Behavior of Intense Relativistic Electron Beams

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A theoretical investigation is made of intense, relativistic, partially neutralized, self-focusing electron beams. Approximate equations are derived that express the time derivatives of the beam radius and mean electron energy as functions of these parameters, the densities of the electrons, ions, and neutral atoms, and the impressed electric and magnetic fields. The equations are numerically integrated for various values of the parameters. It is shown that the radiation damping leads to a shrinkage of the beam radius to a value of order 0.05 mm in a time of order 10^{-2}-10^{-1} second. If the longitudinal electric field is constant the beam approaches the "equilibrium state" discussed by Budker, in which the increase in electron transverse momentum caused by scattering from the ions is compensated by radiative loss, and the energy lost by radiation is restored by the longitudinal field. The various instability problems associated with dense systems of charged particles are not discussed.

I. INTRODUCTION

A RELATIVISTIC electron beam may be "stabilized" by the presence of positive ions in the beam. The electrostatic repulsion between the parallel moving electrons is reduced by a factor of $\gamma^2=(1-v^2/c^2)^{-1}$ by the magnetic attraction; hence even a small percentage neutralization by positive ions may be sufficient to overcome the electrons' mutual repulsion and attract the electrons toward the center of the beam. If the neutralization is not complete (in the laboratory system), the ion-electron attraction dominates the ion-ion repulsion, and the ions are also attracted to the beam center. Thus the electrons and ions each oscillate in a transverse potential "well" determined by the presence of the particles of opposite charge.

The radiation damping associated with the transverse oscillations of such an intense, stabilized beam tends to make the beam radius shrink. Budker[1,2] and Enoch[3] have shown how the parameters that describe the beam must be related if the beam radius and mean electron energy are constant in time. In this "equilibrium" situation the increase in transverse electron momentum caused by scattering from the positive particles is compensated by radiative loss, and the energy lost in radiation is restored by an impressed longitudinal field. If the neutralization is nearly complete, the electron current is about 1500 amperes, and the impressed electric field is about 1 volt/cm, then the equilibrium values of the beam radius and energy are about 3.5×10^{-3} cm and 50 Mev.[1,3] If such a highly pinched electron beam can be produced, it might be quite useful, either for direct use in experiments requiring high current densities, or as a means of providing a guide field in which protons may be accelerated.

Since the equilibrium beam radius is small compared to the beam radius in present-day accelerators, any reasonable scheme for the production of an equilibrium beam must allow time for radial shrinkage. Therefore, it is necessary to know how the beam parameters are related in the nonequilibrium situation, in particular, how the time required for beam shrinkage depends upon the impressed conditions. In this paper approximate equations are derived which express the time derivatives of the beam radius and energy as functions of the instantaneous values of these parameters, the densities of electrons, positive ions, and neutral atoms, and the impressed electric field. In Sec. VIII the time necessary for beam shrinkage is estimated for certain reasonable initial conditions, and compared to the time in which scattering is expected to destroy the beam.[4]

The initial states of the hypothetical beams studied here are taken to be completely unneutralized, relativistic, intense electron beams, focused in the transverse directions by external fields and accelerated in a tube containing a low-density neutral gas. The manner of behavior of such a beam depends on the relative strengths of the different transverse forces felt by the electrons, and thus varies at different stages in the evolution of the beam. Early in the beam's life the external focusing forces and self-focusing forces are both significant, and are both much smaller than the external guide-field force that holds the electron in its closed orbit in the machine. Later, after ionization of the neutral atoms has led to significant beam neutralization, the external focusing forces are smaller than the self forces, and are neglected. In this later stage, the guide-field force may be greater than, comparable to, or less than the self-forces. In Sec. IV it is shown that the effect of the radiation damping depends upon the relative magnitudes of the self-forces and guide-field force. When the beam radius is close to the equilibrium radius, the self-forces are generally dominant, and the beam shrinkage process behaves essentially the same as it would in an enormously long linear accelerator.

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[1] G. J. Budker, *Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956), p. 68.

[2] G. J. Budker, Atomnaya Energ. **5**, 9 (1956).

[3] J. Enoch, Midwestern Universities Research Association Report MURA-311, Madison, Wisconsin, 1957 (unpublished).

[4] An estimate of this shrinkage time for certain special cases has been given by J. D. Lawson, United Kingdom Atomic Energy Authority Research Group Report AERE *GP/M* 200, Harwell, 1957 (unpublished).

II. BASIC ASSUMPTIONS AND PARAMETERS OF THE MODEL

The model to be described here involves many assumptions, some basic, while others are made only for simplicity. Perhaps the most uncertain of the basic assumptions is that the beam can be maintained throughout the shrinkage process. It is known that several types of instabilities exist for certain dense systems of charged particles; it is not known as yet whether or not these problems can be circumvented in the production of a beam satisfying the Budker equilibrium conditions. One difficulty arises because the self-forces lead to a continuously changing transverse oscillation frequency. During the early stages of the shrinking process, before the radiation damping is effective and while the external focusing forces are still important, it might be impossible to maintain the beam as the transverse frequency passes through a resonance region in which it is an integral multiple of the frequency of the circular orbit. Some of these instability problems are discussed briefly in reference 1. They will not be discussed further here, but it is assumed that they can be solved. It is further assumed that the electromagnetic forces do not lead to collective effects which destroy the randomness of the longitudinal positions and transverse phases of the electrons and ions.

The other basic assumptions are more securely founded. Only electrons of energies greater than a few Mev are considered, so that they are relativistic and their longitudinal velocities may be replaced by the velocity of light. The only forces that vary appreciably within an electron's quantum wavelength are exerted in collisions with the ions and neutral atoms. It may be shown that the energy lost by the electrons in such elastic collisions, and in radiative collisions, is small compared to the energy lost in radiation associated with the more slowly varying forces. The important effects of the collisions depend only on the differential cross section for nonradiative scattering. Since this cross section is given accurately by the classical formula, the electrons are assumed to obey classical electrodynamics in all stages of the calculation.

The other important assumptions, made only for simplicity, are listed below. The effects of relaxing some of these assumptions are discussed in Sec. IX. The simplifying assumptions are as follows: (1) The ion velocities are nonrelativistic in the laboratory system. (2) The positive ions are distributed uniformly in a circular beam of sharply defined radius. In Sec. VII a similar assumption is made concerning the electron beam. (3) The longitudinal momenta of all electrons in the beam are the same at any time. (4) Both the self-focusing and external focusing forces are equally strong in the two transverse directions. (5) The radiation damping depends only on the stronger of the guide-field and self-focusing forces. (6) The external fields and initial conditions are independent of the machine azimuth angle (or longitudinal coordinate, if beam curvature is neglected), as viewed in the laboratory Lorentz system. (7) The number of electrons per unit length in the beam remains constant. (8) The neutral gas in the tube is hydrogen at room temperature.

The notation used to represent the principal parameters describing the beam is listed below. The symbols $E_0=mc^2$ and $r_0=e^2/E_0$ denote the electron rest mass and classical radius. $\mathbf{r}$, radius vector in transverse plane from beam center to position of an electron; $\mathbf{p}_t$, p_l, transverse and longitudinal momentum of an electron; E, electron energy ($E\approx p_l c$); A, A_+, amplitudes for the transverse oscillations of an electron and proton, respectively (the transverse amplitude is defined as the square root of the sum of the squares of the amplitudes in the two transverse directions); R, radius of electron beam (R^2 is twice the average over electrons of r^2); R_+, radius of ion beam; $\mathcal{R}$, radius of circular accelerating machine; $\mathcal{E}$, longitudinal electric field, or equivalent inductive field; n, n_+, numbers of electrons and ions per unit beam length; ν, ν_+, numbers of electrons and ions in a beam length equal to r_0 ($\nu=nr_0$, $\nu_+=n_+r_0$); ρ_0, ρ_i, densities of neutral atoms outside, and inside electron beam; K, K_+, force constants for transverse oscillations of electrons and ions, respectively; I, radiated power; Z, an adiabatic invariant for the electron beam, defined by the equation, $Z=K^{\frac{1}{2}}E^{\frac{1}{2}}R^2$; and L_+, L_0, Coulomb logarithms defined in Sec. V. The time derivative of any quantity is denoted by a prime. The subscripts t, l, i, and e are used to denote, respectively, the component of a vector transverse to the beam direction, the longitudinal component of a vector, the initial value of a parameter, and the value of a parameter in the equilibrium state.

In reference 1, many of the equations are derived by referring to the "electron" system, the Lorentz system in which the average electron velocity (at some particular point in the beam) is zero. In this paper, however, all equations are derived in the laboratory system. Surprisingly enough, the motion of the electrons is conceptually simpler in the laboratory system. The reason for this is that the transverse electron momenta are on the order of mc for intense, appreciably neutralized beams. Therefore, in the electron system, the transverse velocities are relativistic. Furthermore, the magnetic force exerted on an electron by the ions is comparable to the ion-electron electrostatic force. The effect of these two forces is to couple the longitudinal and transverse electron momenta together; both oscillate rapidly in time. The fact that this apparently wild electron motion is actually quite simple becomes clear when one studies the situation in the laboratory system, in which the ions are nonrelativistic. In the lab system, the only strong internal force on an electron is the transverse electrostatic force exerted by the ions. The longitudinal electron momentum remains nearly constant during a transverse oscillation, and is much greater than either mc or the transverse momentum. Hence the transverse velocity is small compared to c, and the transverse

motion may be treated in a nonrelativistic manner. A further advantage of considering only the lab system is that by so doing, we avoid introducing the relativistic gamma factor defined in terms of the relative velocities of the two systems. (This factor is confusing because it is not equal to the ratio of the electron's lab system energy and rest energy.)

A natural procedure is to consider those parameters that cannot be directly controlled to depend upon those that can. We consider the parameters describing the electron beam to be the dependent variables, and derive approximate equations expressing E' and $(R^2)'$ in terms of the instantaneous values of R^2, E, $\mathcal{E}$, $\mathcal{R}$, ρ_i, and the ion charge density $n_+/(\pi R_+^2)$. Actually, the ion charge density itself depends on the motion of the electrons, and other conditions. In Sec. VII several alternate assumptions are made concerning the behavior of $n_+/(\pi R_+^2)$, and these assumptions are used to simplify the differential equations describing R^2 and E.

