \quad (30)$$

and after minimizing $\lambda(0)$ with respect to β_1 and β_2 still find approximately the same factor of 15 for He^3 and 7 for nuclear matter.

Another trial function which although not successful in showing the superfluidity in the case of a weak purely attractive potential, might nevertheless be useful for a large repulsive core is

$$g_7(x)=\sin(x-c)e^{-\beta(x-c)}. \quad (31)$$

In this case for appropriate choice of β the ratio by which the potential energy must be multiplied in He^3 is only 6.5. The reason for this is that for He^3, c is such that the function $g(x)$ is large in the region of the potential energy while the additional kinetic energy coming from the wrong phase of $\sin(x-c)$ is more than compensated.

One is thus led to try the trial function

$$g_8(x)=\{[1-e^{-\gamma(x-c)}]\sin x+\alpha e^{-\gamma(x-c)}\sin(x-c)\}\ln\beta x;\quad c\le x\le 1/\beta \quad (32)$$
$$=0;\quad x<c,\quad x>1/\beta,$$

where α, β, γ are variational parameters. This function is large in the region of the potential and then slowly shifts to the correct phase at large r. The calculations with this function, even in the limit of $\beta\to 0$ are extremely complicated. We have only carried them to the extent of seeing that although the ratio may be reduced from 6.5 it will not go below 4 for He^3, while in nuclear matter there is no significant improvement.§

VI. SUMMARY

The basic result of our investigation is the integral equation [Eqs. (18) and (19)] for the correlation function. This leads to the very simple criterion for a superfluid state, expressed as a variational principle [Eq. (24)]. In Sec. V we have used this in a preliminary way to investigate liquid He^3 and nuclear matter, but have not so far succeeded either in finding a trial function which makes these systems superfluid or in proving that they do not become superfluid.

§ *Note added in proof.*—We are grateful to S. A. Moszkowski for pointing out to us the virtues of a trial function which is the solution of a Schrödinger equation with reduced effective mass $\frac{1}{2}m^*$, and a truncated potential so chosen that the solution has zero phase shift. With a modified form of such a trial function one can prove a low-density theorem: if the S-wave solution to the Schrödinger equation, with reduced mass $\frac{1}{2}m$ and the true two-body potential, has positive phase shift at zero energy, then at sufficiently low density the fermion system would have a superfluid state. This suggests the investigation, now in progress, of the properties of He^3 in dilute solution in He^4, as well as a careful study of the finite nucleus, with particular attention to the reduced density at the surface.