III. EQUATIONS OF MOTION OF A BEAM ELECTRON

We consider a partially neutralized, relativistic beam of electrons of energy E circulating in a circular accelerating machine. The force on an electron in this beam may be written as a sum of four parts, resulting from the external fields, the positive ions, scattering from neutral atoms, and radiation damping, i.e.,

$$\mathbf{F}=\mathbf{F}^{\text{ext}}+\mathbf{F}^{\text{ion}}+\mathbf{F}^{\text{neu}}+\mathbf{F}^{\text{rad}}. \tag{1}$$

The force exerted by the other electrons of the beam is neglected, for the reasons discussed in Sec. I. Because of the statistical nature of the ion force $\mathbf{F}^{\text{ion}}$, it is convenient to express it as a sum of two parts, i.e.,

$$\mathbf{F}^{\text{ion}}=\mathbf{F}^{\text{smo}}+\mathbf{F}^{\text{flu}}, \tag{2}$$

where the average or "smooth" ion force $\mathbf{F}^{\text{smo}}$ is defined to be the force that would be exerted by a uniform, continuous, motionless, positively charged beam of radius R_+ and linear charge density n_+e; and the "fluctuation" or ion scattering force $\mathbf{F}^{\text{flu}}$ is defined by Eq. (2). The force exerted by the neutral atoms is also statistical in nature, but need not be written as two terms, since the average of this force vanishes. If the cylindrical coordinates ρ, φ, and z (where the origin is taken at the center of the circular orbit of the beam, and the z axis is perpendicular to the circular orbit) are used to describe the electron motion, the "effective transverse force" $\mathfrak{F}_t$ may be defined in terms of the rate of change of the square of the transverse momentum, i.e.,

$$(p_t^2)'=2\mathbf{p}_t\cdot\mathfrak{F}_t. \tag{3}$$

This effective force is related to the actual transverse force $\mathbf{F}_t$ by the equation, $\mathfrak{F}_t=\mathbf{F}_t+\mathbf{i}_\rho E/\rho$, where $\mathbf{i}_\rho$ is a unit vector in the ρ direction. The centripetal term $\mathbf{i}_\rho E/\rho$ results from the transformation to cylindrical coordinates. If the force $\mathfrak{F}_t^{\text{ext}}$ is defined by the relation $\mathfrak{F}_t^{\text{ext}}=\mathbf{F}_t^{\text{ext}}+\mathbf{i}_\rho E/\rho$, the effective transverse force may be written as the following sum,

$$\mathfrak{F}_t=\mathfrak{F}_t^{\text{ext}}+\mathbf{F}_t^{\text{smo}}+\mathbf{F}_t^{\text{flu}}+\mathbf{F}_t^{\text{neu}}+\mathbf{F}_t^{\text{rad}}. \tag{4}$$

Except during the short time intervals of close collisions with ions or neutral atoms, the scattering and radiation damping forces are small compared with the average value of $\mathfrak{F}_t^{\text{ext}}+\mathbf{F}_t^{\text{smo}}$ in the beam. Hence the force $\mathfrak{F}_t^{\text{ext}}+\mathbf{F}_t^{\text{smo}}$ determines a transverse potential well in which an electron oscillates, while $\mathbf{F}_t^{\text{flu}}$, $\mathbf{F}_t^{\text{neu}}$ and $\mathbf{F}_t^{\text{rad}}$ may be considered as perturbing forces leading to a change in time of the amplitude. The radial oscillations of the electron are characterized by the coordinate $s=\rho-\mathcal{R}$, where the beam orbit radius $\mathcal{R}$ is defined by the relation $\mathfrak{F}_t^{\text{ext}}(z=0,\ \rho=\mathcal{R})=0$. We assume that the center of the uniformly charged ion beam coincides with that of the electron beam. The electrostatic force $\mathbf{F}_t^{\text{smo}}$ on an electron is given by

$$\mathbf{F}_t^{\text{smo}}=-\frac{2n_+e^2}{R_+^2}\mathbf{r}=-\frac{2\nu_+E_0}{R_+^2}\mathbf{r}, \tag{5}$$

where $r=(s^2+z^2)^{\frac{1}{2}}$ is the electron transverse "polar" coordinate, and $\mathbf{r}$ is the polar vector $r\mathbf{i}_r$. We further hypothesize that the external focusing force is of the form $\mathfrak{F}_t^{\text{ext}}=-K^{\text{ext}}\mathbf{r}$, where K^{ext} is a constant. Under these conditions the transverse electron motion is that of a nonrelativistic, two-dimensional harmonic oscillator. The transverse energy equation is

$$\tfrac{1}{2}KA^2=\tfrac{1}{2}p_t^2(c^2/E)+\tfrac{1}{2}Kr^2, \tag{6}$$

where A is the amplitude, and the force constant K is given by the equations,

$$K=K^{\text{ext}}+K^{\text{ion}}, \tag{7}$$

$$K^{\text{ion}}=2\nu_+E_0/R_+^2. \tag{8}$$

The validity of Eq. (8) depends on the assumption that $R_+\geqq R$. This assumption is reasonable for the shrinking beams considered here since, as is seen in Sec. VIII, the electron beam tends to shrink faster than the ion beam.

If the time derivative of Eq. (6) is taken, and use is made of Eq. (3), the result is

$$(A^2)'=\frac{2c^2}{KE}\mathbf{p}_t\cdot\mathfrak{F}_t+2\mathbf{r}\cdot\mathbf{r}'-\frac{c^2p_t^2}{KE^2}E'+\frac{(r^2-A^2)}{K}K'. \tag{9}$$

If the quantity $\mathbf{r}'$ in Eq. (9) is replaced by $(c^2/E)\mathbf{p}_t$ and use is made of the relation $\mathfrak{F}_t^{\text{ext}}=-K^{\text{ext}}\mathbf{r}$ and Eqs. (4) through (8), two of the terms in Eq. (9) cancel, yielding the result,

$$(A^2)'=\frac{2c^2}{KE}\mathbf{p}_t\cdot(\mathbf{F}_t^{\text{flu}}+\mathbf{F}_t^{\text{neu}}+\mathbf{F}_t^{\text{rad}})-\frac{c^2p_t^2}{KE^2}E'+\frac{(r^2-A^2)}{K}K'. \tag{10}$$

Since the energy is approximately equal to cp_t the rate

of change of energy is proportional to the longitudinal force, i.e.,

$$E'=c(\mathcal{E}e+F_l{}^{\rm rad})=\mathcal{E}ec-I, \tag{11}$$

where I is the radiated power. The force $\mathbf{F}^{\rm smo}$ has no longitudinal component, so is omitted from Eq. (11). It is shown in reference 3 that the energy lost in collisions is considerably less than that lost by radiation, for all values of the beam radius. Therefore, the longitudinal scattering forces $F_l{}^{\rm neu}$ and $F_l{}^{\rm flu}$ are also omitted from Eq. (11). The two equations, Eqs. (10) and (11), are the two fundamental equations for the amplitude and energy of an electron in the beam. In order to interpret these equations it is necessary to express the forces in terms of the various beam parameters and to average the equations over the period of a transverse oscillation.

IV. RADIATION DAMPING

The radiation damping force on the relativistic electrons may be written in the form,[5]

$$\mathbf{F}^{\rm rad}=\frac{2r_0E}{3cE_0}\mathbf{p}''+\frac{2r_0E}{3cE_0{}^3}\mathbf{p}(c^2\mathbf{p}\cdot\mathbf{p}''-EE''). \tag{12}$$

Only the strong forces $\mathbf{F}^{\rm ext}$ and $\mathbf{F}^{\rm smo}$ need be considered in computing the right side of Eq. (12). Since the longitudinal momentum is many times greater than mc, the longitudinal component of the first term of Eq. (12) may be neglected. For the intense beams considered here, the average transverse momentum is generally several times greater than mc [see Eq. (50)]. Under these conditions, calculations show that the transverse component of the second term of Eq. (12) is several times larger than that of the first. Hence, for simplicity, we drop the first term, though including it would create no difficulties. If E'' is expressed in terms of E' and derivatives of the momentum, the second term of Eq. (12) becomes,

$$-\frac{2r_0E}{3cE_0{}^3}\mathbf{p}[c^2(\mathbf{p}')^2-(E')^2].$$

The rate of energy loss of the electron due to radiation is given by the formula,[6]

$$E'^{\rm rad}=-I=-\frac{2r_0E^2}{3cE_0{}^3}[c^2(\mathbf{p}')^2-(E')^2]. \tag{13}$$

Hence our approximate expression for the transverse damping force may be written,

$$\mathbf{F}_t{}^{\rm rad}=-(\mathbf{p}_t/E)I. \tag{14}$$

[5] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by Morton Hammermesh (Addison-Wesley Press, Cambridge, 1951), p. 233. Equation (12) above is equivalent to Eq. (9–105) of this reference.

[6] Landau and Lifshitz, see reference 5, p. 211. An extensive discussion of radiation by electrons in accelerators is given by Julian Schwinger, Phys. Rev. **75**, 1912 (1949).

A further simplification may be made in Eq. (13). Since the longitudinal component of $\mathbf{F}^{\rm ext}+\mathbf{F}^{\rm smo}$ is small compared to the average transverse component, $c^2(p_t')^2$ and $(E')^2$ are small compared to $c^2(\mathbf{p}_t')^2$ and Eq. (13) may be approximated by the equation,

$$I=\frac{2r_0cE^2}{3E_0{}^3}(\mathbf{p}_t')^2=\frac{2r_0cE^2}{3E_0{}^3}(\mathbf{F}_t{}^{\rm ext}+\mathbf{F}_t{}^{\rm smo})^2. \tag{15}$$

It should be noted that the force referred to in this equation is the actual transverse force, not the effective transverse force of Eq. (3).

In general both $\mathbf{F}_t{}^{\rm ext}$ and $\mathbf{F}_t{}^{\rm smo}$ contribute to Eq. (15). The behavior of the beam depends on the relative sizes of these two forces. For simplicity we consider only the two cases in which the average over an oscillation of $\mathbf{F}_t{}^{\rm smo}$ is very large or very small compared to $\mathbf{F}_t{}^{\rm ext}$.

Case 1. $|\mathbf{F}_t{}^{\rm ext}|\gg|\mathbf{F}_t{}^{\rm smo}|$

In this case $\mathbf{F}_t\approx-\mathbf{i}_\rho E/\rho$ and, if the transverse amplitude is small compared to the orbital radius, $\mathbf{F}_t$ may be considered equal to $-\mathbf{i}_\rho E/\mathcal{R}$. The radiated power is given by the formula,

$$I=2E^4r_0c/3E_0{}^3\mathcal{R}^2. \tag{16}$$

It should be pointed out that the condition $|\mathbf{F}_t{}^{\rm ext}|\gg|\mathbf{F}_t{}^{\rm smo}|$ does not imply that the external *focusing* forces are stronger than the self-forces.

Case 2. $|\mathbf{F}_t{}^{\rm smo}|\gg|\mathbf{F}_t{}^{\rm ext}|$

In this case the formula for the power radiated may be obtained by substituting Eq. (5) into Eq. (15). The result is

$$I=8E^2r^2\nu_+{}^2r_0c/3E_0R_+{}^4. \tag{17}$$

V. SCATTERING FROM IONS AND NEUTRAL ATOMS

The easiest way to study the effect of $\mathbf{F}^{\rm flu}$ is to first investigate the entire ion force $\mathbf{F}^{\rm ion}$ and then subtract $\mathbf{F}^{\rm smo}$; hence we turn our attention to the quantity $2\mathbf{p}_t\cdot\mathbf{F}_t{}^{\rm ion}=(p_t{}^2)'^{\rm ion}$. The momentum transferred by the passage of a single ion is easily computed. Unless the impact parameter is so small that the electron is knocked clear out of the beam, the momentum transfer is essentially transverse and is given by the formula

$$\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)=2e^2(\mathbf{r}-\mathbf{r}_j)/c(\mathbf{r}-\mathbf{r}_j)^2, \tag{18}$$

where the index j refers to the ion in question, and $(\mathbf{r}-\mathbf{r}_j)$ is the impact vector, defined to be the vector from the ion to the electron at the instant the two are in the same transverse plane.

If $\mathcal{T}$ represents a time interval sufficiently short that the transverse momentum and displacement of an electron change by relatively small amounts in $\mathcal{T}$, the average value of $(p_t{}^2)'^{\rm ion}$ in this interval may be defined by the equation

$$\mathcal{T}(p_t{}^2)'^{\rm ion}=(\mathbf{p}_t+\Delta\mathbf{p}_t)^2-p_t{}^2, \tag{19}$$

where $\Delta\mathbf{p}_t$ is the sum of the momentum transfers from ion collisions, i.e., $\Delta\mathbf{p}_t=\Sigma_j[\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)]$. Although many collisions may occur simultaneously, the assumption of the additivity of the momentum transfers is quite reasonable since the quantity $\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)$ depends only on $(\mathbf{r}-\mathbf{r}_j)$, and since the transverse displacement $\mathbf{r}$ changes only a small amount during a single collision.

Because of the statistical nature of the collision process one cannot obtain $(p_t^2)'^{\text{ion}}$ for an individual electron, but must compute some sort of average. We average over a hypothetical, large random group of electrons whose transverse displacements and momenta are given by $\mathbf{r}$ and $\mathbf{p}_t$, denoting the average with the bracket symbols $\langle\ \rangle$. Although the average is defined in terms of a hypothetical group of electrons, it shall be interpreted as equivalent to an average over all the electrons of coordinates $\mathbf{r}$ and $\mathbf{p}_t$ in the beam under consideration. This assumption will not lead to a significant error, provided that the number of electrons in the beam is large. It is consistent, but not necessary, to regard the quantity $\langle(p_t^2)'^{\text{ion}}\rangle$ as representing an average over a small time interval $\mathcal{T}$ as well as over electrons. In the derivation of this section such a time average is assumed, for convenience.

If the distribution of ions in the beam is random, it may be shown by elementary theorems of statistics that $\langle\Delta\mathbf{p}_t\rangle$ and $\langle(\Delta\mathbf{p}_t)^2\rangle$ satisfy the following equations,[7]

$$\langle\Delta\mathbf{p}_t\rangle=\int d^2r_j[\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)]\mu(\mathbf{r}_j), \tag{20}$$

and

$$\langle(\Delta\mathbf{p}_t-\langle\Delta\mathbf{p}_t\rangle)^2\rangle=\langle(\Delta\mathbf{p}_t)^2\rangle-\langle\Delta\mathbf{p}_t\rangle^2$$
$$=\int d^2r_j[\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)]^2\mu(\mathbf{r}_j), \tag{21}$$

where $d^2r_j\mu(\mathbf{r}_j)$ denotes the mean number of ions with transverse displacement in the area d^2r_j passed by the electron in the time interval $\mathcal{T}$. If Eq. (19) is averaged over electrons and use is made of Eq. (21), the result is

$$\mathcal{T}\langle(p_t^2)'^{\text{ion}}\rangle=2\mathbf{p}_t\cdot\langle\Delta\mathbf{p}_t\rangle+\langle\Delta\mathbf{p}_t\rangle^2$$
$$+\int d^2r_j[\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)]^2\mu(\mathbf{r}_j). \tag{22}$$

From Eqs. (20) and (21) it is seen that if the ion beam were replaced by a uniform, continuous positive charge density, the resulting quantity $\langle\Delta\mathbf{p}_t\rangle$ would be identical to the $\langle\Delta\mathbf{p}_t\rangle$ of Eq. (20) and $\langle(\Delta\mathbf{p}_t)^2\rangle-\langle\Delta\mathbf{p}_t\rangle^2$ would equal zero. Therefore the change in p_t^2 caused by the average ion force $\mathbf{F}_t^{\text{smo}}$ is given by

$$\mathcal{T}(p_t^2)'^{\text{smo}}=\mathcal{T}\langle(p_t^2)'^{\text{smo}}\rangle=2\mathbf{p}_t\cdot\langle\Delta\mathbf{p}_t\rangle+\langle\Delta\mathbf{p}_t\rangle^2. \tag{23}$$

[7] These equations may be derived from Campbell's theorem. For a discussion of this theorem see S. O. Rice, Bell System Tech. J. **23**, 282 (1944). Equations (20) and (21) above may be derived from generalizations of Eqs. (1.2–2) and (1.2–3) of this reference if the function $I(t)$ is chosen to be $\mathbf{p}_t(t)-\mathbf{p}_t(t-\tau)=\Delta\mathbf{p}_t$.

The equation for the average effect of the ion fluctuation force may be obtained by subtracting Eq. (23) from Eq. (22), i.e.,

$$\mathcal{T}\langle(p_t^2)'^{\text{flu}}\rangle=\int d^2r_j[\Delta\mathbf{p}_t(\mathbf{r}-\mathbf{r}_j)]^2\mu(\mathbf{r}_j). \tag{24}$$

The integral in this equation may be evaluated if $\Delta p_t(\mathbf{r}-\mathbf{r}_j)$ is taken from Eq. (18). Since the principal contribution to the integral comes from small values of the impact parameter, we approximate the integral by assuming the transverse cross section of the ion beam to be a circle centered at the electron coordinate $\mathbf{r}$, rather than at the beam center. [Such an approximation could not be used, of course, if the integral of Eq. (20) were under consideration.] In this approximation the function $\mu(\mathbf{r}_j)$ is given by

$$\begin{aligned}&\mu(\mathbf{r}_j)=[n_+/(\pi R_+^2)]c\mathcal{T} &&\text{if}\quad |\mathbf{r}-\mathbf{r}_j|\leqq R_+,\\ &\mu(\mathbf{r}_j)=0 &&\text{if}\quad |\mathbf{r}-\mathbf{r}_j|>R_+.\end{aligned} \tag{25}$$

If Eqs. (18) and (25) are substituted into Eq. (24) and the integral is evaluated, the result may be written in the form,

$$2\langle\mathbf{p}_t\cdot\mathbf{F}_t^{\text{flu}}\rangle=\langle(p_t^2)'^{\text{flu}}\rangle=\frac{8E_0^2r_0^2n_+L_+}{cR_+^2}, \tag{26}$$

where L_+ is defined by the relation $L_+=\log(R_+/R_{\min})$ and $R_{\min}$ is the minimum impact parameter to be considered.

If the force exerted by the neutral atoms on the electrons is written as the sum of an average part and a fluctuation part, as is done for the force $\mathbf{F}^{\text{ion}}$, it is seen that the average force exerted by the atoms is zero. Hence $\mathbf{F}^{\text{neu}}$ is a rapidly fluctuating force resulting from close electron-atom collisions. For impact parameters substantially smaller than an atomic radius, the dependence of the momentum transfer upon impact parameter is identical to that of Eq. (18), while for impact parameters substantially larger than an atomic radius, the momentum transfer is zero. Therefore, the expression for $\langle\mathbf{F}_t^{\text{neu}}\rangle$ may be obtained from the $\langle\mathbf{F}_t^{\text{flu}}\rangle$ expression if two substitutions are made; i.e., the density of atoms within the beam replaces the ion density, and the atomic radius replaces the beam radius in the logarithmic term. The expression is,

$$\langle2\mathbf{p}_t\cdot\mathbf{F}_t^{\text{neu}}\rangle=8\pi E_0^2r_0^2\rho_tL_0/c, \tag{27}$$

where L_0 is given by $L_0=\log(R_{\text{atom}}/R_{\min})$. The beam electron may scatter from the atomic electron as well as from the proton, but this effect is already included (approximately) in Eq. (27) if R_{atom} is appropriately chosen. We choose R_{atom} to be 5×10^{-9} cm, the distance between the electron and proton in the hydrogen atom.

A reasonable choice of $R_{\min}$ is the impact parameter corresponding to a momentum transfer sufficient to knock a beam electron out of the beam into the chamber walls. If the self-focusing forces dominate the external

focusing forces, this criterion leads to the relation[3] $R_{\min}=r_0E_0^{\frac{1}{2}}[E\nu_+\log(R_{\text{tube}}/R)]^{-\frac{1}{2}}$, where R_{tube} is the radius of the accelerating tube. Since L_+ and L_0 depend only weakly on the beam parameters, we simplify the equations by replacing these functions by numbers, chosen to correspond to beam parameters typical of situations in which the scattering is important. We choose the following values,

$$R_{\min}=0.2r_0, \quad L_+=25, \quad L_0=11. \tag{28}$$

VI. EQUATIONS DESCRIBING ELECTRON BEAM PARAMETERS

The results of Secs. IV and V may be substituted into the equations describing the electron motion [Eqs. (10) and (11)], in order that these equations may be expressed in terms of known parameters and parameters describing the ion beam. (The behavior of the ion beam parameters is discussed in Sec. VII.) Since some of the terms in Eqs. (10) and (11) can be expressed in simple form only if certain averages over electrons in the beam are taken, we will not be able to describe the changing parameters of a single electron, but only parameters describing the electron beam as a whole.

Several types of averages are to be taken, so several notations must be used to represent them. As mentioned in Sec. V a single pair of angular brackets $\langle\ \rangle$ is used to represent an average over a large group of electrons whose transverse displacements $\mathbf{r}$ and transverse momenta $\mathbf{p}_t$ are the same at the time in question. A double pair of angular brackets $\langle\langle\ \rangle\rangle$ represents a further average over all electrons described by the same transverse amplitude squared A^2, and may be considered a time average over a period of transverse oscillation as well. Ordinary parenthesis and an asterisk $(\ )^*$ denote an average over all electrons of all amplitudes. For certain of the variables in question, these three degrees of averaging are not all significant, of course. For example, the notations $\langle r^2\rangle$, $\langle p_t^2\rangle$, and $\langle A^2\rangle$ are redundant, since the $\langle\ \rangle$ refer to fixed values of $\mathbf{r}$ and $\mathbf{p}_t$. In a similar manner $\langle\langle A^2\rangle\rangle$ is redundant, and may be denoted simply by A^2.

The equations describing the effects of ion-electron and atom-electron collisions, Eqs. (26) and (27), apply only to an average over electrons described by the same $\mathbf{r}$ and $\mathbf{p}_t$ at some time. Therefore, we can compute only the average of Eq. (10), if we are to account for the scattering. If Eqs. (11), (14), (26), and (27) are substituted into Eq. (10) and the average over electrons of the same $\mathbf{r}$ and $\mathbf{p}_t$ is taken, the resulting equation is

$$\langle(A^2)'\rangle=-\frac{c^2p_t^2}{KE^2}\langle E'\rangle-\frac{2c^2p_t^2}{KE^2}I+\frac{8\pi E_0^2r_0^2c}{KE}\left(\frac{n_+L_+}{\pi R_+^2}+\rho_iL_0\right)+\frac{r^2-A^2}{K}K', \tag{29}$$

where the radiated power I is given by Eq. (16) if $\mathbf{F}_t^{\text{ext}}$ is the dominant force, or by Eq. (17) if $\mathbf{F}_t^{\text{smo}}$ is the dominant force. Since collisions lead to a negligible change in electron energy, the rate of change of energy may be computed for individual electrons, so that $\langle E'\rangle=E'=\mathcal{E}ec-I$.

The terms in Eqs. (11) and (29) depending on r^2 and p_t^2 fluctuate rapidly during the transverse oscillations. Since these fluctuations are not significant for determining the gross properties of the beam, simpler equations may be derived by averaging over all electrons of the same amplitude squared, (the $\langle\langle\ \rangle\rangle$ average discussed above). The relative changes in R_+^2, E, K, and $\mathcal{E}$, and the average relative change in A^2 during the time of a transverse oscillation are small, so that the $\langle\langle\ \rangle\rangle$ average is equivalent to averaging over an oscillation period, as well as over electrons. The averages over a transverse oscillation of r^2 and p_t^2 are given by the expressions,

$$\langle\langle r^2\rangle\rangle=\tfrac{1}{2}A^2, \tag{30a}$$

$$\langle\langle p_t^2\rangle\rangle=\tfrac{1}{2}(KE/c^2)A^2. \tag{30b}$$

The $\langle\langle\ \rangle\rangle$ average of $r^2p_t^2$ depends on the distribution of electrons in transverse states of different ellipticities. We may write

$$\langle\langle p_t^2r^2\rangle\rangle=\tfrac{1}{4}Q_1(KE/c^2)A^4, \tag{30c}$$

where Q_1 is a number between $\frac{1}{2}$ and 1. (If all electrons are in circular orbits Q_1 is one; if all electrons are in linear orbits Q_1 is $\frac{1}{2}$.) We shall assume that Q_1 is equal to $\frac{2}{3}$ for all values of A^2, since this number corresponds to a uniform distribution in the different elliptical orbits.

If use is made of Eqs. (30), the average of $(A^2)'$ and E' for all electrons of a particular amplitude squared may be expressed in terms of A^2 and other parameters. For most terms this process is simple. If the self-forces are dominant, however, the $\langle\langle\ \rangle\rangle$ averages of the terms involving E' and I in Eq. (29) are rather complicated, so these terms are listed below.

$$\frac{2c^2}{KE^2}\langle\langle p_t^2I\rangle\rangle=\tfrac{4}{3}Q_1\frac{EA^4}{E_0R_+^4}\nu_+^2r_0c=Q_1\frac{A^2}{E}\langle\langle I\rangle\rangle, \tag{31a}$$

$$\frac{c^2}{KE^2}\langle\langle p_t^2E'\rangle\rangle=\frac{A^2}{2E}(\mathcal{E}ec-Q_1\langle\langle I\rangle\rangle). \tag{31b}$$

In order to write the equations of motion for the mean square beam radius, it is necessary to average over electrons of different amplitude; i.e., over all electrons in the beam. [The average denoted by $(\ )^*$]. To compute this average we use the relation,

$$(A^2)^*=2(r^2)^*=R^2, \tag{32a}$$

which follows from Eq. (30a) and the definition of the electron beam radius R, and the relation

$$(A^4)^*=4Q_2[(r^2)^*]^2=Q_2R^4, \tag{32b}$$

where Q_2 is a number depending on the distribution of the electrons in orbits of different amplitudes.

If the average over all electrons of Eqs. (11) and (29) is taken, and use is made of Eqs. (30) and (32), the results are,

$$E' = \mathcal{E}ec - (I)^*, \tag{33}$$

$$(R^2)' = -\tfrac{1}{2}\frac{R^2}{E}E' - \frac{R^2}{E}Q(I)^* + \frac{8\pi E_0^2 r_0^2 c}{KE}\left(\frac{n_+ L_+}{\pi R_+^2} + \rho_i L_0\right) - \tfrac{1}{2}\frac{R^2}{K}K', \tag{34}$$

where the energy spread of the electrons has been neglected, i.e., $(E)^*$ has been set equal to E. The quantities Q and $(I)^*$, in the two cases of dominant guide-field force, and dominant self-force, are given by the expressions,

Case I. ($\mathbf{F}_t^{\text{ext}}$ *dominant*)

$$Q_1 = 1,$$
$$(I)^* = I = 2E^4 r_0 c/3E_0^3 \mathcal{R}^2. \tag{35}$$

Case II. ($\mathbf{F}_t^{\text{smo}}$) *dominant*

$$Q = \tfrac{1}{2}(1+Q_1Q_2),$$
$$(I)^* = 4E^2R^2\nu_+^2 r_0 c/3E_0 R_+^4. \tag{36}$$

The value of Q_2 depends on the distribution of electrons in orbits of different amplitudes. In principle this distribution may be computed from a diffusion equation and a knowledge of the initial distribution. Such a procedure is complicated, however, and often requires a more detailed knowledge of the initial conditions than is available, so we shall simply estimate a reasonable range of values for Q_2. We define the distribution function $f(A^2,t)$ so that $f(A^2,t)d(A^2)$ represents the fraction of all electrons whose amplitudes squared are in the range $d(A^2)$ at time t. In the case that the radiated energy is independent of amplitude, so that the quantity $\langle\langle p_t^2 I\rangle\rangle$ is proportional to A^2, it can be shown that an exponential distribution of the form $f(A^2,t) = \alpha^{-2}\exp(-A^2/\alpha^2)$ is a stable, time-independent solution to the diffusion equation.[8] In such a case the actual distribution approaches this equilibrium distribution after sufficient time has elapsed. In the dominant self-force case of the present work the quantity $\langle\langle p_t^2 I\rangle\rangle$ is proportional to A^4 [see Eq. (31a)], implying that large amplitude oscillations are rapidly damped so that the distribution tends to be less broad than an exponential distribution. Therefore, it is reasonable to expect that Q_2 is intermediate between the values corresponding to exponential distribution ($Q_2=2$) and a sharply peaked distribution ($Q_2=1$). If, as assumed above, Q_1 is equal to $\frac{2}{3}$, the quantity $\frac{1}{2}(1+Q_1Q_2)$ is in the range,

$$\tfrac{5}{6} < \tfrac{1}{2}(1+Q_1Q_2) < 7/6. \tag{37}$$

An alternate, useful form of Eq. (34) is the expression

$$(K^{\frac{1}{3}}E^{\frac{1}{3}}R^2)' = -R^2\left(\frac{K}{E}\right)^{\frac{1}{3}}Q(I)^* + \frac{8\pi E_0^2 r_0^2 c}{(KE)^{\frac{1}{3}}}\left(\frac{n_+ L_+}{\pi R_+^2} + \rho_i L_0\right). \tag{38}$$

It is seen from this equation that the product $Z = K^{\frac{1}{3}}E^{\frac{1}{3}}R^2$ changes only because of radiation damping and scattering from ions and neutral atoms. This result is well known and is usually derived from the adiabatic theorem; it results from the fact the period of transverse oscillation is short compared to the times in which the force constant and the energy change appreciably. The puantity Z is termed the action integral since it is proqortional to the average over electrons of the sum over the two transverse directions of the action integral $\int p dq$.

In order to use the equations derived in this section to investigate the behavior of a hypothetical beam, it is necessary to express K and K' in terms of other parameters. This is done in the next section.

VII. BEHAVIOR OF THE IONS AND OF K^{ion}

The force constant K defined in Eqs. (7) and (8) depends only on the external fields and ion charge density $n_+/(\pi R_+^2)$. The external focusing force constant K^{ext} is known, of course, throughout the experiment. The control exerted by the experimenter on the ion charge density is limited, and depends primarily on his ability to control the vacuum, and on the shrinkage rate of the beam. Ions are continuously produced within the acceleration chamber by ionization collisions between electrons and neutral gas atoms. If the neutralization is not complete (in the laboratory system) the newly produced positive ions are held within the beam by electrostatic attraction, while the freed electrons are repelled to the walls of the chamber. This is the only neutralization process considered here.

The quantity K^{ion} changes in time because of two effects, the generation of new ions and the change of radius of the ion beam, i.e.,

$$(K^{\text{ion}})' = \frac{2E_0 r_0}{R_+^2}n_+' - \frac{2E_0 r_0 n_+}{R_+^4}(R_+^2)'. \tag{39}$$

The effects of ion radiation damping, ion-ion collisions, and electron-ion collisions on the ion radius are small, so that R_+ may be considered to change only because the changing electron radius leads to a changing potential

[8] One-dimensional cases of this nature are discussed by J. M. Greenberg and T. H. Berlin, Rev. Sci. Instr. 22, 293 (1951), and by G. K. O'Neill and J. A. Ball, Palmer Physical Laboratory (Princeton University) Report NYO-8015, 1957 (unpublished).

for the ion motion. Hence we may write

$$(R_+^2)' = (dR_+^2/dR^2)(R^2)'. \tag{40}$$

Since the period of the ion transverse motion is short compared to the time in which the electron beam changes appreciably, the change in R_+ is adiabatic, and dR_+^2/dR^2 may be determined from the condition that the amplitude for each ion varies so that the transverse phase integrals remain constant.

For any given initial conditions, approximate values of K^{ion} and $(K^{\text{ion}})'$ may be calculated at all stages of the shrinking beam. The method of calculation must be different at different stages, however, so we discuss briefly here the different conditions that may apply concerning the behavior of K^{ion}.

Case A. Neutralization Fraction Small

If the ratio of ions to electrons in the beam is small the generation of new ions by ion-atom collisions may be neglected. The effect of ion-ion forces on the changing ion radius may also be neglected. If the electron distribution is nearly uniform within the beam, and if the radius of the ion beam is comparable to that of the electron beam, so that the amplitudes of most of the ions are smaller than or approximately equal to the electron radius, the potential felt by the ions may be approximated by a harmonic oscillator potential. The force constant for the ion motion depends on the inverse square electron radius, i.e., $K_+ = CR^{-2}$, where C is a constant. The adiabatic theorem implies that the amplitude of any ion changes with the electron radius in the manner, $dA_+^2/dR^2 = \frac{1}{2}A_+^2/R^2$. Since R_+^2 is proportional to the average over ions of A_+^2, this relation implies the relation,

$$(R^2/R_+^2)(dR_+^2/dR^2) = \tfrac{1}{2}. \tag{41}$$

If the generation of new ions during the shrinkage process is appreciable, it is inconsistent to consider the ion density as uniform throughout the ion beam at all stages of the process, as may be seen by the following argument. The ion beam shrinks more slowly than the electron beam so that, in general, the ion radius is larger than the electron radius (many ions oscillate through the electron beam). New ions are generated only within the electron beam, however, so that the charge density within the electron beam must become greater than that outside. It is still consistent to consider the ion density as constant within the *electron* beam, however, though this density must decrease with radius at radii larger than the electron beam radius. Fortunately, constancy within the electron beam is all that is necessary in order that the electrons may be considered to be moving in a harmonic oscillator potential. Furthermore, the formula for $(K^{\text{ion}})'$, Eq. (39), may be put in a form independent of R_+ by the following procedure. The ion radius R_+ is replaced by R in the ion generation term, since new ions are created only within the electron beam. The equations, Eqs. (8), (40) and (41) are then used to express the $(R_+^2)'$ term in terms of R^2, $(R^2)'$ and K^{ion}. The resulting equation is,

$$(K^{\text{ion}})' = \frac{2E_0 r_0}{R^2} n_+' - \tfrac{1}{2}\frac{K^{\text{ion}}}{R^2}(R^2)'. \tag{42}$$

A further complication arises if the radius of the ion beam becomes many times greater than that of the electron beam, so that most of the ions spend most of the time outside the electron beam. In such a case the ion potential increases with radius considerably less rapidly than does an oscillator potential, and the adiabatic theorem may be used to show that Eqs. (41) and (42) do not represent a valid approximation. A more accurate approximation is to assume that the ion oscillations are not affected by the shrinking electron beam, so that the ion radius remains constant, i.e.,

$$(dR_+^2/dR^2) = 0. \tag{43}$$

This argument implies that in some cases one should keep track of the density of ions at different values of the radii for values exceeding the electron radius. One could do this by regarding the beam as a sum of component beams, each component consisting of ions generated in a certain time interval. The ion charge density within each component could be regarded as uniform, but the radii of the different component beams would be different. Calculations show that, with reasonable accuracy, the radius of each component could be regarded as following the law of Eq. (41) as long as this radius were no more than 3 or 4 times the electron radius; after that, Eq. (43) would be more appropriate.

Since the density of neutral atoms within the beam may be less than the density without, the calculation of $n_+' = r_0\nu_+'$ is not as simple as one might at first imagine. It is necessary to know not only the ionization cross section, but also the rate of entry of atoms into the electron beam. In order to calculate the rate of entry of atoms, we restrict ourselves to cases in which the electron current is less than 17 000 amperes ($\nu < 1$), the neutral hydrogen density is less than 10^{16} atoms/cc everywhere, and the beam is contained in a tube large enough so that the total number of atoms in the tube exceeds the number of beam electrons by a factor of at least ten. Under these conditions, the following argument shows that the density of atoms at all points outside the beam may be considered as unaffected by the presence of the beam. The mean free path λ for atom-atom scattering collisions is larger than the radius of the electron beam. The number of atoms in an imaginary tube of radius equal to λ is many times larger than the number of electrons in the beam. Therefore, the paths of most atoms entering the beam originate in atom-atom collisions at radii where the atom density is essentially unaffected by the presence of the beam. Thus the atom density outside may be considered a constant, denoted by ρ_0. The path length λ may be greater than the radius of the accelerating tube; the

conclusion reached above remains valid in this case, however, since the number of atoms in the tube exceeds the number of electrons in the beam.

The number N_1 of atoms entering the beam per unit time per unit beam length may be computed by standard methods of kinetic theory[9] and is given by the formula,

$$N_1=(2\pi/3)^{\frac{1}{2}}\rho_0 R\bar{v}, \tag{44}$$

where $\bar{v}$ is the root mean square thermal velocity of the atoms [i.e., $\bar{v}=(3kT/M)^{\frac{1}{2}}$]. It should be noted that a simple diffusion equation cannot be used to compute N_1, since the mean free path λ is long.

If the atom density within the beam is equal to that outside, the number N_2 of ions formed per unit time per unit beam length is given by the expression,

$$N_2=\rho_0\sigma_i cn, \tag{45}$$

where σ_i is the cross section for ionization by the electrons.

A simple estimate of the ionization rate n_+' may be made by setting it equal to the smaller of N_1 and N_2, Eqs. (44) and (45). The fact that this is a reasonable procedure may be seen by noting that the ratio of N_2 to N_1 is equal to the ratio of the mean time T_2 for an atom to remain in the beam before being carried out by its thermal velocity, and the mean lifetime T_1 before ionization of an atom in the beam. The expressions for these two times are: $T_2=(3\pi/2)^{\frac{1}{2}}(R/\bar{v})$, and $T_1=\pi R^2/(n\sigma_i c)$. If $T_1\gg T_2$, in which case $N_1\gg N_2$, most atoms entering the beam pass through it, so that the atom density within the beam is nearly equal to that outside and the ionization rate is nearly equal to N_2. If $T_2\gg T_1$, in which case $N_2\gg N_1$, most atoms entering the beam are ionized, so that the ionization rate is nearly equal to N_1. If T_1 and T_2 are nearly equal, the actual ionization rate is somewhat less than either N_1 or N_2.

Case B. Neutralization Fraction Large but Less than One

If the ratio of ions to electrons in the beam is appreciable, several additional effects are important. Ionization collisions between ions and neutral atoms must be considered; this process may be as important as ionization by electrons. It is not necessary to estimate this effect in order to make the calculations of Sec. VIII, so we do not include such an estimate here; however the calculations of de Packh and Godlove,[10] concerning the adiabatic shrinking of partially neutralized beams, include the effect of ionization by ions.

In an appreciably ionized beam the force exerted by the ions on each other is comparable to the electron-ion force, and decreases the magnitude of the attractive potential felt by the ions. Consequently the ion beam is widened. If there are sufficient ionization collisions the linear ion density may become equal to the linear electron density. When this condition is first reached the ion radius is generally larger than the electron radius. As further ions are produced within the electron beam, ions oscillating with large amplitudes escape, leading to a decrease in mean ion radius. Eventually the electron beam may become neutralized in all regions by an ion beam of radius equal to the electron radius.

Case C. Complete Neutralization

If the electron beam is completely neutralized by an ion beam of equal radius, each newly formed ion will release another ion from the beam and will not affect the charge density. If the electron beam shrinks the ion beam shrinks less rapidly, so that the neutralization tends to decrease, but ion generation may be sufficient to maintain complete neutralization. In such a case the ion beam parameters may be expressed simply in terms of the electron parameters, i.e.,

$$n_+=n, \tag{46a}$$

$$R_+=R, \tag{46b}$$

$$K^{\text{ion}}=2E_0\nu/R^2, \tag{46c}$$

$$(K^{\text{ion}})'/K^{\text{ion}}=-(R^2)'/R^2. \tag{46d}$$

The above discussion makes it clear that if the beam is completely neutralized, the rate of change of K^{ion} may be considered equal to the smaller of the rates computed from Eqs. (39) and (46d).

VIII. SOLUTION OF THE EQUATIONS FOR SOME HYPOTHETICAL BEAMS

If values of the initial conditions and external fields are chosen the development of the preceding sections may be used to determine the radius and energy of an intense, relativistic electron beam as functions of time. We shall carry out this procedure for certain sets of parameters, chosen so that the beam shrinkage takes place in a reasonably short time.

A. The Budker Equilibrium State

The ultimate result of the shrinkage process is the "Budker equilibrium state," in which R^2 and E are constant. Since a perfect vacuum cannot be achieved, the neutralization in the lab system must be essentially complete in the equilibrium state. We assume that the self-focusing forces dominate the guide-field force and that ion-electron scattering dominates atom-electron scattering as equilibrium is approached. In this case K is equal to K^{ion}, and is given by Eq. (46c). If Eqs. (46a) through (46d) are substituted into Eq. (34), and the neutral scattering term is dropped, the result is

$$\frac{(R^2)'}{R^2}=-\frac{E'}{E}-\frac{2}{E}Q(I)^*+\frac{8E_0 r_0 cL_+}{ER^2}, \tag{47}$$

[9] J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (John Wiley and Sons, New York, 1950), p. 17.

[10] D. de Packh and T. F. Godlove, Bull. Am. Phys. Soc. Ser. II, **3**, 181 (1958).

where $(I)^*$ is given by

$$(I)^*=\frac{4E^2\nu^2r_0c}{3E_0R^2}, \tag{48}$$

and $E'=\mathcal{E}ec-(I)^*$ [Eq. (33)]. We take the number Q equal to one. The equilibrium conditions, obtained by setting $(R^2)'$ and E' equal to zero, are

$$\nu E_e/E_0=(3L_+)^{\frac{1}{2}}\approx 9, \tag{49a}$$

and

$$\mathcal{E}e\approx 100E_0r_0/R_e^2. \tag{49b}$$

If the symbols J, $R_{e,\,\mathrm{cm}}$ and $\mathcal{E}_v$ are used to represent the current in amperes, equilibrium electron beam radius in centimeters, and electric field in volts/cm, Eqs. (49a) and (49b) are equivalent to the expressions,

$$E_e/E_0\approx 150\,000/J,$$
$$R_{e,\,\mathrm{cm}}\approx 3.5\times 10^{-3}/\mathcal{E}_v^{\frac{1}{2}}.$$

The quantity in Eq. (49a) is closely related to the electron transverse momentum in the case of complete neutralization and negligible external focusing forces. If Eq. (30b) is averaged over all electrons, and use is made of Eqs. (32a) and (46c), the result is

$$(p_t^2)^*c^2/E_0^2=\nu E/E_0. \tag{50}$$

It is seen from this equation and from Eq. (49a) that the average electron transverse momentum is independent of beam radius, and is relativistic, provided that the energy is comparable to or larger than the equilibrium energy for the electron density in question.

B. Factors that Limit the Shrinkage Time

We now return to the description of the dynamics of the beam prior to the attainment of equilibrium. It is seen from Eq. (38) that the processes influencing the electron beam radius may be grouped into two classes, adiabatic processes that leave the action integral Z constant, and the nonadiabatic radiation and scattering processes. If the initial force constant and energy are low, it is possible to increase the product KE rapidly, so that the adiabatic shrinkage may proceed rapidly. This may be accomplished by injecting a low energy, weakly focused beam into a neutral gas of density on the order of 10^{12}-10^{14} atoms/cc, and applying a longitudinal electric field. On the other hand, the radiation damping leads to a slow shrinkage, unless the electron energy is quite high. Unfortunately, unless the external focusing forces are increased a fantastic amount, there is a definite limit to the amount of beam shrinkage obtainable adiabatically for any fixed, final energy, since the ion part of the force constant cannot be made larger than that corresponding to a completely neutralized beam.

In Parts I and III of Table I, the values of Z corresponding to certain specified initial conditions are compared to the Z values corresponding to the Budker equilibrium states for some hypothetical beams. In the cases considered the equilibrium Z is 15 to 100 times smaller than the initial Z. Since $Z=(2E_0\nu)^{\frac{1}{2}}E^{\frac{1}{2}}R$ for a completely neutralized beam, a decrease of 15–100 in Z corresponds to a decrease of 15–100 in the beam radius, a decrease that must be effected by the radiation damping forces.

TABLE I. Parameters descriptive of neutralization stage and of equilibrium state for some hypothetical beams. Part I refers to the initial conditions and external fields, Part II refers to the instant complete neutralization is attained, and Part III refers to the equilibrium conditions. The initial state is taken to be completely unneutralized with a betatron wavelength of 100 cm resulting from external focusing forces. The value of the action integral during the neutralization stage is then given by $Z_i=Z_n=2\pi E_iR_i^2/100$ cm.

	Parameter	Beam 1	Beam 2	Beam 3	Beam 4
I.	E_i (Mev)	2	2	8	2
	R_i (cm)	1	2.5	0.5	2
	Z_i (Mev-cm)	0.13	0.77	0.13	0.50
	$\mathcal{E}$ (volt/cm)	1	1	10	4
	ν	0.9	0.09	0.09	0.009
	ρ_0 (atoms/cc)	10^{13}	10^{12}	10^{13}	10^{12}
II.	T_n (10^{-5} sec)	2.5	30	2.6	30
	E_n (Mev)	2.7	11	16	37
	R_n (cm)	0.08	0.72	0.11	0.75
III.	E_e (Mev)	5	50	50	500
	R_e (cm)	0.0035	0.0035	0.0012	0.0018
	Z_e (Mev-cm)	0.0077	0.0077	0.0025	0.0038

It is seen from Eq. (38) and the equations for $(I)^*$ that the action integral Z may be decreased most rapidly if the force constant and energy are maintained as high as possible. Therefore, it is desirable that the beam become neutralized rapidly. A rough estimate of the time necessary for neutralization is given by the "primary neutralization time," defined by the equation,

$$T_p=(\rho_0\sigma_ic)^{-1}. \tag{51}$$

This is the mean time necessary for an electron to make an ionization collision if the neutral hydrogen density within the electron beam is equal to that outside. If σ_i is taken to be 2×10^{-19} cm^2, then T_p is given by the relation $T_p=1.6\times 10^{-3}$ seconds $\times(10^{11}$ per cc$/\rho_0)$. (At room temperatures $\rho_0=3\times 10^{11}$/cc corresponds to a pressure of about 10^{-5} mm Hg.) It is shown in Sec. VIII D that the time necessary for the radiative effect to shrink the beam appreciably is generally longer than 10^{-2} second, hence much longer than the neutralization time if the gas density is 10^{11} atoms/cc or more.

We will consider only beams in which the ionization is rapid enough so that the beam becomes completely neutralized in 10^{-3} second or less and remains neutralized thereafter. The evolution of the beam will be described in two successive stages, a short "neutralization" stage and a longer "radiation" stage, defined to begin when complete neutralization is attained.

C. The Neutralization Stage

Under the conditions considered here, the neutralization stage is short enough that the effect of radiation may be neglected. The effects of scattering from the ions and atoms may also be neglected, unless the energy is below a few Mev and the density of atoms or ions is quite high. Hence, we consider the action integral Z to be conserved in this stage.

Since the time necessary for neutralization is short compared to the total shrinkage time, we neglect some of the complicated contributions to K'. We assume that K^{ext} is constant, and that $(K^{\text{ion}})'$ is given by Eq. (42), with n_+' taken as equal to the smaller of the two rates N_1 and N_2, as discussed in Sec. VII. The principal effect omitted by this procedure is ionization by the ions, so that the calculations of this section should lead to an overestimate of the time necessary for neutralization.

The three equations of motion for R^2, E, and K^{ion} are Eqs. (33), (34), and (42), with the nonadiabatic scattering and radiation terms neglected. If these three equations are solved for E', $(R^2)'$, and $K'=(K^{\text{ion}})'$, and if K^{ion} is replaced by $K-K^{\text{ext}}$, the results are,

$$E'=\mathcal{E}ec, \tag{52}$$

$$\frac{(R^2)'}{R^2}\left(1+\tfrac{1}{3}\frac{K^{\text{ext}}}{K}\right)=-\frac{4E_0\nu_+'}{3KR^2}-\frac{2\mathcal{E}ec}{3E}, \tag{53}$$

$$\frac{K'}{K}\left(1+\tfrac{1}{3}\frac{K^{\text{ext}}}{K}\right)=\frac{8E_0\nu_+'}{3KR^2}+\left(1-\frac{K^{\text{ext}}}{K}\right)\frac{\mathcal{E}ec}{3E}. \tag{54}$$

The results of integrating Eqs. (52) through (54) for certain assumed values of the initial conditions and external fields are listed in Parts I and II of Table I. The actual neutralization time T_n, defined as the time in which K^{ion} becomes equal to $2E_0\nu/R^2$, and the values of beam electron radius and energy at the instant of neutralization are listed.

In order to investigate the factors that speed up or slow down the neutralization process, it is helpful to define the fractional neutralization of the beam, $F=K^{\text{ion}}R^2/(2E_0\nu)$. The equation of motion for F, determined from its definition, and Eqs. (7), (8), (52), (53), and (54), is

$$F'=\left[1+\tfrac{1}{3}\frac{K^{\text{ext}}}{K}\right]^{-1}\left[\frac{\nu_+'}{\nu_+}\left(\tfrac{2}{3}+\tfrac{2}{3}\frac{K^{\text{ext}}}{K}\right)-\frac{F\mathcal{E}ec}{3E}\right]. \tag{55}$$

If the electric field remains constant, the energy is given by $E=E_i+\mathcal{E}ecl=\mathcal{E}ec(t+T_E)$, where the time $T_E=E_i(\mathcal{E}ec)^{-1}$ is the time necessary for the energy to increase by an amount equal to the original energy.

If the quantity N_2 [Eq. (44)] is less than N_1 [Eq. (45)] the ionization rate is given by $n_+'=N_2=n/T_p$. Comparison of Eqs. (44) and (45) implies that $N_2<N_1$ whenever R is greater than the critical radius $R_{\text{cr}}=[3/(2\pi)]^{\frac{1}{2}}(n\sigma_i c/\bar{v})$. If the neutral gas is hydrogen at room temperature, this relation may be written, $R_{\text{cr}}=0.062\nu$ cm. In all the cases listed in Table I the beam radius is larger than the radius R_{cr} during the entire neutralization process. Hence we may replace the ratio $(n_+'/n)=(\nu_+'/\nu)$ by the reciprocal of the primary neutralization time T_p. If the replacements $\nu_+'/\nu=T_p^{-1}$ and $\mathcal{E}ec/E=(t+T_E)^{-1}$ are made, and if the quantity K^{ext}/K (which is generally small except very early in the neutralization process) is dropped, Eq. (55) becomes independent of K, R, and E. In this case the solution to the equation may be written in the form,

$$F(t)=\left(F_i-\tfrac{1}{2}\frac{T_E}{T_p}\right)\left(\frac{T_E}{t+T_E}\right)^{\frac{1}{3}}+\tfrac{1}{2}\frac{t+T_E}{T_p}, \tag{56}$$

where F_i is the value of F at time zero.

The actual neutralization time T_n may be computed from Eq. (56) by setting $F(T_n)$ equal to one. Although the resulting equation for T_n is complicated in general, it is very simple in the extreme limits corresponding to very large and very small values of the ratio T_E/T_p, i.e.,

$$\text{As } T_E/T_p\to\infty \quad T_n\to\tfrac{3}{2}(1-F_i)T_p, \tag{57a}$$

$$\text{As } T_E/T_p\to 0 \quad T_n\to 2T_p. \tag{57b}$$

It may be seen from the differential equation, Eq. (55), that F' is a monotonic increasing function of T_E; hence, for any T_E the ratio T_n/T_p lies between the above two limits, i.e., $\frac{3}{2}(1-F_i)\leqq(T_n/T_p)<2$.

The limit of Eq. (57a) corresponds to the case in which there is no energy change during the neutralization process. The factor of $\frac{3}{2}$ in this equation results from the fact that the ion beam shrinks less rapidly than the electron beam, so that the number of ions effective for neutralization at a given time (the average number of ions within the electron beam) is less than the total number produced prior to that time. In this case, approximately $\frac{1}{3}$ of the produced ions are later lost to the beam. If T_E is much smaller than T_p, the relative energy increase is large, and the additional beam shrinkage resulting from this energy increase causes most of the ions present early in the neutralization process to be lost to the electron beam, thus inhibiting the process and leading to a longer neutralization time.

An independent calculation of the behavior of intense electron beams has been made by de Packh and Godlove.[10] These authors are particularly concerned with the beam behavior during the neutralization process.

It is concluded that one may easily obtain complete neutralization of the electron beam in a time of less than a millisecond by introducing the beam into a neutral gas of sufficient density.

D. The Radiation Stage

After neutralization is attained, but before radiation is important, the proper choice of F' is the smaller of the

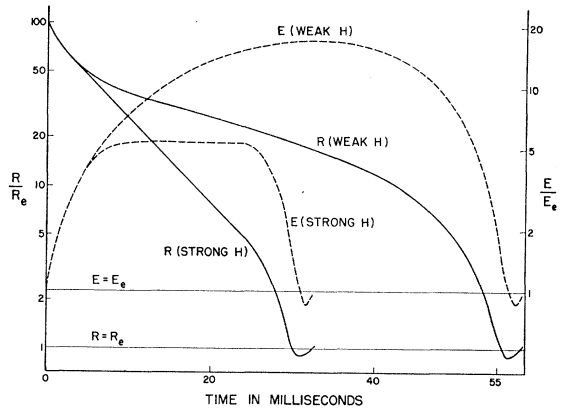

FIG. 1. Behavior of a completely neutralized, 1500 ampere electron current in an electric field of one volt/cm. The curves labeled "Weak H" refer to a beam circulating in a machine of radius greater than 10 meters; the curves labeled "Strong H" correspond to a machine of radius 88 cm.

values zero and the value obtained from Eq. (55). Unless the electric field is increased to such a high value that the expression of Eq. (55) becomes negative, the neutralization remains complete. The effects of radiation and scattering are sufficiently weak that they do not alter the conclusion that once complete neutralization is attained, the beam remains neutral unless the electric field is greatly increased or the neutral atom density is greatly decreased. We assume that neutralization remains complete.

It may be seen from Eq. (38) and the expressions for $(I)^*$ that the action integral Z decreases most rapidly if the energy is maintained at a high value. Hence, we consider only methods of approaching equilibrium in which the electrons initially are accelerated to an energy well above the equilibrium energy E_e, and the energy remains well above E_e until near the end of the shrinkage process. Under these conditions the effect of scattering is important only after the beam radius has shrunk to a size comparable to the equilibrium radius. Since the ion density is much higher than the atom density for neutralized beams of small radii, the effect of atom scattering may be neglected throughout the radiation stage. The equations of motion for E and R^2 are given by Eqs. (33) and (47), where the average radiated power $(I)^*$ is taken as the larger of the values computed from Eqs. (35) and (48). [Equation (35) results in the larger value if the guide-field forces dominate the self-forces.] The value of Q is taken to be one and L_+ is assumed to be 25 [see Eq. (28)].

If values for the initial conditions and external fields are assumed, Eqs. (33) and (48) may be numerically integrated in order to obtain R^2 and E as functions of time. The curves of Fig. 1 represent the results of such a procedure for a typical case in which the longitudinal field remains constant throughout the process. The values of $\mathcal{E}$, ν, R_e and E_e corresponding to these curves are those of Beam 2 of Table I. The initial conditions are $R=0.35$ cm, $E=50$ Mev$=E_e$. These conditions correspond to an initial value of the action integral Z of 0.77 Mev-cm, and thus represent Beam 2 of Table I if, after complete neutralization is attained, the beam is accelerated from 11 Mev to 50 Mev. In order for this beam to reach equilibrium, it is necessary for the radiation to lead to a decrease of the action integral to 1/100 its original value.

The curves labeled "Weak H" apply if the beam circles in a machine of radius sufficiently large that the effect of the guide field on the radiation damping may be neglected. In this case the electron energy rises to 860 Mev before the radiated power exceeds the power fed in by the longitudinal field. The entire shrinkage process requires about 0.06 second. An extension of the curves to the right would show oscillations of rapidly decreasing amplitudes about the equilibrium values R_e and E_e.

The curves labeled "Strong H" apply if the beam circles at such a radius ($\mathcal{R}\approx 88$ cm) that the guide magnetic field required is 10 000 gauss when the beam energy is at its maximum. In this case the beam energy increases to 275 Mev, at which point the radiated power resulting from the guide-field force is equal to the power fed in. The energy remains at this plateau value until the radius shrinks to such a point that the average self-force is equal to the guide-field force, after which time the radiation associated with the self-forces leads to decreases in both the energy and radius until equilibrium is approached. If the contributions to $(I)^*$ of both the guide force and the self-force were included at all times, rather than just the larger of the two, the strong field energy curve would start decreasing from the plateau value two or three milliseconds sooner, and the time required for the whole process would be decreased by 10 or 15%.

One may conclude from the curves of Fig. 1 that the presence of a strong guide field shortens the shrinkage time, and that a constant electric field during the shrinkage process results in a large fluctuation in beam energy, so that regulation of the guide field is a difficult problem.

An alternate procedure is to vary the electric field, holding it at a low enough value that the beam energy remains near the equilibrium energy, and increasing it to the desired final field strength near the end of the shrinkage process. In order for the effect of radiation to dominate the beam-widening effect of scattering from ions, it is necessary that the energy be held somewhat higher than the equilibrium energy, but one can require that this energy remain below a certain value. Unfortunately, this procedure results in a longer shrinkage time. For example, if a beam satisfying the initial conditions of Fig. 1 is circulating in a large machine, so that the radiation associated with the guide field is negligible, and the electric field is varied so that the beam energy rises to 100 Mev and remains there until near the end of the process, the total time required is about two seconds. If the machine radius is such (≈ 32 cm) that a guide

field of 10 000 gauss corresponds to 100 Mev, and if the electric field is varied so that the beam energy remains at 100 Mev until near the end of the process, the shrinkage takes place in about 0.1 second.

If a strong guide-field is used, the behavior of R^2 during the period when the energy is nearly constant at the plateau value may be determined by setting E' equal to zero and Q equal to one in Eq. (47) and computing $(I)^*$ from Eq. (35). If the scattering is neglected during this period the radius varies exponentially with time, i.e., $R=R_0\exp(-t/T_H)$, where T_H is given by

$$T_H=\tfrac{3}{8}\left(\frac{E_0}{E}\right)^3\frac{\Re^2}{r_0c}=\tfrac{3}{8}\frac{E_0^2}{EH^2r_0^2c}, \tag{58}$$

and H is the magnetic field at the orbit. If the guide-field strength is 10 000 gauss, the relaxation time T_H may be written $T_H=1.1(E_0/E)$ seconds. Since the time required for beam shrinkage depends primarily on the maximum energy attained, rather than on the equilibrium energy, this equation leads to a rough estimate of the beam-shrinkage time T as a function of $E_{\max}$ for the case of a strong guide field. If $H=10\,000$ gauss, the result of this estimate is

$$T\approx 1.1(E_0/E_{\max})\log(Z_i/Z_e)\text{ seconds}.$$

This time may be compared to the time in which ion-electron collisions are expected to destroy the beam. The effects of multiple scattering are compensated by the radiation damping, so we shall estimate the single-scattering lifetime, defined as the mean time necessary for an electron to undergo an ion collision of sufficiently small impact parameter to knock it into the wall of the accelerating tube.[11] This criterion implies that the single-scattering lifetime T_s is given by the relation $T_s=R_+^2(n_+cR_{\min}^2)^{-1}$, where $R_{\min}$ is the minimum impact parameter. If the value of $R_{\min}$ is taken from Eq. (28) and the beam is completely neutralized, the criterion becomes $T_s=25(R^2/\nu cr_0)$. For the beam considered in Fig. 1, the lifetime is about 0.4 second after equilibrium is reached and much longer while the beam radius is still large.

Fortunately, the parameters may be so chosen that the time required for beam shrinkage is less than the single-scattering lifetime. However, as is pointed out in Sec. II, it is by no means certain that the beam can be protected against all instabilities for a time sufficient for the equilibrium to be reached.

IX. MODIFICATIONS OF THE MODEL AND CONCLUSIONS

The derivations of this work could be generalized so that it would not be necessary to make most of the simplifying assumptions listed in Sec. II, and at various other points of the paper. On the other hand, many aspects of this problem are inherently so complicated that one does not expect the calculations to be accurate to better than a factor of two or so, even if no unnecessary assumptions are made. Thus it is not worthwhile to generalize the equations; instead we will discuss qualitatively the effects of removing some of the assumptions.

The assumption of nonrelativistic ion velocities may be invalid, if the ions are accelerated for sufficient time in the longitudinal field. On the other hand, the transverse ion motion is always nonrelativistic, since the transverse kinetic energies are on the order of $E_0\nu$ or less, i.e., in the range of 1 to 500 kilovolts for the cases discussed here. The principal effect of the longitudinal ion velocity on the electrons is equivalent to an increased ion density in the laboratory system, as can be seen by transforming into the Lorentz system in which the ions are at rest.

The assumption that the linear electron density n remains constant implies that no new electrons are added to the beam and that the orbital radius of the beam remains constant. The effect on n of a changing orbit radius could be included in the equations easily; the effect of added electrons depends on the distribution of transverse states entered by the electrons.

The intense ring currents discussed here produce not only transverse potential wells for the particles, but also corrections to the guide field force; i.e., the radial force at the center of the beam is affected by the ring current. This effect is not included here because the results do not depend on the origin of the force, only on the magnitude. The references in Sec. VIII to guide fields of 10 000 gauss should be interpreted as applying to the total force at beam center, including both the guide field and the ring current correction.

The equations could be modified to be in accordance with more realistic ion and electron radial distributions than the uniform distributions assumed here. A derivation of the form of the radial distribution resulting from collisions (in the absence of radiation damping) is given by Bennett.[12]

Transverse cross sections of many beams are elliptical, rather than circular, and the focusing forces are often different in the two transverse directions. If the guide field force is the strongest force, the radiation damping tends to preserve the transverse shape of the beam, provided the vertical and radial focusing forces vary with energy in the same manner. If the scattering forces are unimportant the beam shape remains nearly constant. In such a case the behavior of the mean square transverse radius is given correctly by the appropriate formulas of Secs. VI and VIII. If the dominant forces are the self-forces the description of an elliptical beam is more complicated. In sufficient time, however, ion-electron collisions will result in a nearly circular beam.

[11] This criterion for the single-scattering lifetime is identical to that used in references 1 and 3.

[12] W. H. Bennett, Phys. Rev. **98**, 1584 (1955). Many important properties of self-focusing beams are derived in that paper.

The energy spread of the beam is neglected in Secs. III–VIII. If the radiation damping depends only on the guide-field force, the energy change of an electron is independent of amplitude, so the equations may be applied to any group of electrons of similar energy. If the radiation depends on the self-forces the situation is quite complicated. The electrons with larger transverse amplitudes radiate more rapidly; hence an energy spread is induced in the beam. The width of the energy spread is limited by the scattering, which induces transitions between different transverse states, and by the fact that the radiation rate increases with increased energy. Furthermore, the width of the spread depends upon the manner of application of the longitudinal field.

The principal effects of an energy spread in the dominant self-force case are two-fold. First, the equations should be averaged over energy as well as over amplitude. Such a procedure would lead to several quantities similar in nature to the quantity Q_2 of Sec. VI, quantities that depend on the distribution of electrons in states of different amplitudes and energies. The second effect of an energy spread is a slight broadening of the beam. If ΔE represents the difference between the energy of a particular electron and the mean electron energy, the effective transverse force exerted on the electron is greater than that on the average electron by a radial force of magnitude $\Delta E/\mathcal{R}$, so that the transverse potential felt by the electron is displaced radially from the beam center. The adiabatic theorem implies that the amplitude of oscillation of the electron is not changed by this displacement. The beam is broadened, however, because electrons of different energy are oscillating about circular orbits of different radii. The relative change in beam width caused by the energy spread is small, even if the width of the spread is comparable to the mean electron energy, because the strong self-forces lead to a high value of the transverse force gradient within the beam.

If the actual electron energy and equilibrium energy are both below 10 Mev and the beam is appreciably neutralized, the equations should be modified to account for the fact that the transverse momentum is comparable to the longitudinal momentum [see Eq. (50)].

The maintenance of an inductive betatron field over the distances and times discussed here would be extremely difficult. On the other hand, if the acceleration took place at one or more accelerating gaps, the longitudinal field would no longer be independent of the azimuth angle. The consequent bunching of electrons and ions would greatly complicate the relation between the electron and ion beams.

Although many assumptions are made in Secs. II–VIII, the results should all be of the correct order of magnitude, provided that all instability problems can be overcome. Furthermore, several firm, semiquantitative conclusions can be made; these are listed below.

(1) Essentially complete neutralization of the electron beam may be produced in a time of less than a millisecond if a neutral gas of density about 10^{12} or 10^{13} atoms/cc is present in the tube. Considerable beam shrinkage may take place during the neutralization process, but this shrinkage is insufficient to produce a highly pinched beam satisfying the Budker equilibrium conditions. Further beam shrinkage must be effected by the radiation damping; the time required to reach equilibrium by this process is in the range of 10^{-2} to 10 seconds, depending upon the values of the various parameters.

(2) A strong guide field can be quite useful in speeding up the radiative shrinkage process, even though this field plays no role in determining the beam energy and radius in the final equilibrium state.

(3) If the beam energy is maintained close to the equilibrium energy during the radiative shrinkage process, the shrinkage time is long. The time is considerably shortened if the energy is held at a high value during most of the process. Thus one must put up with either a long shrinkage time, or a large variation in energy. This conclusion may be put in quantitative form if reference is made to Eq. (47). Since the energy increase and radiated energy are both supplied by the electric field, it is seen from this equation that, once complete neutralization is attained, it is necessary to feed in at least as much energy as the beam already has in order to reduce the radius by a factor of two.

In conclusion, the production of an electron beam satisfying the equilibrium condition of Budker would be enormously difficult. However, many of the effects discussed in the present article will occur with several types of intense beams of charged particles.

Note added in proof.—It has been pointed out to the author by Judd and Smith[13] that electron-ion pairs produced in ionization collisions may both stick in the beam if the neutralization is nearly complete, and that such ion pair capture may be an important effect. Though we are not familiar with the calculations of these authors, we have made a rough estimate of the effect, and agree with their conclusions. If the ratio of electron-to-ion density in the laboratory system is slightly greater than one, and the freed electron is given a velocity in the direction of the electron beam, the magnetic attraction of the electron to the beam center may be stronger than the electrostatic repulsion. In such a case both members of the ion pair are bound. These particles may be scattered out of the beam before attaining appreciable velocities (scattering from the beam ions should be more important than scattering from the beam electrons, because of the low relative velocity in the former case). A rough estimate reveals, however, that the newly formed ion pairs have a good chance of remaining in the beam until accelerated to velocities comparable to

[13] D. L. Judd and Lloyd Smith (private communication).

those of the other beam particles, unless the beam radius is almost as small as the Budker equilibrium radius.

This effect may be minimized if the neutral atom density is small enough so that the neutralization time is at least a tenth as long as the time necessary to reach equilibrium in the radiation stage. In such a case the results of Sec. VIII D, should still represent the approximate behavior of the beam after the neutralization fraction has become large.

Although this possibility of electron capture makes an accurate calculation of the beam behavior difficult, the effect may be useful in the production of an intense beam. In fact, appreciable electron capture may be necessary to compensate for losses. (The single-scattering loss mechanism discussed in the present work and in references 1 and 3 very likely leads to an underestimate of the rate of electron loss.)

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$LiH_3(SeO_3)_2$: New Room-Temperature Ferroelectric*

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$LiH_3(SeO_3)_2$ crystallizes in the monoclinic system and exhibits useful ferroelectric properties in the entire temperature range −196°C to 90°C. The spontaneous polarization and coercive field at room temperature are 15.0 μcoulombs/cm² and 1400 v/cm, respectively. X-ray examination reveals that the crystals belong to the space group *Pn*, with cell dimensions $a=6.25_8$A, $b=7.88_6$A, $c=5.43_3$A, $\beta=105.2°$, and two formula units per unit cell. The polar direction is perpendicular to the (001) plane. No Curie temperature could be observed. $NaH_3(SeO_3)_2$ is not isomorphous, but is ferroelectric below −79°C.

INTRODUCTION

A SURVEY of various recently-discovered ferroelectrics reveals the importance of the near-planar configuration of the glycine molecule, and of the tetrahedral ions $(NH_4)^+$ and $(SO_4)^{--}$, in ferroelectric transitions. Now we have found that hydrogen-bonded crystals containing pyramidal ions like $(SeO_3)^{--}$ also exhibit interesting phase transitions. For example, optical examination reveals that $KHSeO_3$ exhibits a beautiful transition at −39°C; however, no dielectric anomaly could be detected at this temperature. $NaH_3(SeO_3)_2$, which crystallizes in the monoclinic system and undergoes a transition at −79°C, exhibits useful ferroelectric properties from this temperature down to liquid nitrogen temperature. Detailed measurements on this crystal will be reported elsewhere. $LiH_3(SeO_3)_2$, on the other hand, is ferroelectric at room temperature and possesses many features of practical importance. The results of x-ray crystallographic, dielectric, and thermal measurements of this crystal are reported here.

PREPARATION AND PROPERTIES OF $LiH_3(SeO_3)_2$

$LiH_3(SeO_3)_2$, also described as $LiHSeO_3 \cdot H_2SeO_3$, can easily be crystallized from aqueous solution of lithium hydroxide or lithium carbonate in selenious acid in stoichiometric proportions: one mole equivalent of LiOH to two moles of H_2SeO_3. The crystals obtained are fairly stable at room temperature and do not require protective coating. X-ray measurements using single-crystal Weissenberg photographs with Cu$K\alpha$ radiation reveal that the crystal belongs to the monoclinic system, with space group *Pn* and with the cell dimensions $a=6.25_8$A, $b=7.88_6$A, $c=5.43_3$A, and $\beta=105.2°$. There are two formula units per unit cell. The crystal possesses an imperfect cleavage plane perpendicular to the *b* axis. A detailed crystal structure analysis is in progress in this laboratory; this will be followed by a neutron analysis at Brookhaven National Laboratory.

The ferroelectric direction is perpendicular to the (001) plane and is close to the $[40\bar{1}]$ direction, thus again demonstrating that in monoclinic ferroelectric crystals the polar direction can lie along any general direction perpendicular to the *b* axis, or along **b**.

Figure 1 represents the temperature variation of the small-field dielectric constant, measured at a frequency of 10 kc/sec and with a field of 5 v/cm applied along the polar direction. The dielectric constant, which has a value of 30 at room temperature, rises very rapidly above 80°C, with a corresponding increase in dielectric loss due possibly to increasing ionic conductivity. To determine whether this rise in dielectric constant might be due to a phase transition at higher temperature,

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